Retractable and Coretractable Modules

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Abstract

In the paper, we study mod-retractable modules, CSL-modules, fully Kasch modules, and their interrelations. Right fully Kasch rings are described. It is proved that for a module M of finite length, the following conditions are equivalent. 1) In the category $\sigma(M)$, every module is retractable. 2) In the category $\sigma(M)$, every module is coretractable. 3) M is a CSL-module. 4) $\mathrm{Ext}^1_R(S_1, S_2) = 0$ for any two simple nonisomorphic modules $S_1, S_2 \in \sigma(M)$. 5) M is a fully Kasch module.

Key words: retractable module, coretractable module, CC-ring, mod-retractable ring, Kasch ring, semi-Artinian ring, perfect ring, CSL ring

1. Introduction

All rings are assumed to be associative and with nonzero identity element; all modules are assumed to be unitary.

A module M is said to be *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for every nonzero submodule N of M. If every module in the category $\sigma(M)$ is retractable, then the module M is said to be *mod-retractable*. A ring R is said to be *right mod-retractable* if every right R-module is retractable. In [1, Theorem 3.5], it is proved that the class of SV-rings coincides with the class of regular mod-retractable rings R such that every primitive image of R is Artinian. In the papers [1] and [11], it is proved that the class of commutative mod-retractable rings coincides with the class of commutative semi-Artinian rings.

A module M is said to be *coretractable* if $\mathrm{Hom}_R(M/N, M) \neq 0$ for every proper submodule N of the module M. If every module in the category $\sigma(M)$ is coretractable, then the module M is called a *CC module*. A ring R is called a *right CC ring* if every right R-module is coretractable. Coretractable modules are studied in the paper [5]. In the papers [1] and [18], right CC rings are described.

If every simple module in the category $\sigma(M)$ can be embedded in the module M, then the module M is called a *Kasch module*. If every module in the category $\sigma(M)$ is a Kasch module, the the module M is called a *fully Kasch module*. Kasch modules were introduced in the paper [4]. The same paper contains the following open question: Describe fully Kasch rings and modules.

A module M is called a CSL *module* if every module N in $\sigma(M)$, such that $\mathrm{End}_R(N)$ is a division ring, is a simple module. A ring R is called a *right* CSL *ring* if the module R_R is a CSL module. In the paper [15], it is

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proved that the class of commutative CSL rings coincides with the class of commutative rings R such that the Krull dimension of R is equal to zero. Perfect CSL rings are described in the paper [3]. Semi-Artinian CSL rings are described in the paper [1].

In the presented paper, we study interrelations between the above-mentioned classes of rings and modules. In Section 3, we consider mod-retractable rings and modules. In Section 4, we study CC rings and fully Kasch rings. In Corollary 4.6, we describe right fully Kasch rings. In Theorem 4.9, we prove that the class of fully Kasch rings coincides with the class of CC rings. In Section 5, we consider CSL modules and their interrelations with mod-retractable modules.

2. Preliminaries

For two modules M and N, the module N is said to be *M-subgenerated* if N is isomorphic to a submodule of a homomorphic image of some direct sum of copies of M. In the category of all right R-modules, the full subcategory, consisting of all M-subgenerated modules, is denoted by $\sigma(M)$; it is called the *Wisbauer category* of the module M.

Theorem 2.1 ([14, Proposition 2.2].) *Let M be a right R-module and let $M = \bigoplus_{i\in I} M_i$. Then the following conditions are equivalent.*

1) *For any two distinct subscripts i, j in I, the modules M_i and M_j do not have isomorphic nonzero subfactors.*

2) *For any two distinct subscripts i, j in I, we have the relation $\sigma(M_i) \cap \sigma(M_j) = 0$.*

3) *For an arbitrary module $N \in \sigma(M)$, there exist uniquely defined modules $N_i \in \sigma(M_i)$, $i \in I$, such that $N = \bigoplus_{i\in I} N_i$.*

Lemma 2.2. *Let M be a right R-module and let $S \in \sigma(M)$ be a simple module. Then the module S is isomorphic to the socle of some factor module of the module M.*

Proof. The assertion directly follows from the property that the injective hull of the module S in the category $\sigma(M)$ is generated by the module M. $\square$

For a module M, the *Loewy series* of M is the ascending chain

$$0 \subset \mathrm{Soc}_1(M) = \mathrm{Soc}(M) \subset \ldots \subset \mathrm{Soc}_\alpha(M) \subset \mathrm{Soc}_{\alpha+1}(M) \subset \ldots,$$

where $\mathrm{Soc}_\alpha(M)/\mathrm{Soc}_{\alpha-1}(M) = \mathrm{Soc}(M/\mathrm{Soc}_{\alpha-1}(M))$ for every nonlimit ordinal number α and $\mathrm{Soc}_\alpha(M) = \bigcup_{\beta<\alpha} \mathrm{Soc}_\beta(M)$ for every limit ordinal number α. We denote by $L(M)$ the submodule of the form $\mathrm{Soc}_\xi(M)$, where ξ is the the least ordinal number with $\mathrm{Soc}_\xi(M) = \mathrm{Soc}_{\xi+1}(M)$. A module M is semi-Artinian if and only if $M = L(M)$. In this case, ξ is called the *Loewy length* of the module M; it is denoted by $\mathrm{Loewy}(M)$. A ring R is said to be *right semi-Artinian* if the module R_R is semi-Artinian. For any ring R, we denote by $L(R)$ and $\mathrm{Soc}(R)$ the ideals $L(R_R)$ and $\mathrm{Soc}(R_R)$, respectively. A ring A is called a *right V-ring* if the following equivalent conditions hold: 1) all simple right A-modules are injective; 2) in A, every proper right ideal is the intersection of maximal right ideals. A right semi-Artinian right V-ring is called a *right* SV-*ring*.

Module M is called I_0 – *module*, if every nonsmall submodule of module M contains nonzero direct summand of M.

Lemma 2.3 ([2, Lemma 3].) *Let M be a right R-module. If M is an I_0-module and N is a submodule in M such that $(N + J(M))/J(M)$ is a simple submodule in $M/J(M)$, then N has a local direct summand mR of M such that $(N + J(M))/J(M) = (m + J(M))R$.*

Lemma 2.4. *For a semi-Artinian right R-module M, the following conditions are equivalent*

1) *every nonsmall submodule of module M contains local direct summand of M.*

2) *M is an I_0-module.*

Theorem 2.5. *Let R be a ring and let P be a finitely generated quasi-projective semi-Artinian right R-module.*

1) $\mathrm{End}_R(P)$ *is a right semi-Artinian ring.*

2) *P is a module with finite exchange property.*

3) *P is an I_0-module.*

Proof. 1). We set $\alpha = \mathrm{Loewy}(P)$. It is clear that $\mathrm{Hom}_R(P, \mathrm{Soc}_\beta(P))$ is an ideal of the ring $\mathrm{End}_R(P)$ for every ordinal number $\beta \leqslant \alpha$. Since P is a finitely generated quasi-projective module,

$$\mathrm{Hom}_R(P, \mathrm{Soc}_{\beta+1}(P))/\mathrm{Hom}_R(P, \mathrm{Soc}_\beta(P)) \cong \mathrm{Hom}_R(P, \mathrm{Soc}_{\beta+1}(P)/\mathrm{Soc}_\beta(P))$$

is a semisimple right $\mathrm{End}_R(P)$-module for every ordinal number $\beta \leqslant \alpha$. Since P is finitely generated, we have that for every limit ordinal number $\gamma < \alpha$, we have the relation $\mathrm{Hom}_R(P, \mathrm{Soc}_\gamma(P)) = \bigcup_{\beta<\gamma} \mathrm{Hom}_R(P, \mathrm{Soc}_\beta(P))$. Therefore, it follows from [8, 3.12] that the ring $\mathrm{End}_R(P)$ is right semi-artinian.

2). The assertion follows from 1) and [7, 11.17], [6, Theorem 1.4].

3). Let N be a nonsmall submodule of the module P. Since P is a finitely generated quasi-projective module, we have that for some homomorphism $f \in \mathrm{End}_R(P)$, which is not contained in $J(\mathrm{End}_R(P))$, we have the inclusion $f(P) \subset N$. Since each right semi-Artinian ring is an I_0-ring, it follows from 1) that $\mathrm{End}_R(P)$ is an I_0-ring. Therefore, for some homomorphism $g \in \mathrm{End}_R(P)$, we have that fg is a nonzero idempotent of the ring $\mathrm{End}_R(P)$ and $fg(P) \subset N$. □

Module M is called *semilocal*, if $M/J(M)$ is semisimple module.

Corollary 2.6. *If M is a finitely generated, quasi-projective, semi-Artinian, semilocal right R-module, then M is a finite direct sum of local modules.*

3. mod-**Retractable Modules**

Lemma 3.1. *For a semiprimitive I_0-ring R, the following conditions are equivalent.*

1) *Every nonsingular right R-module is retractable.*

2) *Every nonzero nonsingular right R-module contains a nonzero injective submodule.*

3) *Every nonzero right ideal of the ring R contains a nonzero injective submodule of the module R_R.*

4) *Every nonzero submodule of any projective right R-module contains a nonzero injective submodule.*

Proof. The implications 2) $\Rightarrow$ 1) and 4) $\Rightarrow$ 3) are directly verified.

3) $\Rightarrow$ 2). Let M be a nonzero nonsingular right R-module and let m be a nonzero element of M. Since the right ideal $\operatorname{Ann}(m)$ of the ring R is not essential, the submodule mR contains a nonzero submodule which is isomorphic to some submodule of the module R_R. Therefore, the module M contains a nonzero injective submodule.

1) $\Rightarrow$ 4). Let P_0 be a nonzero submodule of a projective right R-module P. Since R is a semiprimitive I_0-ring, $E(P)$ is a nonsingular module. Therefore, there exists a nonzero homomorphism $f \in \operatorname{Hom}_R(E(P), P_0)$. It follows from [10, Theorem 3.2] that $\operatorname{Im} f$ contains a nonzero direct summand A of the module P. Let π be the projection from P onto A. Then the kernel of the homomorphism $\pi_{|P_0} f$ is a direct summand in $E(P)$. Therefore, the submodule P_0 contains a nonzero injective submodule. □

A module M is called a max-*module* if every nonzero module in the category $\sigma(M)$ has a maximal submodule.

Lemma 3.2. *Let M be a* mod-*retractable module. Then we have the following assertions.*

1) *M is a* max-*module.*

2) *M is a* CSL *module.*

3) *If M is quasi-projective, then M is a self-generated module.*

Proof. 1) and **2).** The assertions are directly verified.

3). It follows from Lemma 2.2 that M generates every simple module in the category $\sigma(M)$. Therefore, it follows from [16, 18.5] that M is a self-generated module.□

Theorem 3.3. *Let M be a projective semiperfect module in the category $\sigma(M)$. If M is a semi-Artinian module, then the following conditions are equivalent.*

1) *M is a* mod-*retractable module.*

2) *$M = \oplus_{i\in I} M_i$, where $\sigma(M_i) \cap \sigma(M_j) = 0$ for $i \neq j$, all simple subfactors of the module M_i are isomorphic to each other, and M_i is the direct sum of pairwise isomorphic local* max-*modules for every $i \in I$.*

3) *The category $\sigma(M)$ has a projective generator of the form $\oplus_{i\in I} P_i$, where $\sigma(P_i) \cap \sigma(P_j) = 0$ for $i \neq j$, and P_i is a local* max-*module such that all simple subfactors of P_i are isomorphic to each other for every $i \in I$.*

Proof. 1) ⇒ 2). By [16, 42.5], the module M can be represented in the form $M = \oplus_{i\in I} M_i$, where M_i is a direct sum of pairwise isomorphic local modules and the modules M_i and M_j do not have pairwise isomorphic local direct summands for $i \neq j$. We assume that $\sigma(M_i) \cap \sigma(M_j) \neq 0$. Therefore, it follows from Lemma 2.2 that for some distinct subscripts $i, j \in I$ and some local direct summands L_i, L_j of the modules M_i, M_j, respectively, we have the isomorphism $\mathrm{Soc}(L_i/N_i) \cong \mathrm{Soc}(L_j/N_j)$, where L_i/N_i and L_j/N_j are uniform nonzero modules. Then either $L_i/J(L_i) \ncong \mathrm{Soc}(L_j/N_j)$ or $L_j/J(L_j) \ncong \mathrm{Soc}(L_j/N_j)$. This contradicts to the assumption of 1). Therefore, $\sigma(M_i) \cap \sigma(M_j) = 0$. Finally, it is directly verified that, all simple subfactors of the module M_i are isomorphic to each other, for every $i \in I$.

2) ⇒ 3). The implication follows from [16, 18.5].

3) ⇒ 1). Let $N \in \sigma(M)$ and let S be a simple submodule of the module N. It follows from Theorem 2.1 that $N = \oplus_{i\in I} N_i$, where $N_i \in \sigma(P_i)$ for every $i \in I$, and $S \subset \sigma(N_{i_0})$ for some $i_0 \in I$. Since all simple modules in the category $\sigma(N_{i_0})$ are isomorphic to each othe, and M is a max-module, we have $\mathrm{Hom}_R(N_{i_0}, S) \neq 0$. Therefore, $\mathrm{Hom}_R(N, S) \neq 0$. □

Corollary 3.4. *Let M be a projective semiperfect module in the category $\sigma(M)$. If M is a finitely generated semi-Artinian module, then the following conditions are equivalent.*

1) *M is a* mod *-retractable module.*

2) *$M = \oplus_{i\in I} M_i$, where $\sigma(M_i) \cap \sigma(M_j) = 0$ for $i \neq j$, all simple subfactors of the module M_i are isomorphic to each other, and M_i is the direct sum of pairwise isomorphic local* max*-modules for every $i \in I$.*

3) *The category $\sigma(M)$ is equivalent to the category of modules over a ring R which is a finite direct product of full matrix rings over perfect local rings.*

Corollary 3.5. *For a left perfect ring R, the following conditions are equivalent.*

1) *R is a right* mod*-retractable ring.*

2) *The ring R is a finite direct product of full matrix rings over perfect local rings.*

For a ring R, an indecomposable factor ring R/B is called a *maximal indecomposable factor* of R if for every ideal B' which is properly contained in the ideal B, the factor ring R/B' is not an indecomposable ring. A module M is said to be *regular* if every cyclic submodule of M is a direct summand in M. A ring R is said to be *strongly regular* if $a \in a^2R$ for each element $a \in R$. A ring R with Jacobson radical J is said to be *semiregular* if R/J is a regular ring and all idempotents of R/J can be lifted to idempotents of R.

Theorem 3.6. *If R is a semiregular ring and every primitive image of R is Artinian, then the following conditions are equivalent.*

1) *R is a* mod*-retractable ring.*

2) *R is a semi-Artinian* CSL *ring.*

3) *R is a semi-Artinian ring and every maximal indecomposable factor of R is a full matrix ring over a perfect local ring.*

Proof. The implications 2) ⇒ 3) and 3) ⇒ 1) follow from [1, Theorem 3.3].

1) ⇒ 2). It is clear that each mod-retractable ring is a CSL-ring. We prove that the ring R is semi-Artinian. It follows from [1, Theorem 3.5] that $R/J(R)$ is a SV-ring. Since the ring R is a max-ring by Lemma 3.2, it follows from [13, Lemma 26.2] that $J(R)$ is a t-nilpotent ideal. By [13, Remark 21.3]. the ring R is a semi-Artinian ring. □

Theorem 3.7. *Let P be a finitely generated quasi-projective module such that every primitive image of the ring* $\mathrm{End}_R(P)$ *is Artinian. Then the following conditions are equivalent.*

1) *P is a* mod-*retractable regular module.*

2) *P is an* SV*-module.*

Proof. The implication 2) ⇒ 1) follows from [1, Theorem 3.10].

1) ⇒ 2). It follows from [1, Theorem 3.10], [1, Lemma 1.10], and [16, 46.2] that the category $\sigma(P)$ is equivalent to the category of all right modules over the regular ring $\mathrm{End}_R(P)$. Therefore, it follows from [1, Theorem 3.5] that $\mathrm{End}_R(P)$ is a SV-ring. Therefore, P is an SV-module. □

4. Kasch Rings and CC Rings

Lemma 4.1. *If M is a generator in $\sigma(M)$, then the following conditions are equivalent.*

1) *M is a fully Kasch module.*

2) *For every submodule M_0 of the module M, the module M/M_0 is a Kasch module.*

3) *For every fully invariant submodule M_0 of the module M, the module M/M_0 is a Kasch module.*

Proof. The implications 1) ⇒ 2) and 2) ⇒ 3) are directly verified.

3) ⇒ 1). Let $N \in \sigma(M)$ and let $M_0 = \bigcap_{f \in \mathrm{Hom}_R(M,N)} Ker(f)$. It is easy to see that M_0 is a fully invariant submodule of the module M. Since $N = \mathrm{Hom}_R(M,N)M$, we have $N = \mathrm{Hom}_R(M/M_0, N)(M/M_0)$. Therefore, $\sigma(N) \subset \sigma(M/M_0)$. Let $S \in \sigma(N)$ be a simple module. Then there exists an element $m \in M$ such that $(m + M_0)R$ is a simple submodule of the module M/M_0 which is isomorphic to the module S. Since the element m is not contained in the submodule M_0, there exists a homomorphism $f \in \mathrm{Hom}_R(M,N)$ such that $f(m) \neq 0$. Since $M_0 \subset Ker(f)$, the homomorphism f induces the homomorphism $\overline{f} \in \mathrm{Hom}_R(M/M_0, N)$ with $\overline{f}((m + M_0)R) \neq 0$. Therefore, the module S is isomorphic to some simple submodule of the module N. □

Lemma 4.2. *For an arbitrary right R-module M, we have the following assertions.*

1) *If M is a finitely generated Kasch module, then M is a coretractable module.*

2) *If M is a self-generated, quasi-projective, coretractable module, then M is a Kasch module.*

Proof. 1). The assertion is directly verified.

2). Let S be an arbitrary simple module in the category $\sigma(M)$. Then $S \cong A/B$, where A is a finite direct sum of isomorphic copies of the module M and B is a submodule of the module A. It follows from [5, Proposition 2.6] that $\mathrm{Hom}_R(A/B, A) \neq 0$. Therefore, $\mathrm{Hom}_R(S, M) \neq 0$. □

Lemma 4.3. *For an arbitrary right R-module M, the following assertions are true.*

1) *If M is a fully Kasch module, then M is a semi-Artinian module.*

2) *If M is a finitely generated, quasi-projective, fully Kasch module, then M is a semiperfect module in the category $\sigma(M)$.*

Proof. 1). The assertion is directly verified.

2). By [16, 18.2], the module $M/J(M)$ is quasi-projective. If N is a maximal and essential submodule of the module $M/J(M)$, then it follows from the assumption that $(M/J(M))/N$ is an $M/J(M)$-projective simple module; it is clear that this is impossible. Therefore, $M/J(M)$ is a semisimple finitely generated module. Now, it follows from 1) and Corollary 2.6 2.3 that M is a finite direct sum of local modules. Therefore, it follows from [16, 42.5] that M is a semiperfect module in the category $\sigma(M)$. □

Proposition 4.4. *If M is a projective semiperfect module in the category $\sigma(M)$, then the following conditions are equivalent.*

1) *M is a fully Kasch module.*

2) *In the category $\sigma(M)$, every cyclic module is coretractable.*

3) *In the category $\sigma(M)$, every finitely generated module is coretractable.*

4) *$M = \oplus_{i\in I} M_i$, where $\sigma(M_i) \cap \sigma(M_j) = 0$ if $i \neq j$, all simple subfactors of the module M_i are isomorphic to each other, and M_i is a direct sum of pairwise isomorphic local semi-Artinian modules for every $i \in I$.*

5) *The category $\sigma(M)$ has a projective generator of the form $\oplus_{i\in I} P_i$, where $\sigma(P_i) \cap \sigma(P_j) = 0$ for $i \neq j$, and P_i is a local semi-Artinian module such that all simple subfactors of P_i are isomorphic to each other for every $i \in I$.*

Proof. The proof of the implication 1) ⇒ 4) is similar to the proof of the implication 1) ⇒ 2) in Theorem 3.3.

The implications 3) ⇒ 2) and 1) ⇒ 3) are directly verified.

4) ⇒ 5). The implication directly follows from [16, 18.5].

5) ⇒ 1). Let $N \in \sigma(M)$ and let $S \in \sigma(N)$ be a simple module. By Theorem 2.1, $N = \oplus_{i\in I} N_i$, where $N_i \in \sigma(P_i)$ for every $i \in I$. It follows from Lemma 2.2 that $S \cong A_{i_0}/B_{i_0}$, where A_{i_0}, B_{i_0} are submodules of the module N_{i_0} for some $i_0 \in I$. Since all simple modules in $\sigma(P_{i_0})$ are isomorphic to each other and $\mathrm{Soc}(N_{i_0}) \neq 0$, the module S is isomorphic to some submodule of the module N.

2) $\Rightarrow$ 1). Let $N \in \sigma(M)$ and let $S \in \sigma(N)$ be a simple module. It follows from Lemma 2.2 that $S \cong A/B$, where A, B are submodules of the module N. Without loss of generality, we can assume that A is a cyclic module. Since A is a coretractable module, the simple module S is isomorphic to some submodule of the module N. □

The following assertion directly follows from Lemma 4.3, Proposition 4.4, and [16, 46.2].

Corollary 4.5. *If P is a finitely generated quasi-projective module, then the following conditions are equivalent.*

1) *P is a fully Kasch module.*

2) *In the category $\sigma(P)$, every cyclic module is coretractable.*

3) *In the category $\sigma(P)$, every finitely generated module is coretractable.*

4) *The category $\sigma(P)$ is equivalent to the category of right modules over a ring which is a finite direct product of full matrix rings over left perfect local rings.*

The following assertion directly follows from Lemma 4.1 and Corollary 4.5.

Corollary 4.6. *For a ring R, the following conditions are equivalent.*

1) *R is a right fully Kasch ring.*

2) *Over the ring R, every right finitely generated module is coretractable.*

3) *Over the ring R, every cyclic right module is coretractable.*

4) *Every factor ring of R is a right Kasch ring.*

5) *The ring R is isomorphic to the finite direct product of full matrix rings over left perfect local rings.*

Corollary 4.7. *For a ring R, the following conditions are equivalent.*

1) *R is a right semi-Artinian ring and all simple right R-modules are isomorphic to each other.*

2) *R is a right fully Kasch ring and all simple right modules are isomorphic to each other.*

3) *Every nonzero injective right R-module is a generator in the category of all all right R-modules.*

4) *The ring R is isomorphic to a matrix ring over a local, left perfect ring.*

Proof. The implications 1) $\Rightarrow$ 2) and 4) $\Rightarrow$ 3) are directly verified.

The implication 2) $\Rightarrow$ 4) follows from Corollary 4.6.

3) $\Rightarrow$ 1). Let S be a simple right R-module. We consider an arbitrary nonzero right R-module M. It follows from the assumption that the module S is isomorphic to some simple submodule of the module $E(M)$. Therefore,

R is a right semi-Artinian ring such that all simple right R-modules are isomorphic to the module S. □

The proof of the following assertion is similar to the proof of [5, Proposition 3.2].

Lemma 4.8. *If R is a ring and every free right R-module is coretractable, then the following conditions are equivalent.*

1) *R is a* max*-ring.*

2) *Every submodule of the module R_R has a maximal submodule.*

Theorem 4.9. *For a ring R, the following conditions are equivalent.*

1) *For every ideal I of the ring R, each free right module over the ring R/I is coretractable.*

2) *The ring R is isomorphic to the finite direct product of full matrix rings over perfect local rings.*

Proof. The implication 2) ⇒ 1) is directly verified.

1) ⇒ 2). It is clear that every factor ring of the ring R is a right Kasch ring. Therefore, it follows from Corollary 4.6 and Lemma 4.1 that the ring R is isomorphic to the finite direct product of full matrix rings over left perfect local rings. Therefore, it is sufficient to prove that R is a right max ring. Let I be the sum of all radical submodules of the module R_R. We assume that $I \neq 0$. It is clear that I is an ideal. Since R is a right semi-Artinian ring, then it follows from [6, Theorem 3.1] that $I^2 \neq I$. Then I/I^2 is a nonzero radical right R/I-module. Let A/I be a nonzero submodule of the module $R/I_{R/I}$. Then A contains a maximal submodule M. Since $AJ(R) = J(A)$ and $I = IJ(R)$, we have $I \subset J(A)$. Therefore, $I \subset M$. Therefore, every nonzero submodule of the module $R/I_{R/I}$ contains a maximal submodule. Therefore, it follows from Lemma 4.8 that R/I is a max-ring. On the other hand, the right R/I-module I/I^2 is a nonzero radical module. It follows from this contradiction that $I = 0$. Therefore, it follows from Lemma 4.8 that R is a right max ring. □

The following assertion directly follows from the previous results and [1], [18].

Theorem 4.10. *For a ring R, the following conditions are equivalent.*

1) *R is a fully Kasch ring.*

2) *For every ideal I of the ring R, the factor ring R/I is a Kasch ring.*

3) *Over the ring R, every finitely generated right or left module is coretractable.*

4) *Over the ring R every cyclic right or left module is coretractable.*

5) *R is a right* CC *ring.*

6) *R is a left* CC *ring.*

7) *The ring R is isomorphic to the finite direct product of full matrix rings over perfect local rings.*

5. CSL **Rings and Modules**

Lemma 5.1. *Let R be a ring and let M be a semi-Artinian right R-module such that all simple subfactors of M are isomorphic to each other. Then the following conditions are equivalent.*

1) *M is a* CSL *module which is a* max-*module.*

2) *M is a* mod-*retractable module.*

Proof. 1) ⇒ 2). Let N be a nonzero module in the category $\sigma(M)$. Since M is a max-module, we have the least non-limit ordinal number α such that $\mathrm{Soc}_\alpha(N) \nsubseteq J(N)$. Then $\mathrm{Soc}_\alpha(N)/\mathrm{Soc}_{\alpha-1}(N) \nsubseteq J(N)/\mathrm{Soc}_{\alpha-1}(N)$. Therefore, some simple submodule S of the module $N/\mathrm{Soc}_{\alpha-1}(N)$ is a direct summand in $N/\mathrm{Soc}_{\alpha-1}(N)$. Therefore, $\mathrm{Hom}_R(N/\mathrm{Soc}_{\alpha-1}(N), S) \neq 0$. Since all simple subfactors of the module N are isomorphic to each other, N is a retractable module.

The implication 2) ⇒ 1) is directly verified. □

It is easy to see that for every prime integer p, the $\mathbb{Z}$-module C_{p^∞} is a CSL module, which is not mod-retractable.

Lemma 5.2. *Let P be a finitely generated quasi-projective semi-Artinian* max-*module such that the following conditions hold.*

1) *For all pairwise nonisomorphic simple modules $S_1, S_2 \in \sigma(P)$, we have the relation $\mathrm{Ext}^1_R(S_1, S_2) = 0$.*

2) *$P = P_1 \oplus P_2$, where P_1 is a direct sum of local modules, P_2 is a submodule of the module P, and the modules P_1, P_2 do not have isomorphic nonzero direct summands.*

Then $\mathrm{Hom}_R(P_1, P_2) = 0, \mathrm{Hom}_R(P_2, P_1) = 0$.

Proof. We assume that $\mathrm{Hom}_R(P_2, P_1) \neq 0$. It follows from Theorem 2.3 that $P_0 = \mathrm{Hom}_R(P_2, P_1)P_2 \subset J(P_1)$. Let M be a maximal submodule of the module P_0 and let P' be a $\cap$-complement of the module P_0/M in the module P_1/M. Then $L = (P_1/M)/P'$ is a uniform module such that $\mathrm{Soc}(L) \cong P_0/M$ and $\mathrm{Soc}(L) \subset J(L)$. Since P is quasi-projective, we have that $\mathrm{Hom}_R(P_2, L)P_2 = \mathrm{Soc}(L)$ and $\mathrm{Hom}_R(P_2, L/\mathrm{Soc}(L))P_2 = 0$. Since P is a semi-Artinian module, the module L contains a local submodule L_0 of length two. Since $\mathrm{Hom}_R(P_2, \mathrm{Soc}(L_0)) \neq 0$ and $\mathrm{Hom}_R(P_2, L_0/J(L_0)) = 0$, the simple modules $\mathrm{Soc}(L_0)$, $L_0/J(L_0)$ are not isomorphic to each other, and $\mathrm{Ext}^1_R(\mathrm{Soc}(L_0), L_0/J(L_0)) \neq 0$. This contradicts to the assumption of the lemma. The relation $\mathrm{Hom}_R(P_1, P_2) = 0$ is similarly proved. □

Theorem 5.3. *Let P be a finitely generated semi-Artinian quasi-projective module. If every primitive image of the ring* $\mathrm{End}(P)$ *is Artinian, then the following conditions are equivalent.*

1) *P is a self-generated* CSL *module which is a* max-*module.*

2) *P is a* mod-*retractable module.*

3) *P is a self-generated* max-*module and* $\operatorname{Ext}^1_R(S_1, S_2) = 0$ *for any two simple nonisomorphic modules* $S_1, S_2 \in \sigma(P)$.

4) *The category $\sigma(P)$ is equivalent to the category of right modules over a semi-Artinian ring S such that every maximal indecomposable factor of S is a full matrix ring over a perfect local ring.*

Proof. 1) ⇒ 3). The assertion is directly verified.

2) ⇒ 1). The implication follows from Lemma 3.2.

4) ⇒ 2). The implication follows from [1, Theorem 3.3].

3) ⇒ 4). By [16, 46.2], it is sufficient to prove that $\operatorname{End}_R(P)$ is a semi-Artinian ring such that every maximal indecomposable factor of this ring is a full matrix ring over a perfect local ring. With the use of the transfinite induction, for every ordinal number α, we construct a fully invariant submodule P_α in the module P, as follows. For $\alpha = 0$, we set $P_0 = 0$. If $\alpha = \beta + 1$, then $P_{\beta+1}/P_\beta = \sum_{\pi \in I_\beta} \pi(P/P_\beta)$, where I_β is the set of all nonzero indecomposable central idempotents $\pi \in \operatorname{End}(P/P_\beta)$ such that $\pi(P/P_\beta)$ is a finite direct sum of pairwise isomorphic local modules. If α is a limit ordinal number, we set $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$. It follows from Theorem 2.3, Lemma 5.2, and [1, Lemma 1.2] that $P_\tau = P$ for some ordinal number τ. It is easy to see that for an arbitrary ordinal number β and for every $\pi \in I_\beta$ module $\pi(P/P_\beta)$ is a self-generated quasi-projective module. Since P is a max-module, it follows from [16, 46.2] that $\operatorname{End}_R(\pi(P/P_\beta))$ is a full matrix ring over a perfect local ring. Since P is a quasi-projective finitely generated module, $\operatorname{Hom}(P, P_{\alpha+1})/\operatorname{Hom}(P, P_\alpha)$ is a direct sum of full matrix rings over perfect local rings. Since $\operatorname{End}_R(P) = \bigcup_{\beta \leqslant \tau} \operatorname{Hom}(P, P_\beta)$, we have that $\operatorname{End}_R(P)$ is a semi-Artinian ring such that every maximal indecomposable factor of this ring is a full matrix ring over a perfect local ring. □

Corollary 5.4. *Let P be a projective semiperfect module in the category $\sigma(P)$. If P is a finitely generated semi-Artinian module, then the following conditions are equivalent.*

1) *P is a self-generated* CSL *module which is a* max-*module.*

2) *P is a* mod-*retractable module.*

3) *The category $\sigma(P)$ is equivalent to the category of modules over a ring S which is a finite direct product of full matrix rings over perfect local rings.*

Corollary 5.5. *If R is a right semi-Artinian ring and every primitive image of R is an Artinian ring, then the following conditions are equivalent.*

1) *R is a right* CSL *ring and a right* max-*ring.*

2) *R is a right* mod-*retractable ring.*

3) *R is a semi-Artinian ring such that every maximal indecomposable factor of R is a full matrix ring over a perfect local ring.*

Theorem 5.6. *For a right or left quasi-invariant ring R, the following conditions are equivalent.*

1) *R is a* mod-*retractable ring.*

2) *R is a semi-Artinian* CSL *ring.*

3) *R is a semi-Artinian ring such that every maximal indecomposable factor of R is a local perfect ring.*

Proof. 1) ⇒ 2). It follows from [17, Corollary 2.4] that $R/J(R)$ is a reduced ring. Therefore, it follows from [12, Theorem 3.2] that $R/J(R)$ is a semi-Artinian strongly regular ring. Since R is a max-ring, it follows from [13, Remark 21.3, Lemma 26.2] that the ring R is semi-Artinian. Therefore, R is a semi-Artinian ring such that every primitive image of R is a division ring. Now, it follows from Lemma 3.2 that R is an CSL-ring.

2) ⇒ 3). Since the ring R is right quasi-invariant, the implication directly follows from [1, Theorem 3.3].

3) ⇒ 1). The implication follows from [1, Theorem 3.3]. □

Corollary 5.7. *For a right (or left) invariant ring R, the following conditions are equivalent.*

1) *R is a* mod-*retractable ring.*

2) *R is a semi-Artinian* CSL *ring.*

3) *R is a semi-Artinian ring.*

Theorem 5.8. *Let R be a ring and let M be a right R-module of finite length. Then the following conditions are equivalent.*

1) *Every module N of finite length in the category $\sigma(M)$, such that* $\mathrm{End}(N)$ *is a division ring, is a simple module.*

2) *M is a* CSL *module.*

3) *Every module of finite length in the category $\sigma(M)$ is retractable.*

4) *M is a* mod-*retractable module.*

5) $\mathrm{Ext}^1_R(S_1, S_2) = 0$ *for any two simple nonisomorphic modules* $S_1, S_2 \in \sigma(M)$.

6) *M is a fully Kasch module.*

7) *M is a* CC *module.*

8) *In the category $\sigma(M)$, all simple subfactors of every indecomposable module of finite length are isomorphic to each other.*

9) *In the category $\sigma(M)$, all simple subfactors of every indecomposable module are isomorphic to each other.*

Proof. The implications 2) ⇒ 1), 4) ⇒ 3), 1) ⇒ 5), 2) ⇒ 5), 3) ⇒ 5), 6) ⇒ 7), 7) ⇒ 1), 9) ⇒ 1), 9) ⇒ 2), 9) ⇒ 8) are directly verified.

The implications 9) ⇒ 4), 9) ⇒ 6), 8) ⇒ 9) follow from Theorem 2.1.

5) ⇒ 8). We assume that the category $\sigma(M)$ has an indecomposable module of finite length which has nonisomorphic simple subfactors. Let N be an

indecomposable module of least length in the category $\sigma(M)$ such that N does not satisfy 8). Then $J(N) = N_1 \oplus \ldots \oplus N_k$, where $\sigma(N_i) \cap \sigma(N_j) = 0$ for $i \neq j$, N_i is a nonzero submodule of the module N, and all simple subfactors of the module N_i are isomorphic to each other for every i.

Let $k = 1$ and let the module N be not local. Then the factor module $N/J(N)$ contains a simple submodule S such that $S \notin \sigma(J(N))$. Let N_0 be a submodule of the module N with $N_0/J(N) = S$. It is clear that $\lg(N_0) < \lg(N)$. Consequently, the module N_0 has a decomposition $N_0 = A_1 \oplus \ldots \oplus A_s$ where for every i, we have that A_i is an indecomposable module such that all simple subfactors of it are isomorphic to each other. Since $\mathrm{Soc}(N_0) \subset \mathrm{Soc}(N) \subset J(N)$, we have $\mathrm{Soc}(N_0) \in \sigma(J(N))$. Since $N_0/J(N) = S$ for some subscript i_0, the module A_{i_0} has a simple subfactor which is isomorphic to the module S. Thus, the indecomposable module A_{i_0} has two nonisomorphic simple subfactors; this contradicts to the choice of the module N.

It is easy to see that if N is a local module, then for some submodule N_0 in N, the module N/N_0 is a local module of length two such that $\mathrm{Soc}(N/N_0) \not\cong N/\mathrm{Soc}(N/N_0)$. Therefore, without loss of generality, we can assume that N is a non-local module and $k > 1$. Let $\overline{N_i}$ be the closer of the module N_i in the module N for every i and let $S = N_0/J(N)$ be an arbitrary simple submodule of the semisimple module $N/J(N)$. Since $\lg(N_0) < \lg(N)$, we have that $N_0 = A_1 \oplus \ldots \oplus A_s$, where all simple subfactors of the module A_i are isomorphic to each other for every i, $\sigma(A_i) \cap \sigma(A_j) = 0$ for $i \neq j$, and every simple subfactor of the module A_s is isomorphic to the module S. We take a subscript i_0 such that all simple subfactors of the module N_{i_0} are isomorphic to the module S. Since $\lg(\overline{N_{i_0}}) < \lg(N)$ and all simple submodules of the module $\overline{N_{i_0}}$ are isomorphic to each other, it follows from the choice of the module N that all simple subfactors of the module $\overline{N_{i_0}}$ are isomorphic to each other. Then $\overline{N_{i_0}} + A_s$ is an essential extension of the module N_{i_0}. Therefore, $\overline{N_{i_0}} + A_s = \overline{N_{i_0}}$ and $A_s \subset \overline{N_{i_0}}$. Since $A_1 \oplus \ldots \oplus A_{s-1} \subset J(N)$, we have

$$N_0 = A_1 \oplus \ldots \oplus A_s \subset \overline{N_1} \oplus \ldots \oplus \overline{N_k}.$$

It is clear that the module N can be represented in the form $N = B_1 + \ldots + B_t$, where $J(N) \subset B_i$ and $B_i/J(N)$ is a simple submodule for every i. Therefore, we have $N = \overline{N_1} \oplus \ldots \oplus \overline{N_k}$; this contradicts to the indecomposability of the module N. $\square$

Corollary 5.9. *For artinian module M following conditions are equivalent.*

1) *M is a* mod-*retractable module.*

2) *M is a* CSL *module which is a* max-*module.*

3) *M is module of finite length and $M = M_1 \oplus \ldots \oplus M_n$, where M_i is indecomposable module for every $1 \leqslant i \leqslant n$ and all simple subfactors of the module M_i are isomorphic to each other.*

Proof. The implication 1) $\Rightarrow$ 2) follows from Lemma 3.2. The implication 3) $\Rightarrow$ 2) follows from Theorem 5.8.

2) $\Rightarrow$ 3). For every nonnegative integer n, we inductively deafine a submodule $J^{(n)}(M)$ of the module M such that $J^{(0)}(M) = M$ and $J^{(n+1)}(M) =$

$J(J^{(n)}(M))$. Since M is an Artinian max-module, there exists a nonnegative integer n_0 such that $J^{(n_0)}(M) = 0$ and $J^{(n+1)}(M)/J^{(n)}(M)$ is a semisimple module of finite length for every nonnegative integer n. Thus, M is a module of finite length, and the implication follows from Theorem 5.8. $\square$

The following assertion is similar to [9, Theorem 1.2]. We note that the proof of the previous theorem and the proof of [9, Theorem 1.2] are distinct.

Corollary 5.10. *Let R be a ring and let M be a right R-module. Then the following conditions are equivalent.*

1) *Every module N in the category $\sigma(M)$, such that N is of finite length and* End(N) *is a division ring, is a simple module.*

2) *In the category $\sigma(M)$, every module N of finite length is retractable.*

3) $\mathrm{Ext}^1_R(S_1, S_2) = 0$ *for any two simple nonisomorphic modules S_1, S_2 in $\sigma(M)$.*

4) *In the category $\sigma(M)$, for every indecomposable module N of finite length, all simple subfactors of N are isomorphic to each other.*

Corollary 5.11. *Let P be a quasi-projective module of finite length. Then the following conditions are equivalent.*

1) *P is a* CSL *module.*

2) *P is a* mod-*retractable module.*

3) $\mathrm{Ext}^1_R(S_1, S_2) = 0$ *for any two simple nonisomorphic modules S_1, S_2 in $\sigma(P)$.*

4) *The category $\sigma(P)$ is equivalent to the category of right modules over a semi-Artinian ring S such that every maximal indecomposable factor of S is a full matrix ring over a perfect local ring.*

References

[1] A.N. Abyzov, On some classes of semi-Artinian rings, *Siberian Math. J.* **53**, no. 5 (2012) 763-771.

[2] A.N. Abyzov, Generalized SV-modules, *Siberian Math. J.* **50**, no. 3 (2009) 481-488.

[3] M. Alaoui and A. Haily, Perfect rings for which the converse of Schur's lemma holds, *Publ. Mat.* **45**, no. 1 (2001) 219-222.

[4] T. Albu and R. Wisbauer, Kasch modules, *In: Advances in Ring Theory (Granville, OH, 1996)*, Trends in Mathematics, Birkhauser, Boston, 1997, pp. 1-16.

[5] B. Amini, M. Ershad and H. Sharif, Coretractable modules, *J. Austral.Math. Soc. Ser. A*, **86**, no. 3 (2009) 289-304.

[6] G. Baccella, Exchange property and the natural preorder between simple modules over semi-Artinian rings, *J. Algebra*, **253** (2002), 133-166.

[7] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, *Lifting modules. Supplements and Projectivity in Module Theory* (Frontiers in Math., Boston, Birkhauser, 2006).

[8] N.V. Dung, D.V. Huynh, P.F. Smith and R. Wisbauer, *Extending modules* (Pitman, London, 1994).

[9] M. Greg and S. Markus, Extensions of Simple Modules and the Converse of Schur's Lemma *In: Advances in Ring Theory* (Trends in Mathematics, pp 229-237, 2010).

[10] H. Hamza, I_0-rings and I_0-modules, *Math. J. Okayama Univ.*, **40** (1998), 91–97.

[11] M.T. Kosan, J. Zemlicka, Mod-retractable rings, *Comm. Algebra* **42**, no. 3 (2014), 998-1010.

[12] Y. Tolooei and M. R. Vedadi, On rings whose modules have nonzero homomorphisms to nonzero submodules, *Publ. Mat.* **57**, no. 1 (2013), 1-264

[13] A.A. Tuganbaev, *Rings Close to Regular* (Kluwer Academic Publishers, Dordrecht, Boston, London, 2002).

[14] N. Vanaja, All finitely generated M-subgenerated modules are extending, *Comm. Algebra* **24**, no. 2 (1996), 543-572.

[15] R. Ware and J. Zelmanowitz, Simple endomorphism rings, *Amer. Math. Monthly* **77**, no. 9 (1970), 987-989.

[16] R. Wisbauer, *Foundations of Module and Ring Theory* (Gordon and Breach, Philadelphia, 1991).

[17] H. P. Yu, On quasiduo rings, *Glasgow Math. J.*, **37** (1995), 21-31.

[18] J. Zemlicka, Completely coretractable rings, *Bull. Iran. Math. Soc.*, **39**, no. 3 (2013), 523-528.