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τ-PSEUDOCOMPACT MAPPINGS

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Introduction. In this article we consider the problem of extending the notion of τ-pseudocompactness from spaces to continuous mappings.

Recall that a Tychonoff space X is *pseudocompact* if every continuous function on X is bounded. The product of pseudocompact spaces is not necessarily pseudocompact; however, there are conditions under which pseudocompactness is preserved. Comfort and Ross (see [1]) proved that the product of pseudocompact topological groups is a pseudocompact topological group. Generalizing this assertion, M. G. Tkachenko [2] demonstrated that the product of relatively pseudocompact subsets of topological groups is relatively pseudocompact in the product. V. V. Uspenskiĭ [3] considered a similar fact for relatively pseudocompact subsets of d-spaces. M. G. Tkachenko [4] established the most general fact in this field: If spaces X_α have countably directed lattices of d-open mappings onto Dieudonné complete spaces and if sets C_α are relatively pseudocompact in X_α, $\alpha \in A$, then the product $\prod_{\alpha\in A} C_\alpha$ is relatively pseudocompact in the product $\prod_{\alpha\in A} X_\alpha$.

B. A. Pasynkov posed the problem of further abstracting these assertions. This leads to the notion of τ-pseudocompact space [5] and to a sought generalization for τ-pseudocompact spaces.

Theorem [5]. *If topological spaces X_s, $s \in S$, have τ-directed lattices of d-open mappings onto c-τ-bounded spaces and if sets C_s are relatively τ-pseudocompact in X_s, $s \in S$, then the set $C = \prod\{C_s : s \in S\}$ is relatively τ-pseudocompact in $X = \prod\{X_s : s \in S\}$.*

The problem arises of extending the notion of τ-pseudocompactness from spaces to mappings. Translating the notion of τ-pseudocompactness to mappings, we obtain properties of τ-pseudocompact mappings similar to those of τ-pseudocompact spaces in [5].

Since every space X can be treated as a continuous mapping $f : X \to Y$ into a singleton, we derive corollaries to the multiplicativity theorems for τ-pseudocompactness for spaces which include the above results of [1–5] in particular.

1. τ-Pseudocompact and τ-compact mappings. Let X be a topological space and let τ be an infinite cardinal. A system λ is called *τ-local in X* if every point $x \in X$ possesses a neighborhood Ox such that $|\,\mathrm{St}(Ox, \lambda)| < \tau$.

Definition 1. A continuous mapping $f : X \to Y$ is *τ-pseudocompact* if for every open set O in Y, a point $y \in O$, and a τ-local open system λ in $f^{-1}O$, there is a neighborhood Oy of y such that $Oy \subset O$ and $|\,\mathrm{St}(f^{-1}Oy, \lambda)| < \tau$.

For $\tau = \omega$ the notion of τ-pseudocompact mapping coincides with the notion of o-pseudocompact mapping (see [6]).

Property 1. *Suppose that mappings $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$, and $g : X_1 \to X_2$ are continuous, the mapping g is surjective, and $f_1 = f_2 \circ g$. Then τ-pseudocompactness of f_1 implies τ-pseudocompactness of f_2.*

Proof. Suppose that the mapping $f_2 : X_2 \to Y$ is not τ-pseudocompact. Then there exist an open set O in Y, a point $y \in O$, and a τ-local open system λ_2 in the inverse image $f_2^{-1}O$ such that $\left|\mathrm{St}\left(f_2^{-1}Oy, \lambda_2\right)\right| \ge \tau$ for every neighborhood Oy of y.

Since g is continuous and surjective, the system $g^{-1}\lambda_2$ is τ-local and open in the space $g^{-1}\left(f_2^{-1}O\right) = f_1^{-1}O$. Since f_1 is τ-pseudocompact; for the system $g^{-1}\lambda_2$, there is a neighborhood O_2y of y such

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that $\left|\mathrm{St}\left(f_1^{-1}O_1y, g^{-1}\lambda_2\right)\right| < \tau$. Consequently, $\left|\mathrm{St}\left(g\left(f_1^{-1}O_1y\right), g(g^{-1}\lambda_2)\right)\right| \geq \tau$. However, we have $g\left(f_1^{-1}O_1y\right) = \left(f_2^{-1} \circ f_1\right)\left(f_1^{-1}O_1y\right) = f_2^{-1}O_1y, g(g^{-1}\lambda_2) = \lambda_2$ and $\left|\mathrm{St}\left(f_2^{-1}Oy, \lambda_2\right)\right| \geq \tau$ for every neighborhood Oy of y. This contradiction proves Property 1.

Let $f_\alpha : X_\alpha \to Y$, $\alpha \in A$, be mappings. Henceforth the fiberwise product f of the mappings $f_\alpha : X_\alpha \to Y$, $\alpha \in A$ (see [7]) is called the *product* of $f_\alpha : X_\alpha \to Y$, $\alpha \in A$, and denoted by $f = \prod_{\alpha\in A} f_\alpha$.

Thus, we have some mapping $f : X \to Y$, and

$$f^{-1}y = \prod_{\alpha\in A} f_\alpha^{-1}y \subset \prod_{\alpha\in A} X_\alpha$$

for every $y \in Y$. The following property is well known:

Property 2. *The fiberwise product of perfect mappings is perfect.*

2. Relatively τ-pseudocompact mappings.

Definition 1. Let $f : X \to Y$, $X_1 \subset X$, be a continuous mapping. A submapping $g = f|_{X_1} : X_1 \to Y$ of f is *relatively τ-pseudocompact* in f if for every open set O in Y, a point $y \in O$, and a τ-local open system λ in $f^{-1}O$, there is a neighborhood Oy of y such that $Oy \subset O$ and $|\,\mathrm{St}(g^{-1}Oy, \lambda)| < \tau$.

Remark 1. It is clear that each τ-pseudocompact mapping $f : X \to Y$ is relatively τ-pseudocompact in itself.

Property 1. *The continuous image of a relatively τ-pseudocompact mapping is relatively τ-pseudocompact.*

Proof. Assume that the following diagram commutes:

$$\begin{array}{ccc} X_1' \subset X_1 & \xrightarrow{\xi} & X_2 \supset X_2' \\ g_1\searrow \quad f_1\searrow & & \swarrow f_2 \quad \swarrow g_2 \\ & Y & \end{array}$$

The mapping $g_1 : X_1' \to Y$ is relatively τ-pseudocompact in f_1. Prove that the mapping $g_2 = \xi(g_1)$ is relatively τ-pseudocompact in f_2. Consider an open set O in Y and a point $y \in O$. Let λ_2 be a τ-local open system in the tubular neighborhood $f_2^{-1}O$. Then the system $\lambda_1 = \xi^{-1}\lambda_2$ is open and τ-local in the tubular neighborhood $f_1^{-1}O = \xi^{-1}f_2^{-1}O$. Since the mapping $g_1 : X_1' \to Y$ is relatively τ-pseudocompact in f_1, there is a neighborhood Oy of y such that $Oy \subset O$ and $\left|\mathrm{St}\left(g_1^{-1}Oy, \lambda_1\right)\right| < \tau$. Then $\left|\mathrm{St}\left(\xi\left(g_1^{-1}Oy\right), \xi_1^2\right)\right| < \tau$; consequently, $\left|\mathrm{St}\left(g_2^{-1}Oy\right), \lambda_2\right| < \tau$.

Property 2. *Suppose that a mapping $f_1 : X_1 \to Y$, $f_1 = f|_{X_1}$ is relatively τ-pseudocompact in $f : X \to Y$, $X_1 \subset X$, and $X_2 \subset X_1$. Then the mapping $f_2 = f|_{X_2} : X_2 \to Y$ is relatively τ-pseudocompact in f.*

Proof. Consider an open set O in Y, a point $y \in O$, and a τ-local open system λ in the tubular neighborhood $f^{-1}O$. Since the mapping $f_1 = f|_{X_1}$ is relatively τ-pseudocompact in f, there is a neighborhood $Oy \subset O$ of y such that $\left|\mathrm{St}\left(f_1^{-1}Oy, \lambda\right)\right| < \tau$. Since $f_2^{-1}Oy \subset f_1^{-1}Oy$, we have $\left|\mathrm{St}\left(f_2^{-1}Oy, \lambda\right)\right| < \tau$.

Property 3. *Suppose that $X_2 \subset X_1 \subset X$, where $f_2 = f|_{X_2} : X \to Y$ is a mapping, and $f : X \to Y$ is relatively τ-pseudocompact in $f_1 : X_1 \to Y$, $f_1 = f|_{X_1}$. Then f_2 is relatively τ-pseudocompact in f.*

Proof. Consider an open set O in Y, a point $y \in O$, and a τ-local open system λ in $f^{-1}O$. Then the system $\lambda \cap f_1^{-1}O$ is open and τ-local in $f_1^{-1}O$; consequently, there is a neighborhood $Oy \subset O$ of y such that $\left|\mathrm{St}\left(f_2^{-1}O, \lambda \cap f_1^{-1}O\right)\right| < \tau$. Since $f_2^{-1}O \subset f_1^{-1}O$, we also have $\left|\mathrm{St}\left(f_2^{-1}Oy, \lambda\right)\right| < \tau$.

3. c-τ-Bounded mappings.

Definition 1. A continuous mapping $f : X \to Y$ is *c-τ-bounded* if every closed relatively τ-pseudocompact submapping $g = f|_{X_1}$, where X_1 is closed in X, is a perfect mapping.

Consider some examples of c-τ-bounded mappings.

Recall [8] that $f : X \to Y$ is a *T_0-mapping* if, for arbitrary two points x and $x' \neq x$ such that $fx = fx'$, at least one of the points x and x' has a neighborhood in X that does not contain the other point, and f is *completely regular* if, for every point $x \in X$ and a closed set F in X ($F : x \notin F$), there is a neighborhood O of fx such that x and F are functionally separated in the inverse image $f^{-1}O$. A *Tychonoff mapping* is a completely regular T_0-mapping.

Recall also [8] that a mapping $bf : X \to Y$ is a *bicompactification of a mapping* $f : X \to Y$ if $X \subseteq X$, $[X] = X$, $bf|_X = f$, and bf is a bicompact (perfect) mapping. Given two bicompactifications $b_1 f : X_1 \to Y$ and $b_2 f : x_2 \to Y$ of a mapping $f : X \to Y$, we write $b_2 f \geq b_1 f$ if there is a natural mapping from X_2 into X_1. Every Tychonoff mapping $f : X \to Y$ possesses at least one Tychonoff bicompactification and among all Tychonoff bicompactifications of f there is a maximal bicompactification $\beta f : \beta_f X \to Y$.

Generalizing B. A. Pasynkov's definition of a Dieudonné tubular complete mapping [9], we arrive at the following definition:

Definition 2. A Tychonoff mapping $f : X \to Y$ is called *Dieudonné complete in the extended sense* if, for every point $x \in \beta_f X \setminus X$, there exist a neighborhood U of $(\beta f)x$ in Y and a locally finite (in $f^{-1}U$) open covering λ of the tubular neighborhood $f^{-1}U$ such that $x \notin U[\lambda]^{-1}_{(\beta f)U} \equiv \cup\{[O](\beta f)^{-1}U : O \subset \lambda\}$.

Note that the Dieudonné tubular complete mappings as well as $\mathbb{R}$-complete mappings [10] are Dieudonné complete in the extended sense.

Lemma 1. *If a mapping $f : X \to Y$ is Dieudonné complete in the extended sense, where Y is a T_1-space, then f is a c-ω-bounded mapping.*

Proof. Consider a relatively pseudocompact closed submapping $f_1 = f|_{X_1}$ of f, where X_1 is closed in X. Prove that the mapping $f_1 : X_1 \to Y$ is perfect. Take $y \in Y$. Show that the inverse image $f_1^{-1}y$ is bicompact.

Since Y is a T_1-space and the set $\{y\}$ is closed in Y, the set $f_1^{-1}y$ is closed in X_1. Since f is Dieudonné complete in the extended sense; for every $x \in \beta_f X \setminus X$ such that $(\beta f)x = y$, there exist a neighborhood O of y in Y and an open locally finite covering λ of the tubular neighborhood $f^{-1}O$ such that $x \notin \cup[\lambda]_{(\beta f)^{-1}O}$. Since the mapping $f_1 : X_1 \to Y$ is relatively pseudocompact in f, there is a neighborhood $O_1 y$ of y in O such that $\left|\mathrm{St}\left(f_1^{-1}O_1 y, \lambda\right)\right| < \omega$. Recalling that $x \notin \cup[\lambda]_{\beta_f X}$, we hence obtain

$$f_1^{-1}y \subset f_1^{-1}Oy \subset B(x) = \cup\left[\mathrm{St}\left(f_1^{-1}O_1 y, \lambda\right)\right]_{\beta_f X} \cap X_1$$
$$= \left[\cup\,\mathrm{St}\left(f_1^{-1}O_1 y, \lambda\right)\right]_{\beta_f X} \cap X_1 \not\ni x.$$

The set $B(x)$ is closed in $\beta_f X$ and $f_1^{-1}y \subset B(x)$ for every $x \in \beta_f X \setminus X$ such that $(\beta f)x = y$. Since the mapping $\beta f : \beta_f X \to Y$ is perfect, the set $(\beta f)^{-1}y$ is bicompact; moreover, $f^{-1}y \subset (\beta f)^{-1}y$. Then $B = (\cap\{B(x) = x \in \beta_f X \setminus X, (\beta f)x = y\}) \cap (\beta f)^{-1}y$ is a bicompact set such that $f_1^{-1}y \subset B \subset (\beta_f X \setminus ((\beta_f X) \setminus X)) \cap X_1 = X_1$. Thereby $f_1^{-1}y$ is bicompact, since it is a closed subset of the bicompact set B. The mapping f_1 is perfect for it is closed.

Corollary 1. *A tubular $\mathbb{R}$-complete mapping $f : X \to Y$, where Y is a T_1-space, is a c-ω-bounded mapping.*

Corollary 2. *A Dieudonné complete mapping $f : X \to Y$, where Y is a T_1-space, is a c-ω-bounded mapping.*

It is well known [11, Theorem 3.1.1] that a space X is bicompact if and only if X is pseudocompact and $\mathbb{R}$-complete.

Theorem 1. *A closed Tychonoff mapping $f : X \to Y$ is perfect if and only if f is pseudocompact* [6] *and $\mathbb{R}$-complete.*

Proof. 1. A perfect mapping is pseudocompact and $\mathbb{R}$-complete [6].

2. A pseudocompact closed $\mathbb{R}$-complete mapping is perfect (see [5]).

Theorem 2. *A mapping $f : X \to Y$ is perfect if and only if f is pseudocompact, c-ω-bounded, and closed.*

Proof. 1. A complete mapping is pseudocompact, closed, and c-ω-bounded. Indeed, suppose that X_1 is closed in X. Then $f_1 = f|_{X_1} : X_1 \to Y$ is perfect.

2. A pseudocompact closed c-ω-bounded mapping is perfect (by definition).

4. Multiplicativity of c-τ-boundedness of mappings.

Proposition 1. *Suppose $f_\alpha : X_\alpha \to Y$ are closed c-τ-bounded mappings for $\alpha \in A$. Then their fiberwise product $f : X \to Y$ is c-τ-bounded as well.*

Proof. Let $f_1 : X_1 \to Y$ be a relatively τ-pseudocompact closed submapping of f, with X_1 a closed set in $X_0 \subset \prod\{X_\alpha : \alpha \in A\}$. (Observe that $X_0 \neq \prod\{X_\alpha : \alpha \in A\}$; see [7].) Prove that f_1 is perfect.

Since the mappings $\pi_\alpha|_{X_1} : X_1 \to X_\alpha$, where $\pi_\alpha : X_0 \to X_\alpha$ is the projection, are surjective and continuous for all $\alpha \in A$; the mappings $f_\alpha|_{\pi_\alpha} X_1 : X_1 \to Y$ are relatively τ-pseudocompact in f_α for all $\alpha \in A$ (Property 1 of Section 2).

Since the set X_1 is closed in X_0 and $X_1 = \prod_{\alpha\in A} \pi_\alpha X_1$, the sets $\pi_\alpha X_1$ are closed in X_α for all $\alpha \in A$ [11]. Consequently, the mappings $f_\alpha|_{\pi_\alpha} X_1$ are closed for all $\alpha \in A$.

Since $f_\alpha : X_\alpha \to Y$ are c-ω-bounded mappings, the mappings $f_\alpha|_{\pi_\alpha X_1} : X_1 \to Y$ are perfect for all $\alpha \in A$. Then the mapping $\prod_{\alpha\in A}(f_\alpha|_{\pi_\alpha X_1}) \equiv f_1 : X_1 \to Y$ is perfect by Property 2.

5. Lattices of continuous morphisms over mappings.

Definition 1. Suppose that $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are mappings. A morphism $\varphi : f_1 \to f_2$ is called an *embedding* if the continuous mapping $\varphi : X_1 \to X_2$ is an embedding ($f_2|_{\varphi X_1} \equiv f_2$).

Remark. Here and in the sequel, it is convenient to use the same symbol φ for a morphism $\varphi : f_1 \to f_2$ and the corresponding continuous mapping $\varphi : X_1 \to X_2$, where $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$.

Definition 2. Let $f : X \to Y$ be a continuous mapping. We say that a system $L = \{\varphi_\alpha, \varphi_{\beta\alpha}; A\}$ of a directed set A, continuous surjective morphisms φ_α of f, $\varphi_\alpha : f \to f_\alpha$, where $f_\alpha : X_\alpha \to Y$, $f_\alpha = \varphi_\alpha f$, $\alpha \in A$, and continuous surjective morphisms $\varphi_{\beta\alpha} : \varphi_\beta \circ f \to \varphi_\alpha \circ f$, $\alpha, \beta \in A$, $\alpha < \beta$, is a *lattice of continuous morphisms over* $f : X \to Y$ if the following are satisfied:

(1) $\Delta = \Delta_{\alpha\in A}\varphi_\alpha : f \to \prod_{\alpha\in A} f_\alpha$ is an embedding;

(2) $\varphi_\alpha = \varphi_{\beta\alpha} \circ \varphi_\beta$, $\alpha, \beta \in A$, $\alpha < \beta$.

Recall [2] that a mapping $f : X \to Y$ is called d-open if the image of every open set in X is dense in some open set in Y.

Definition 3. The lattice L is *τ-directed* if the set A is τ-directed, and L is *d-open* if all morphisms φ_α are d-open.

Remark. To each lattice $L = \{\varphi_\alpha, \varphi_{\beta\alpha}; A\}$ of continuous morphisms over a mapping $f : X \to Y$ there corresponds the lattice $L_0 = \{\varphi_\alpha, \varphi_{\beta\alpha}; A\}$ of continuous mappings of X, where $\varphi_\alpha : X \to X_\alpha$ are continuous surjective mappings of X for all $\alpha \in A$ and $\varphi_{\beta\alpha} : \varphi_\beta X \to \varphi_\alpha X$ are mappings of the images of X under φ_β and φ_α, $\alpha, \beta \in A$.

If a lattice L is τ-directed (d-open) then the corresponding lattice L_0 is τ-directed (d-open) as well.

Definition 4. Let $\{\varphi_\alpha : f_\alpha \to g_\alpha,\ \alpha \in A\}$ be a system of morphisms. The *product $\varphi = \prod_{\alpha\in A} \varphi_\alpha$ of morphisms* is the morphism taking the product $f = \prod_{\alpha\in A} f_\alpha$ of mappings into the product $f = \prod_{\alpha\in A} g_\alpha$ of mappings such that $f = g \circ \varphi$.

6. Multiplicativity theorems for relatively τ-pseudocompact mappings.

Proposition 1. *Let $f : X \to Y$ be a continuous mapping and let $L = \{\varphi_\alpha, \varphi_{\beta\alpha}; A\}$ be a τ-directed lattice of d-open morphisms over f. A submapping $f_1 : X_1 \to y$, $X_1 \subset X$, is relatively τ-pseudocompact in f if and only if its image $\varphi_\alpha \circ f_1 = f_{1\alpha}$ is relatively τ-pseudocompact in $f_\alpha = \varphi_\alpha \circ f$ for every $\alpha \in A$.*

PROOF. 1. If $f_1 : X_1 \to Y$ is relatively τ-pseudocompact in $f : X \to Y$ then its image $f_{1\alpha} = X_{1\alpha} = \varphi_\alpha X_1 \to Y$ is relatively τ-pseudocompact in f_α by Property 1 of Section 2.

2. Suppose that $f_{1\alpha} : X_{1\alpha} \to Y$ is relatively τ-pseudocompact in f_α for each $\alpha \in A$. Assume that $f_1 : X_1 \to Y$ is not relatively τ-pseudocompact in f. Then there exist an open set O in Y, a point $y \in O$, and a τ-local system $\lambda = \{U_\gamma\}_{\gamma\in\Gamma}$ in $f^{-1}O$ constituted by elements of the base of X such that $|\Gamma| \geq \tau$ and $\left|\mathrm{St}\left(f_1^{-1}Oy, \lambda\right)\right| \geq \tau$ for every neighborhood Oy of y in O. Since the lattice I_0 is τ-directed, we have

$$\forall U_\gamma \in \lambda\ \exists W_\gamma \subset X_{\alpha 0} = \varphi_{\alpha(0)} X : \varphi^{-1}_{\alpha(0)} W_\gamma = U_\gamma.$$

We obtain the open system $\nu = \{W_\gamma\}_{\gamma\in\Gamma}$ in $X_{\alpha(0)}$ (here $\alpha(0) \in A$ is an index such that $\alpha(0) > \alpha(\gamma)$ for every $\gamma \in \Gamma$).

Since the system ν is open in $\varphi_{\alpha(0)}(f^{-1}O)$, the mapping $\varphi_{\alpha(0)}$ is d-open, and the system $\varphi^{-1}_{\alpha(0)}\nu = \lambda$ is τ-local in $f^{-1}O$; it follows that the system ν is τ-local in $\varphi_{\alpha(0)}(f^{-1}O)$ [5]. As soon as $f = f_{\alpha[0]} \circ \varphi_{\alpha(0)}$, we obtain $f^{-1}O = (f_{\alpha(0)} \circ \varphi_{\alpha(0)})^{-1}O = \varphi^{-1}_{\alpha(0)}\left(f^{-1}_{\alpha(0)}O\right)$; i.e., the system ν is τ-local in $f^{-1}_{\alpha(0)}O$. Since the mapping $f_{\alpha(0)} : X_{\alpha(0)} \to Y$ is relatively τ-pseudocompact in $f_{\alpha(0)}$, for $y \in O$ and the system ν there is a neighborhood Oy of y such that $Oy \subset O$ and $\left|\mathrm{St}\left(f^{-1}_{1\alpha(0)}Oy, \nu\right)\right| < \tau$; consequently, $\left|\mathrm{St}\left(f_1^{-1}Oy, \lambda\right)\right| < \tau$. This contradiction proves the proposition.

Lemma 1. *Suppose that a mapping $f : X \to Y$ is d-open and $X_1 \subset X$. Then the submapping $f_1 = f|_{X_1} : X_1 \to Y$ is d-open.*

PROOF. It is well known [2] that a mapping $f : X \to Y$ is d-open if and only if $[f^{-1}V] = f^{-1}[V]$ for every open set V in Y. Consider an open set V in Y. We have $f_1^{-1}[V]_{X_1} = f^{-1}[V] \cap X_1 = [f^{-1}V] \cap X_1 = \left[f_1^{-1}V\right]_{X_1}$. Consequently, the mapping $f_1 : X_1 \to Y$ is d-open.

Lemma 2. *Suppose that a mapping $f : X \to Y$ is c-τ-bounded and X_1 is closed in X. Then the submapping $f_1 = f|_{X_1} : X_1 \to Y$ is c-τ-bounded.*

PROOF. Let $f_2 : X_2 \to Y$, where X_2 is closed in X_1, be a mapping relatively τ-pseudocompact in f_1 and closed. We have $f_2 = f_1|_{X_2} = f|_{X_2}$. Since X_2 is closed in X_1 and X_1 is closed in X, X_2 is closed in X [11]. Since $X_2 \subset X_1$, by Property 3 of Section 2 the mapping $f_2 : X_2 \to Y$ is relatively τ-pseudocompact in f.

In view of c-τ-boundedness of $f : X \to Y$, the mapping $f_2 : X_2 \to Y$ is perfect.

Theorem 1. *If mappings $f_1^s : X_1^s \to Y$ are closed, c-τ-bounded, and relatively τ-pseudocompact in $f^s : X^s \to Y$, where $f_1^s = f^s|_{X_1^s}$ and $X_1^s \subset X^s$, $s \in S$, then the mapping $f_1 = \prod_{s\in S} f_1^s$ is relatively τ-pseudocompact in $f = \prod_{s\in S} f^s$.*

PROOF. Consider an open set O in Y, a point $y \in O$, and an open τ-local system $\lambda = \{U_\alpha\}_{\alpha\in A}$ in $f^{-1}O$.

The mapping f_1^s is perfect for every $s \in S$. Hence, the mapping $f_1 = \prod_{s\in S} f_1^s$ is perfect. Consequently, the inverse image $f_1^{-1}y$ of y is bicompact.

Since the system λ is τ-local in $f^{-1}O$, every point $x \in f_1^{-1}y$ has a neighborhood Ox in $f^{-1}O$ such that $|\,\mathrm{St}(\lambda, Ox)| < \tau$. Since $f_1^{-1}y$ is bicompact, from the open covering $\mu = \left\{Ox \wedge f_1^{-1}y,\ x \in f^{-1}y\right\}$ of $f_1^{-1}y$ we can extract a finite subcovering μ' such that $f^{-1}y \subset \cup\mu'$. Then there exist $x_1, \ldots, x_k \in f^{-1}y$ such that $f^{-1}y \subset \bigcup_{j=1}^k Ox_j = V$. Since f_1 is closed, there is a neighborhood Oy of y such that $f_1^{-1}Oy \subset V$. Thus, we obtain $|\,\mathrm{St}(\lambda, V)| < \tau$; consequently, $\left|\mathrm{St}\left(\lambda, f_1^{-1}Oy\right)\right| < \tau$. Hence, f_1 is relatively τ-pseudocompact in f.

Theorem 2. *Suppose that mappings $f^s : X^s \to Y$ are closed and over them there are τ-directed lattices $L^s = \{\varphi_{\alpha(s)}, \varphi_{\beta(s)\alpha(s)}; A(s)\}$, $s \in S$, of d-open morphisms onto c-τ-bounded mappings $f_{\alpha(s)}$: $f_{\alpha(s)} = \varphi_{\alpha(s)} \circ f^s$, $s \in S$, $\alpha(s) \in A(s)$. Suppose also that the mappings $f_1^s = f^s|_{X_1^s}$, $f_1^s : X_1^s \to Y$, where X_1^s is closed in X^s, are closed and relatively τ-pseudocompact in f^s, $s \in S$. Then the product $f_1 = \prod_{s\in S} f_1^s$ is relatively τ-pseudocompact in $f = \prod_{s\in S} f^s$.*

Proof. By Proposition 1 of Section 6, the mapping $f^1_{\alpha(s)} = \varphi_{\alpha(s)} \circ f_1^s$ is relatively τ-pseudocompact in $f_{\alpha(s)} = \varphi_{\alpha(s)} \circ f^s$ for every $\alpha(s) \in A(s)$.

Take $A \in \prod_{s\in S} A(s)$. Consider the lattice $L = \{\varphi_\alpha, \varphi_{\beta\alpha}; A\}$ over f. Here

$$\varphi_\alpha = \prod_{s\in S} \varphi_{\alpha(s)},\ \varphi_\alpha : f \to f_\alpha, \quad f_\alpha = \prod_{s\in S} f_{\alpha(s)}, \quad \alpha \in A,\ s \in S,$$

$$\varphi_{\beta\alpha} = \prod_{s\in S} \varphi_{\beta(s)\alpha(s)},\ \beta > \alpha,\ \beta \in A, \quad \varphi_{\beta\alpha} : f \to f_\alpha.$$

We order A as follows: given $\alpha = \{\alpha(s) : s \in S\}$ and $\beta = \{\beta(s) : s \in S\}$, with $\alpha(s), \beta(s) \in A(s)$, $s \in S$, we put $\beta > \alpha$ if $\beta(s) > \alpha(s)$ for every $s \in S$. The set A with this order is τ-directed, since the set $A(s)$ is τ-directed for every $s \in S$.

Since the morphisms $\varphi_{\alpha(s)}$ are d-open for all $s \in S$, the products $\varphi_\alpha = \prod_{s\in S} \varphi_{\alpha(s)}$ are d-open as products of d-open morphisms. Moreover, since $\varphi_{\alpha(s)} = \varphi_{\beta(s)\alpha(s)} \circ \varphi_{\beta(s)}$ for every $s \in S$, we have $\varphi_\alpha = \varphi_{\beta\alpha} \circ \varphi_\beta$, $\beta > \alpha$, $\beta, \alpha \in A$.

Since $\Delta\{\varphi_{\alpha(s)} : \alpha(s) \in A(s)\} : f_s \to \prod_{\alpha(s)\in A(s)} f_{\alpha(s)}$ is an embedding for every $s \in S$; therefore, $\Delta\{\varphi_\alpha : \alpha \in A\} : f \to \prod_{\alpha\in A} f_\alpha$ is an embedding too.

Thus, L is a τ-directed lattice of d-open morphisms over the mapping $f = \prod_{s\in S} f^s$ and the mappings $f_1 = \prod_{s\in S} f_1^s$ are relatively τ-pseudocompact in the product $f_\alpha = \prod_{s\in S} f_{\alpha(s)}$ by Theorem 1 of Section 6.

By Proposition 1 of Section 6, the mapping f_1 is relatively τ-pseudocompact in f.

As a consequence of the Theorem 2 for $\tau = \omega$, we obtain the following assertion:

Theorem 3. *Suppose that, over closed mappings $f^s : X^s \to Y$, there are countably directed lattices of d-open morphisms onto mappings Dieudonné complete in the extended sense* (*in particular, on Dieudonné complete or $\mathbb{R}$-complete mappings*) *and the mappings $f^s : X^s \to Y$ and $f_1^s = f^s|_{X_1^s}$, $X_1^s \subset X^s$ are relatively pseudocompact in f_s, $s \in S$. Then $f_1 = \prod_{s\in S} f_1^s$ is relatively pseudocompact in $f = \prod_{s\in S} f^s$.*

7. Corollaries to the multiplicativity theorem of τ-pseudocompactness for spaces. Every space X can be viewed as a continuous mapping $f : X \to Y$ into a singleton. Since f is closed and the space $Y = \{y\}$ is locally bicompact, we obtain the following corollaries to the multiplicativity theorems of τ-pseudocompactness for spaces [5].

7.1. c-τ-Bounded spaces. Recall [5] that a set $B \subset X$ is relatively *τ-pseudocompact* in X if $|\operatorname{St}(\lambda, B)| < \tau$ for every τ-local open system λ in X.

For $\tau = \omega$, relative τ-pseudocompactness of a set B in a Tychonoff space X is equivalent to its boundedness or relative pseudocompactness [2], i.e., boundedness of every continuous function $\varphi = X \to \mathbb{R}$ on B.

From the definition of relative τ-pseudocompactness we derive the following properties:

Property 1. *If a mapping $f : X \to Y$ is continuous and a set B is relatively τ-pseudocompact in X then the image $f(B)$ is relatively τ-pseudocompact in Y.*

Property 2. *If a set B is relatively τ-pseudocompact in X then its closure $[B]$ is relatively τ-pseudocompact in X.*

Property 3. *If B is a relatively τ-pseudocompact subset of a subspace Y of a space X then B is relatively τ-pseudocompact in X.*

Definition 1. A space X is *c-τ-bounded* if the closure of every relatively τ-pseudocompact subset in X is bicompact.

Remark 1. A closed subspace Y of a c-τ-bounded space X is c-τ-bounded.

Proposition 1. *The class of c-τ-bounded spaces is multiplicative.*

Proof. Suppose that X_α are c-τ-bounded spaces for all $\alpha \in A$ and B is a relatively τ-pseudocompact set in $X = \prod\{X_\alpha : \alpha \in A\}$. For every $\alpha \in A$, the set $\pi_\alpha B$, where π_α is the projection of X onto X_α, is relatively τ-pseudocompact in X_α. Therefore, the closure $[\pi_\alpha B]$ is bicompact. Then the product $C = \prod\{[\pi_\alpha B] : \alpha \in A\}$ is bicompact too, and $[B] \subset C$. Consequently, the closure $[B]$ is bicompact.

Lemma 1. *A mapping f of a c-bounded space X into a singleton $\{y\}$ is c-τ-bounded.*

Proof. Given a submapping $f_1 = f|_{X_1}$, where X_1 is closed in X, for a point y we have $\left|\mathrm{St}\left(f_1^{-1}y, \lambda\right)\right| < \tau$. The space X_1 is relatively τ-pseudocompact in X; moreover, since X_1 is closed in X, the space $[X_1] = X_1$ is bicompact. Consequently, the mapping f_1 is perfect.

Thus, c-τ-bounded spaces are particular instances of c-τ-bounded mappings.

7.2. c-ω-Bounded spaces. We indicate some classes of c-ω-bounded spaces.

Definition 2 (B. A. Pasynkov). A Tychonoff space X is called *Dieudonné complete in the extended sense* if, for every point $x \in \beta X \setminus X$, there is an open locally finite covering ω of X such that $x \notin \cup[\omega]_{\beta X} \equiv \cup\{[O]_{\beta X} : O \in \omega\}$.

Proposition 2. *A space Dieudonné complete in the extended sense is c-ω-bounded* [5].

Corollary. *Dieudonné complete spaces are c-ω-bounded.*

Lemma 2. *If a space X is normal and a set B is relatively pseudocompact in X and closed in X then the space B is countably compact* [5].

Corollary 1. *The closure of a relatively pseudocompact subset B of a normal space X is countably compact.*

Recall that a space X is called *isocompact* if every countably compact closed subspace of X is bicompact.

Corollary 2. *A normal isocompact space is c-ω-bounded.*

Corollary 3. *A closed subspace of the product of normal isocompact spaces is c-ω-bounded.*

Corollary 4. *A closed subspace of the product of normal weakly paracompact spaces is c-ω-bounded.*

Other examples of c-ω-bounded spaces can be found in [4, 12].

Assertion 6 [4]. *Suppose that a space X condenses on a metrizable space and a set B is bounded in X. Then $[B]_X$ is a compact set.*

Hence, spaces condensing on metrizable spaces are c-ω-bounded.

Definition [4]. A subgroup H of a topological group G is called *admissible* if there is a sequence $\{U_n : n \in \mathbb{N}\}$ of open neighborhoods of the identity in G such that $U_n^{-1} = U_n$, $U_{n+1}^3 \subseteq U_n$ for every $n \in \mathbb{N}$, and $H = \cap\{U_n : n \in \mathbb{N}\}$.

It is well known [4] that every admissible subgroup of a group G is closed in G.

Assertion 3 [4]. *Let H be an admissible subgroup of a topological group G. Then G/H condenses on a metrizable space.*

Corollary. *If H is an admissible subgroup of a topological group G then the space G/H is c-ω-bounded.*

In [12] a μ-space is defined as follows:

Definition [12]. If A is a compact set for every bounded $A \subset X$ then X is a μ-space.

It is clear that the classes of μ-spaces and c-ω-bounded spaces coincide. Some properties of μ-spaces are listed in [12]:

Theorem 1 [12]. *If X is a μ-space and Y is closed in X then the canonical mapping $i_Y : F(Y) \to F(X)$ is a k-mapping (i.e., $i^{-1}y(\Phi)$ is a compact set for every compact set $\Phi \subset F(X)$).*

We have considered several classes of c-ω-bounded spaces. We can construct some classes of corresponding c-ω-bounded mappings.

Recall [12] that a mapping $f : X \to Y$ is *d-open* if, for every open set $O \subset X$, there is an open set V in Y such that $fO \subset V \subset [fO]$.

Property 1. *The product of d-open mappings is d-open [2].*

Theorem 1 [5]. *If topological spaces X_s, $s \in S$, have τ-directed lattices of d-open mappings onto c-τ-bounded spaces and if sets C_s are relatively τ-pseudocompact in X_s, $s \in S$, then the set $C = \prod\{C_s : s \in S\}$ is relatively τ-pseudocompact in $X = \prod\{X_s : s \in S\}$.*

Proof. The assertion follows from Theorem 1 of Section 1 and Lemma 1.

Using the above examples of c-ω-bounded spaces, we obtain the following consequences of the above theorem for $\tau = \omega$:

Theorem 2. *Assume that topological spaces X_s, $s \in S$, have countably directed lattices of d-open mappings onto c-τ-bounded spaces, in particular, onto*

(1) *spaces Dieudonné complete in the extended sense;*
(2) *Dieudonné complete spaces;*
(3) *normal isocompact spaces;*
(4) *closed subspaces of normal isocompact spaces;*
(5) *closed subspaces of normal weakly paracompact spaces;*
(6) *spaces condensing on metrizable spaces;*
(7) *quotient spaces of topological groups by admissible subgroups of these groups;*
(8) *free topological groups over μ-spaces.*

Assume further that sets C_s are relatively pseudocompact in X_s, $s \in S$. Then the set $C = \prod\{C_s : s \in S\}$ is relatively pseudocompact in $X = \prod\{X_s : s \in S\}$.

Corollary 1 [1]. *The product of pseudocompact topological groups is a pseudocompact topological group.*

Corollary 2 [3]. *A subproduct of relatively pseudocompact subsets of d-spaces is relatively pseudocompact in the product.*

Remark. Assertion (2) of Theorem 2 was proven in [4].

Thus, in this article we have considered the notion of τ-pseudocompact mapping, some properties of such a mapping similar to those of a pseudocompact space, and consequences of the above assertions for spaces.

All problems were posed by B. A. Pasynkov.

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