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Matiyasevich Formula for Chromatic and Flow Polynomials and Feynman Amplitudes

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Abstract—Matiyasevich formula which expresses the chromatic polynomial of an arbitrary graph through a linear combination of flow polynomials of subgraphs of the original graph is generalized by using the Feynman amplitudes technique. The article presents a formula expressing a flow polynomial through a linear combination of chromatic polynomials of constricted graphs. This proof is obtained by using the Feynman amplitudes technique. A simple proof of Matiyasevich formula and its consequences are derived by using the same technique.

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1. INTRODUCTION

Our work is connected with two directions that have been developing independently for a long time: the theory of Feynman amplitudes related to mathematical physics and the theory of Tutte polynomials, in particular, chromatic and flow polynomials related to combinatorics.

As is well known, the notion of the chromatic polynomial for a graph was introduced by George Birkhoff in 1912 in connection with the 4 color problem. Next William Tutte considered in details the relationship between the chromatic polynomial and its dual object the flow polynomial.

In the case of a planar connected graph the normalized chromatic polynomial (divided by the number of colors) coincides with the flow polynomial for the dual graph. The flow polynomial counts the number of nonzero flows everywhere in $\mathbb{Z}_m$ (here $\mathbb{Z}_m$ is ring of integers modulo m). Tutte's hypotheses about the existence of a nonzero 5-flow is formulated for all graphs (except trivial ones containing bridges), in contrast, to the problem of 4-colors which is true only for planar graphs (i.e., graphs that do not contain subgraphs homeomorphic to K_5 or K_{33}). Thus, the 5-flow hypothesis is more universal than the 4-colour problem. So far it has been possible to prove the statement about the existence of an everywhere non-zero 6-flow [1].

The 3-flow hypothesis is no less intriguing. It says that there is a non-zero 3-flow for any 4-connected graph. A relatively recent attempt to prove it [2] was completed by asserting its validity for a 6-connected graph.

In this paper we establish a connection between flow polynomials, chromatic polynomials and vacuum Feynman amplitudes (FA). Using the FA technique we generalize the Matiyasevich formula expressing the chromatic polynomial of an arbitrary graph through a linear combination of flow polynomials of subgraphs of the original graph. Also, we show that using the FA technique gives us possibility to derive both the Matiyasevich formula and the dual formula expressing the flow polynomial through the chromatic one.

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The history of connection between FA and Tutte polynomials is quite recent. In December 1997 Kontsevich, speaking at the Gelfand seminar at Rutgers University about FA, hypothesized that the formal analogue of the so-called α-representation (known in the theory of Feynman amplitudes) is a polynomial of the number of elements of this field in the case of a finite field. After some time this hypothesis was refuted. In our work [3] we have presented the correct version of Kontsevich's formula. It turned out that the α-representation gives the flow polynomial of the graph. It is important to note that the Fourier transform technique (it is characteristic of FA theory) is used to derive the α-representation. The same technique turned out to be useful in [4] when we derive explicit formulas for chromatic polynomials of some series-parallel graphs.

In the late 70s (1977) Yu. V. Matiyasevich revealed a formula expressing the chromatic polynomial of an arbitrary graph through a linear combination of flow polynomials of subgraphs of the original graph [5]. Fourier transform technique over a ring modulo m was implicitly used to derive this formula.

The goal of our paper is to show that the Matiyasevich formula as well as its dual formula expressing the flow polynomial through a linear combination of chromatic polynomials of subgraphs of the graph G is a consequence of the connection between FA in coordinate and momentum spaces.

This article has the following structure. The Section 2 introduces the basic definitions and designations. Next the concepts of Feynman amplitude in coordinate space and momentum space are considered. In Section 4 the Fourier transform and the main theorems are introduced to help to obtain the main result of this work, namely, the expression of a flow polynomial through a linear combination of chromatic ones. In Section 5 we show that the Matiyasevich formula is a simple consequence of the formula that represents the connection between FA in coordinate and momentum representations. Also, we discuss other variants of the formula expressing the flow polynomial through a linear combination of chromatic.

2. DESIGNATIONS

Let $\mathbb{Z}_m$ is a ring of integers modulo m; G is a multigraph with a set of vertices V and set of edges E (with loops and multiple edges).

Everywhere except the beginning the subsection (5.1) we assume that the original graph G is connected, although, in essence, this does not affect the generality of reasoning. If we are not talking about the original graph, but about its minor H, then we will denote the set of its vertices and edges as $E(H)$ and $V(H)$. In the case of the original graph G we use E and V without arguments to denote its set of vertices and edges.

$H(W)$ is a subgraph induced by the set of vertices W, where $W \subseteq V$ [6];

$P_G(m)$ is the chromatic polynomial of graph G;

$P'_G(m)$ is a reduced (divided by m) chromatic polynomial;

$F_G(m)$ is the flow polynomial of graph G;

V' is a set of vertices; $V \setminus \{v_0\}$, where v_0 is some given vertex (nothing depends on its choice);

$i(\ell), o(\ell)$ are two ends of the edge ℓ, "beginning" and "end" in the oriented case;

$f(\cdot)$, $\widetilde{f}(\cdot)$ are so-called propagators in momentum and coordinate representations, even functions from $\mathbb{Z}_m \to \mathbb{C}$;

$\mathcal{F}_{G,f}(m)$ is FA in the momentum representation in the vacuum case with the propagator f (over the ring $\mathbb{Z}_m$), for more details see section (3.2);

$\widetilde{\mathcal{F}}_{G,\widetilde{f}}(m)$ is FA in the coordinate representation in the vacuum case with the propagator $\widetilde{f}$ (over the ring $\mathbb{Z}_m$), for more details see section (3.1).

Since the ring $\mathbb{Z}_m$ is fixed, we will mainly use abbreviations: $\mathcal{F}_{G,f}$ and $\widetilde{\mathcal{F}}_{G,\widetilde{f}}$.

All variables x_v, k_ℓ, y, z take values in $\mathbb{Z}_m$, and summation is carried out in this area, unless otherwise is specified.

3. CHROMATIC AND FLOW POLYNOMIALS AND THEIR CONNECTION WITH FEYNMAN AMPLITUDES

3.1. FA in Coordinate Representation and its Relation To the Chromatic Polynomial

In this section, we introduce a definition of vacuum FA in coordinate space, which is close to the definition of partition function of the Potts model (see [8] and other reference in this subsection). All definitions are similar to the case of real FA [7] only with the replacement of the field $\mathbb{R}$ by $\mathbb{Z}_m$ and, accordingly, with the replacement of integration by summation with some other technical details.

FA in coordinate representation with an arbitrary propagator $\widetilde{f}$ for a connected graph G has the form

$$\widetilde{\mathcal{F}}_{G,\widetilde{f}}(m) = \frac{1}{m} \sum_{x_v, v\in V} \prod_{\ell\in E} \widetilde{f}(x_{i(\ell)} - x_{o(\ell)}). \tag{1}$$

We assume that $\widetilde{f}$ is an even function and therefore the orientation of the edges is not important, when we define $\widetilde{\mathcal{F}}_{G\widetilde{f}}$ (because $\widetilde{f}(x_1 - x_2) = \widetilde{f}(x_2 - x_1)$).

The partition function of the Potts model, which is a special case of $\widetilde{\mathcal{F}}_{G,\widetilde{f}}$ with propagator $\widetilde{f}$ takes only two values $\widetilde{f}(0) = \alpha$ and $\widetilde{f}(x) = \beta$, when $x \neq 0$ is considered in Biggs' book [9] (see also "a bad coloring polynomial" in the book [10] of Dominic Welsh). In order to work with such partition functions (they are also called Biggs interactions) a special technique has been developed that allows to express the partition function of one model through the partition function of another model.

The multiplier $\frac{1}{m}$ in the definition of (1) disappears if the summation is carried out over the set V' and not over the set V.

Proposition 1. To find $\widetilde{\mathcal{F}}_{G,\widetilde{f}}$ can be used following formula

$$\sum_{x_v, v\in V'} \prod_{\ell\in E} f(x_{i(\ell)} - x_{o(\ell)}), \tag{2}$$

where $x_{v_0} = 0$.

Proof. Let us make a replacement in the expression (1): $y_v = x_v - x_{v_0}$ for all $V \setminus \{v_0\}$. Then, $x_v - x_{v'} = y_v - y_{v'}$, when $v, v' \neq v_0$. Otherwise $x_v - x_{v_0} = y_v$.

We get that the summable expression does not depend on x_{v_0}. Summation over a variable v_0 in (1) gives us expression in m times larger than the expression (2) and so m in the numerator and denominator is reduced. □

Remark 1. The direct analogue of the vacuum FA over a real field is given by the formula (2) (which in our case is equivalent to the formula (1)). Integration (summation) is carried out over all vertices except one and the result does not depend on the selected vertex.

Let us consider how the chromatic polynomial is related to FA in coordinate space. Let $\widetilde{f} = \Delta$, where

$$\Delta(z) = \begin{cases} 1, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases} \tag{3}$$

The propagator $\Delta(\cdot)$ is a trivial norm $||\cdot||$ on $\mathbb{Z}_m$.

If we take propagator Δ formula (1) looks like

$$\widetilde{\mathcal{F}}_{G,\Delta} = \sum_{x_v, v\in V'} \prod_{\ell\in E(G)} \Delta(x_{i(\ell)} - x_{o(\ell)}).$$

Note that the chromatic polynomial is the number of proper colorings in m colors. Recall that the coloring is called proper if all vertices are colored in different colors. It follows from the Proposition 1 that the number of correct colorings for a connected graph is a multiple of m (nothing depends on the choice of the color of the first vertex). Thus, the reduced chromatic polynomial of a connected graph can be defined by the formula $P'(G, m) = \frac{P(G,m)}{m}$ is an integer function.

If we match the number x_v to the colors in which the vertex v is colored then the product $\prod_{\ell\in E}\Delta(x_{i(\ell)}-x_{o(\ell)})$ is zero for incorrect colorings and turns into one for correct ones. When we sum up, we get

$$\widetilde{\mathcal{F}}_{G,\Delta}=\sum_{x_v,v\in V'}\prod_{\ell\in E}\Delta(x_{i(\ell)}-x_{o(\ell)})=P'_G(m).$$

3.2. FA in Momentum Representation and its Relation To the Flow Polynomial

In this section we consider the momentum representation of FA in the vacuum case. We need the orientation of the edges (although it does not affect the result) to set the momentum representation, unlike the coordinate representation. The variable $k_\ell\in\mathbb{Z}_m$ is on each oriented edge (note that in the coordinate representation we mapped variables to vertices not edges). We consider the elements of the incidence matrix to account the orientation

$$\varepsilon_{\ell v}=\begin{cases}+1, & \text{if } f(\ell){=}v;\\ -1, & \text{if } i(\ell){=}\ v;\\ 0, & \text{if } \ell \text{ is nonincident to } v.\end{cases}$$

We must consider the zero flow condition rule for graph vertices

$$\sum_{\ell}\varepsilon_{\ell v}k_\ell=0,\quad \text{for any vertice } v\in V. \tag{4}$$

The Feynman amplitude in the momentum representation is determined by the formula

$$\mathcal{F}_{G,f}=\sum_{k_\ell,\ell\in E}\prod_{\ell\in E}f(k_\ell)\prod_{v}\delta(\sum_{\ell}\varepsilon_{\ell v}k_\ell). \tag{5}$$

Here, δ is Kronecker's symbol:

$$\delta(z)=\begin{cases}0, & \text{if } z\neq 0;\\ 1, & \text{if } z=0.\end{cases}$$

Note that the result of summing (5) does not depend on the orientation of the edges. Indeed, if a particular edge e is oriented in the opposite direction and at the same time we replace the variable k_ℓ with $-k_\ell$, then the summable expression will not change (recall that f is an even function). Since summation is over all possible values of k_ℓ, so the replacement of variables does not affect the summation result.

Recall that the flow polynomial of the variable m is the number of nowhere-zero flows with the values in $\mathbb{Z}_m$:

$$\mathcal{F}_{G,\Delta}=F_G(m). \tag{6}$$

Therefore, when we choose a specific propagator $f=\Delta$ (see the formula (3)) we get a flow polynomial $F_G(m)$.

There is a way to take into account (5) when we calculate (4) and to get rid of unnecessary variables. Since the graph G is connected, it has a spanning tree T (a tree that includes all vertices). Edges that are not included in the spanning tree are called chordal. For variables that are mapped to these edges we use the term chord momenta. Their set is denoted by $\underline{k}$.

There is an algorithm to express all other (non-chordal) variables through chordal momenta. Each tree has a leaf w and an incident edge e. Since w is incident only to the chord edges of the graph G except the edge e. The variable k_e can be expressed using chord momenta from the condition (4) for $v=w$. We denote this expression by $q_e(\underline{k})$. Next by removing this edge e from the tree T we get a new tree T'. We can apply the same procedure to it. The new expression for the leaf variable $q_{e'}$ (through e' we denote the edge incident to the leaf in the tree T') can be included in q_e through other variables. At the end everything is expressed in terms of variables $\underline{k}$. We will repeat this procedure $|E(T)|$ times (as many times as there are edges in the spanning tree). Since $|E(T)|=|V|-1$ we will take into account all the conditions of (4) except one. But this last condition turns into the identity $0=0$. Indeed, adding up all the equalities (4) we will take into account each edge with both a plus sign and a minus sign (since each edge is both incoming and outgoing for some vertices). So when adding identities (4) we get the equality $0 = 0$ the system is linearly dependent.

So, instead of summing by $|E|$ variables we will sum by $\underline{k}$ variables, there are $|E| - |V| + 1$ pieces of them. The expression that we sum is

$$I(m, \underline{k}) = \prod_{e \in E(T)} f(q_e(\underline{k})) \prod_{\ell \in E(G) \setminus E(T)} f(k_\ell).$$

Note that the result of summation does not depend on the choice of T. As a result of summation, we will get the value of the Feynman amplitude (5) in the determination of which the spanning tree T did not participate.

4. THE CONNECTION BETWEEN FA IN THE MOMENTUM AND COORDINATE SPACE

4.1. Fourier Transform for $\mathbb{Z}_m$

The function $\widehat{g}(z)$ is called the Fourier transform of the function $g(y)$ on the group $\mathbb{Z}_m$. It is defined by the formula

$$\widehat{g}(z) = \sum_{y \in \mathbb{Z}_m} e^{\frac{2\pi i y z}{m}} g(y).$$

Here, the symbol $\widehat{}$ means the result of the Fourier transform. Let us recall a well-known fact about the Fourier transform on $\mathbb{Z}_m$ (see, for example, [11]).

Lemma 1. *Let $\widehat{f}$ be Fourier transform from even function f. Then, $\widehat{\widehat{f}} = m\, f$*

Proof. Note that we obtain an even function $\widehat{f}$ as a result of the Fourier transform of an even function f. Indeed, we made a replacement $-k = z$ in the formula of Fourier transform the above.

$$\widehat{f}(-k) = \sum_{x \in \mathbb{Z}_m} e^{\frac{-2\pi i k x}{m}} f(x) = \sum_{z \in \mathbb{Z}_m} e^{\frac{2\pi i k z}{m}} f(-z) = \sum_{z \in \mathbb{Z}_m} e^{\frac{2\pi i k z}{m}} f(z) = \widehat{f}(k).$$

Thus, we need to prove that

$$\sum_{k \in \mathbb{Z}_m} e^{\frac{-2\pi i k y}{m}} \widehat{f}(k) = m\, f(y).$$

We have

$$\sum_{k \in \mathbb{Z}_m} e^{\frac{-2\pi i k y}{m}} \widehat{f}(k) = \sum_{x,k \in \mathbb{Z}_m} e^{2\pi i \frac{(kx - ky)}{m}} f(x) = \sum_{x \in \mathbb{Z}_m} f(x) \sum_{k \in \mathbb{Z}_m} e^{2\pi i \frac{k(x-y)}{m}}. \tag{7}$$

The last sum can be interpreted as Fourier transform of one. It can be calculated directly by considering the trivial case $k = 0$ and the sum of the geometric progression at $k \neq 0$. The result is a "delta function" multiplied by m:

$$\sum_{z \in \mathbb{Z}_m} e^{2\pi i \frac{kz}{m}} = m\, \delta(k). \tag{8}$$

Therefore, the right part in the formula (7) can be written as

$$\sum_{x \in \mathbb{Z}_m} f(x) \delta(x - y)\, m = m\, f(y).$$

□

We obtain the following equality as a consequence

$$m^{|E|} \mathcal{F}_{G,f} = \mathcal{F}_{G,\widehat{\widehat{f}}}. \tag{9}$$

Theorem 1 (about the connection of vacuum FA in coordinate and momentum representation). *There are relations*

$$\mathcal{F}_{G,f} \equiv \frac{1}{m^{|V|-1}} \widetilde{\mathcal{F}}_{G,\widehat{f}}\,, \tag{10}$$

$$\widetilde{\mathcal{F}}_{G,f} \equiv \frac{1}{m^{|E|-|V|+1}} \mathcal{F}_{G,\widehat{f}}\,. \tag{11}$$

Proof. Let us prove the statement (10). Recall that the FA in the momentum representation has the form (5) containing δ-functions. Let us substitute representation (8) for the delta-function into the formula (5)

$$m^{|V|}\mathcal{F}_{G,f} = \sum_{k_\ell} \prod_{\ell\in E(G)} f(k_\ell) \sum_{x_v, v\in V} e^{2\pi i \sum_v \sum_\ell \frac{\varepsilon_{\ell v} k_\ell x_v}{m}}.$$

Let us change the order of summation

$$\sum_{x_v,v\in V} \sum_{k_\ell,\ell\in E} \prod_{\ell\in E(G)} f(k_\ell) \prod_{\ell\in E(G)} e^{2\pi i \sum_v \frac{\varepsilon_{\ell v} k_\ell x_v}{m}} = \sum_{x_v,v\in V} \prod_{\ell\in E(G)} \left(\sum_{k_\ell} f(k_\ell) e^{2\pi i \sum_v \frac{\varepsilon_{\ell v} k_\ell x_v}{m}} \right)$$

$$= \sum_{x_v,v\in V} \prod_{\ell\in E(G)} \left(\sum_{k_\ell} f(k_\ell) e^{2\pi i \frac{k_\ell (x_{i(\ell)} - x_{o(\ell)})}{m}} \right).$$

Then, we obtain

$$\mathcal{F}_{G,f} = \frac{1}{m^{|V|}} \sum_{x_v,v\in V} \prod_{\ell\in E} \widehat{f}(x_{i(\ell)} - x_{o(\ell)}) = \frac{\sum_{x_v,v\in V'} \prod_{\ell\in E} \widehat{f}(x_{i(\ell)} - x_{o(\ell)})}{m^{|V|-1}}.$$

The formula (10) is proved.

Let us prove that (11) is true. According to (10) we have

$$\mathcal{F}_{G,\widehat{f}} = \frac{1}{m^{|V|-1}} \widetilde{\mathcal{F}}_{G,\widehat{\widehat{f}}}.$$

The equality (9) allows us to replace the propagator $\widehat{\widehat{f}}$ in the right part with f. As a result, we get

$$\mathcal{F}_{G,\widehat{f}} = m^{|E|} \widetilde{\mathcal{F}}_{G,f} \frac{1}{m^{|V|-1}},$$

so

$$\widetilde{\mathcal{F}}_{G,f} = \frac{1}{m^{|E|-|V|+1}} \mathcal{F}_{G,\widehat{f}}.$$

$\square$

Further we will use the proven theorem in the case of the propagator $f = \Delta$. Let us calculate the Fourier transform of the propagator Δ:

$$\widehat{\Delta}(k) = \sum_{x\in\mathbb{Z}_m} e^{\frac{2\pi i k x}{m}} \Delta(x). \tag{12}$$

We add and subtract 1 in the right part. Then, the expression (12) takes the form

$$\widehat{\Delta}(k) = \sum_{z\in\mathbb{Z}_m} e^{2\pi i \frac{kz}{m}} - e^{2\pi i \frac{k\cdot 0}{m}}.$$

We have

$$\sum_{z\in\mathbb{Z}_m} e^{2\pi i \frac{kz}{m}} = \begin{cases} 0, & \text{if } \ k \neq 0; \\ m, & \text{if } \ k = 0. \end{cases}$$

Thus, we get a new propagator for the Fourier transform

$$\widehat{\Delta}(k) = \begin{cases} -1, & \text{if } k \neq 0; \\ m-1, & \text{if } k = 0. \end{cases} \tag{13}$$

5. THE MATIYASEVICH FORMULA AND THE DERIVATION OF A FORMULA EXPRESSING A FLOW POLYNOMIAL THROUGH A LINEAR COMBINATION OF CHROMATIC POLYNOMIALS

5.1. The Main Theorem

The Matiyasevich formula [5] looks like this in our notation

$$P_G(m) = \frac{(m-1)^{|E(G)|}}{m^{|E(G)|-|V(G)|}} \sum_{H \subseteq G} \frac{F_H(m)}{(1-m)^{|E(H)|}}. \tag{14}$$

It considers an arbitrary undirected graph G, in the general case with multiple edges and loops. Here, $H \subseteq G$ means that a graph H can be obtained from a graph G by removing a certain number of edges in other terminology this is called a subgraph of the graph G. Note that the formula (14) is sufficient to consider for a connected graph G in the case of a disconnected graph, everything is factorized.

There is a chromatic polynomial on the left and there is a linear combination of flow polynomials on the right in Matiyasevich formula. The following theorem gives an inverse relationship: a flow polynomial through a linear combination of chromatic polynomials.

Theorem 2 (A theorem on the expression of a flow polynomial through a linear combination of chromatic polynomials). *Let the set of vertices V be divided into parts $V_1, V_2, \ldots, V_k$, k is arbitrary and the subgraphs $H(V_i)$ (induced subgraph), including all edges with ends in the set V_i, are connected. Denote by H the union of subgraphs $H(V_i)$. There is a formula*

$$F_G(m) = \frac{(-1)^{|E|}}{m^{|V|}} \sum_H P_{G/H}(m)(1-m)^{|E(H)|},$$

where G/H denotes the contraction of the graph G by H.

Proof. According to the formula (6) we replace $F_G(m)$ in the left to FA $\mathcal{F}_{G,\Delta}$. Let us now use the formula (10) and the relation (13) for the propagator Δ. We have

$$F_G(m) = \frac{1}{m^{|V|-1}} \frac{1}{m} \sum_{x_v \in \mathbb{Z}_m, v \in V} \prod_{\ell \in E(G)} \widehat{\Delta}(x_{i(\ell)} - x_{f(\ell)}).$$

We transform the propagator $\widehat{\Delta}(k)$ by making (-1), we put

$$\widehat{\Delta}'(y) = \begin{cases} 1, & \text{if } y \neq 0; \\ 1-m, & \text{if } y = 0, \end{cases}$$

$$\sum_{x_v \in \mathbb{Z}_m, v \in V} \prod_{\ell \in E(G)} \widehat{\Delta}(x_{i(\ell)} - x_{o(\ell)}) = (-1)^{|E(G)|} \sum_{x_v \in \mathbb{Z}_m, v \in V} \prod \widehat{\Delta}'(x_{i(\ell)} - x_{o(\ell)}). \tag{15}$$

According to the second line in the formula (15) for $x_{i(\ell)} = x_{o(\ell)}$ we have to replace the propagator $\widehat{\Delta}'(x_{i(\ell)} - x_{o(\ell)})$ by $(1-m)$. At the same time two variables mapped to the vertices $i(\ell)$ and $o(\ell)$ match, so we can pull these two vertices into one.

The graph from the set of contractible edges is denoted by H. Note that each connectivity component of this graph contains all edges induced by a set of vertices of this connectivity component (due to the transitivity of the equality relation). As a result, we get the formula

$$F_G(m) = \frac{(-1)^{|E|}}{m^{|V|}} \sum_H P_{G/H}(m)(1-m)^{|E(H)|},$$

where summing is over all subgraphs H □

Let us demonstrate this formula with an example. Consider the following graph

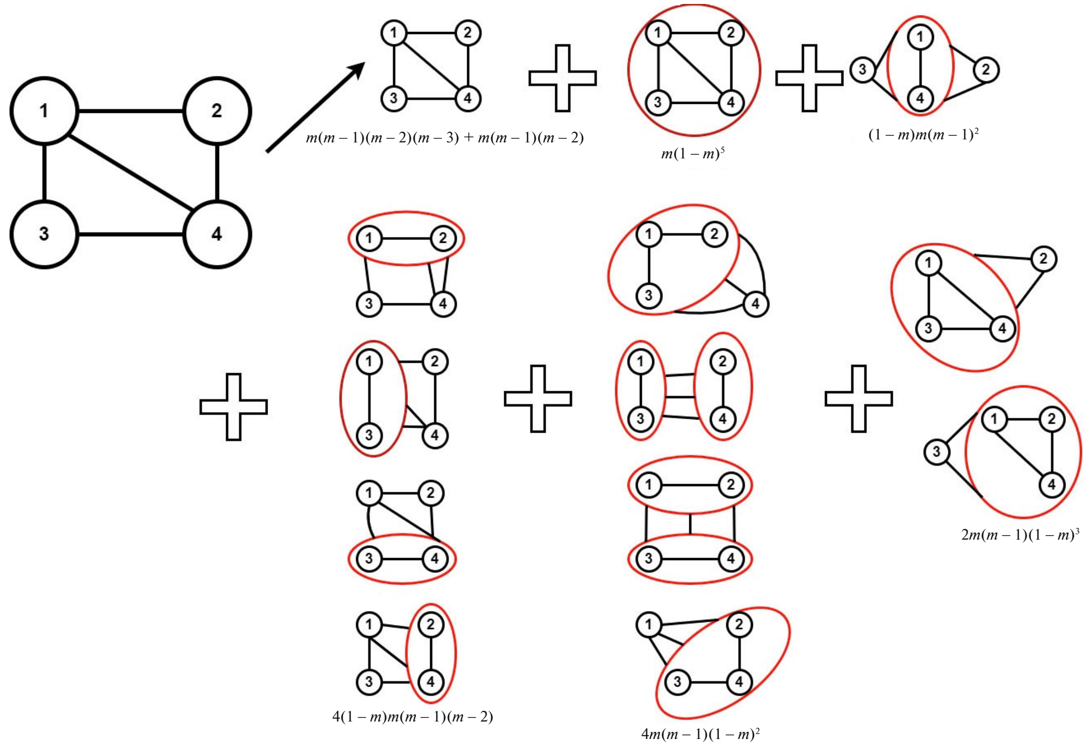

Fig. 1. Illustration of the Theorem 2.

5.2. Derivation of the Matiyasevich Formula Using the FA Technique

In this section, we will show that the Matiyasevich formula (14) is a simple consequence of the formula (10).

Indeed, we get the following form of the formula (10) if we substitute $f = \Delta$ into the formula (10), using (13) and (15):

$$P_G(m) = \frac{1}{m^{|E|-|V|+1}}(-1)^{|E|}\mathcal{F}_{G,\widehat{\Delta}}.$$

In $\mathcal{F}_{G,\Delta}$ the momenta of k_ℓ can take both zero and non-zero values. Note that when we consider the condition (4) zero momenta can be ignored. In our case zero momenta contribute to the product inside (5): $(1-m)$ to the degree equal to their number. Denote by H the subgraph of the graph G corresponding to nonzero momenta when finding $\mathcal{F}_{G,\Delta}$. We have

$$P_G(m) = \frac{(-1)^{|E|}}{m^{|E|-|V|}}\sum_H (1-m)^{|E(G)|-|E(H)|}\mathcal{F}_{H,\Delta}.$$

According to (6) in the last expression $\mathcal{F}_{H,\Delta}$ can be replaced by $F_H(m)$. It is easy to see that the latter formula is equivalent to the formula (14):

$$\frac{(m-1)^{|E(G)|}}{m^{|E(G)|-|V(G)|}}\sum_{H\subseteq G}\frac{F_H(m)}{(1-m)^{|E(H)|}} = (-1)^{|E(G)|}\sum_{H\subseteq G}\frac{(1-m)^{|E(G)-E(H)|}F_H(m)}{m^{|E(G)|-|V(G)|}}.$$

and, thus, Matiyasevich formula is proved.

Note that the formula (14) can be interpreted as the expression: $P_G(m)\frac{m^{|E(G)|-|V(G)|}}{(m-1)^{|E(G)|}}$ as the sum of $\frac{F_H(m)}{(1-m)^{|E(H)|}}$, where H runs through all subsets of the edge set. Applying the Mobius inversion formula on the lattice of all possible subsets of the set [12] (which in this case coincides with the inclusion-exclusion formula) we immediately get the inverse expression

$$\frac{F_G(m)}{(1-m)^{|E(G)|}} = \sum_H (-1)^{|E(G)|-|E(H)|} \frac{P_H(m) m^{|E(H)|-|V(H)|}}{(m-1)^{|E(H)|}}.$$

The last formula can be rewritten as

$$F_G(m) = (m-1)^{|E(G)|} \sum_H P_H(m) \frac{m^{|E(H)|-|V(H)|}}{(1-m)^{|E(H)|}},$$

which coincides with the formula (2.5) from [13]. This formula from [13] is obtained based on the properties of the Biggs decomposition and the Tutte polynomial.

6. CONCLUSION

In this paper a formula that expresses a flow polynomial through a linear combination of chromatic ones is obtained. To derive this formula the Fourier transform technique was used, as well as the relationship between the Feynman amplitude in the coordinate and momentum representations in the vacuum case. Matiyasevich formula that expresses the chromatic polynomial as the sum of the flow polynomials is also obtained in this paper in a similar way.

Note that the implicit use of the Fourier transform by Matiyasevich in the derivation of his formula was at the same time when Biggs also used the Fourier transform when he considered the relationship between two objects: flow and chromatic polynomials ([14], pp. 44–47).

The use of the FA technique, when we consider chromatic and flow polynomials, was first performed in the article [15] [section 3]. They appeared there when we described the procedure for calculating the FA from a p-adic argument, in which the propagator is the corresponding norm to the power of λ. Note that the chromatic and flow polynomials can also be obtained as limit expressions of the FA from the p-adic argument, so that the results of our work can be obtained as limit expressions for the results of [15]. Namely, the Theorem A.1 of this work is a generalization of Matiyasevich's Theorem, and the Theorem A.2 is a generalization of Theorem 2 of this work. The latter are exactly obtained after taking the limit $\lambda \to \infty$ and $\lambda \to 0$ respectively.

Note that non-vacuum amplitudes were generally considered in the ([15], section 3). Non-vacuumity in coordinate space means that some vertices have a fixed coloring. The non-vacuumity in the momentum space fixes some values of the variables k_ℓ.

Further we are going to establish a connection between these "generalized" flow and chromatic polynomials. This connection will generalize Matiyasevich formula and Theorem 2. It would be interesting to have properties of the coefficients of such polynomials similar to the properties of the coefficients of chromatic and flow polynomials [16].

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