Integral Equation Method in the Theory of Dielectric Waveguides

E. M. Karchevskiy
Kazan State University, Russia

Abstract— The eigenvalue problems for generalized natural modes of an inhomogeneous dielectric waveguide without a sharp boundary and a step-index dielectric waveguide with smooth boundary of cross-section are formulated as problems for the set of time-harmonic Maxwell equations with partial radiation conditions (Sveshnikov radiation conditions) at infinity in the cross-sectional plane. The original problems by integral equations method are reduced to nonlinear spectral problems with Fredholm integral operators. Theorems on spectrum localization are proved, and then it is proved that the sets of all eigenvalues of the original problems can only be some sets of isolated points on the Reimann surface, ant it also proved that each eigenvalue depends continuously on the frequency and dielectric permittivity and can appear and disappear only at the boundary of the Reimann surface. The Galerkin method for numerical calculations of the generalized natural modes are proposed, and the convergence of the method is proved. Some results of numerical experiments are discussed.

1. INTRODUCTION

Optical fibers are dielectric waveguides (DWs), i.e., regular dielectric rods, having various cross sectional shapes, and where generally the dielectric permittivity may vary in the waveguide's cross section [1]. Although existing technologies often result in a dielectric permittivity that is anisotropic, frequently it is possible to assume that the fiber is isotropic [2], which is the case investigated in this work. The study of the source-free electromagnetic fields, called natural modes, that can propagate on DWs necessitates that longitudinally the rod extend to infinity. Since often DWs are not shielded, the medium surrounding the waveguide transversely forms an unbounded domain, typically taken to be free space. This fact plays an extremely important role in the mathematical analysis of natural waveguide modes, and brings into consideration a variety of possible formulations. Each different formulation can be cast as an eigenvalue problem for the set of time-harmonic Maxwell equations, but they differ in the form of the condition imposed at infinity in the cross-sectional plane, and hence in the functional class of the natural-mode field. This also restricts the localization of the eigenvalues in the complex plane of the eigenparameter [3]. All of the known natural-mode solutions (i.e., surface guided modes, leaky modes, and complex modes) satisfy the partial radiation conditions at infinity (see, for example, [4]), which firstly were originally introduced by A. G. Sveshnikov [5] for a scattering problem. The partial radiation conditions in waveguiding problems are connected with the fact that propagation constants β may be complex and may be generally considered on the appropriate logarithmic Reimann surface [6, 7]. For real propagation constants on the principal ("proper") sheet of this Reimann surface, one can reduce the partial radiation conditions to either the Sommerfeld radiation condition or to the condition of exponential decay. The partial radiation conditions may be considered as a generalization of the Sommerfeld radiation condition and can be applied for complex propagation constants. This conditions may also be considered as the continuation of the Sommerfeld radiation condition from a part of the real axis of the complex parameter β to the appropriate logarithmic Reimann surface [8].

2. GENERALIZED NATURAL MODES OF A STEP-INDEX DIELECTRIC WAVEGUIDE

Let the three-dimensional space be occupied by an isotropic source-free medium, and let the dielectric permittivity be prescribed as a positive real-valued function $\varepsilon = \varepsilon(x)$ independent of the longitudinal coordinate and equal to a constant $\varepsilon_\infty > 0$ outside a cylinder. In this section we consider the generalized natural modes of an step-index optical fiber and suppose that the dielectric permittivity is equal to a constant $\varepsilon_+ > \varepsilon_\infty$ inside the cylinder. The axis of the cylinder is parallel to the longitudinal coordinate, and its cross section is a bounded domain Ω_i with a twice continuously differentiable boundary γ (see Fig. 1). The domain Ω_i is a subset of a circle with radius R_0. Denote by Ω_e the unbounded domain $\Omega_e = \mathbb{R}^2 \setminus \overline{\Omega}_i$. Denote by U the space of complex-valued continuous and continuously differentiable in $\overline{\Omega}_i$ and $\overline{\Omega}_e$, twice continuously differentiable in Ω_i and Ω_e functions. Denote by Λ the Riemann surface of the function $\ln \chi_\infty(\beta)$, where $\chi_\infty = \sqrt{k^2\varepsilon_\infty - \beta^2}$. Here $k^2 = \omega^2\varepsilon_0\mu_0$, ω is a given radian frequency; ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. Denote by Λ_0 the principal ("proper") sheet of this Riemann surface, which is specified by the condition $\mathrm{Im}\chi_\infty(\beta) \geq 0$.

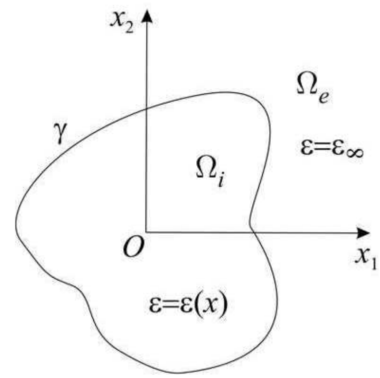

Figure 1: A schematic waveguide's cross-section.

A nonzero vector $\{\mathrm{E}, \mathrm{H}\} \in U^6$ is referred to as generalized eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [9]:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2 \setminus \gamma, \tag{1}$$

$$\nu \times \mathrm{E}^+ = \nu \times \mathrm{E}^-, \quad x \in \gamma, \tag{2}$$

$$\nu \times \mathrm{H}^+ = \nu \times \mathrm{H}^-, \quad x \in \gamma, \tag{3}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{4}$$

Here differential operator rot_β is obtained from usual operator by replacing generating waveguide line derivative with $i\beta$ multiplication; and $H_l^{(1)}(z)$ is the Hankel function of the first kind and index l. The conditions (4) are the the partial radiation conditions.

Theorem 1 (see [9]). *The imaginary axis $\mathbb{I}$ and the real axis $\mathbb{R}$ of the sheet Λ_0 except the set $G = \left\{\beta \in \mathbb{R} : k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\right\}$ are free of the eigenvalues of the problem* (1)–(4)*. Surface and complex eigenmodes correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenmodes correspond to complex eigenvalues β belonging to an "improper" sheet of Λ for which* $\mathrm{Im}\chi_\infty(\beta) < 0$.

The results of the theorem 1 generalize the well known results on the spectrum localization of the step-index circular dielectric waveguide, which were obtained by separation of variables method (see, for example, [10]).

We use the representation of the eigenvectors of problem (1)–(4) in the form of the single-layer potentials u and v:

$$\mathrm{E}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_2} + \beta\frac{\partial u}{\partial x_1}\right), \quad \mathrm{E}_2 = \frac{-i}{k^2\varepsilon - \beta^2}\left(\mu_0\omega\frac{\partial v}{\partial x_1} - \beta\frac{\partial u}{\partial x_2}\right), \quad \mathrm{E}_3 = u, \tag{5}$$

$$\mathrm{H}_1 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_1} - \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_2}\right), \quad \mathrm{H}_2 = \frac{i}{k^2\varepsilon - \beta^2}\left(\beta\frac{\partial v}{\partial x_2} + \varepsilon_0\varepsilon\omega\frac{\partial u}{\partial x_1}\right), \quad \mathrm{H}_3 = v, \tag{6}$$

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \frac{i}{4}\int_\gamma H_0^{(1)}\left(\sqrt{k^2\varepsilon_{+/\infty} - \beta^2}\,|x-y|\right)\begin{bmatrix} f_{+/\infty}(y) \\ g_{+/\infty}(y) \end{bmatrix} dl(y), \quad x \in \Omega_{i/e}, \tag{7}$$

where unknown densities $f_{+/\infty}$ and $g_{+/\infty}$ belong to the space of Hölder continuous functions $C^{0,\alpha}$. The original problem (1)–(4) by single-layer potential representation (5)–(7) is reduced [9] to a nonlinear eigenvalue problem for a set of singular integral equations at the boundary γ. This problem has the operator form

$$A(\beta)w \equiv (I + B(\beta))w = 0, \tag{8}$$

where I is the identical operator in the Banach space $W = (C^{0,\alpha})^4$ and $B(\beta) : W \to W$ is a compact operator consists particularly of the following boundary singular integral operators:

$$Lp = -\frac{1}{2\pi}\int_0^{2\pi} \ln\left|\sin\frac{t-\tau}{2}\right| p(\tau)d\tau, \quad t \in [0, 2\pi], \quad L : C^{0,\alpha} \to C^{1,\alpha}, \tag{9}$$

$$Sp = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ctg}\frac{\tau - t}{2} p(\tau)d\tau + \frac{i}{2\pi}\int_0^{2\pi} p(\tau)d\tau, \quad t \in [0, 2\pi], \quad S : C^{0,\alpha} \to C^{0,\alpha}. \tag{10}$$

The original problem (1)–(4) is spectrally equivalent [9] to the problem (8). Namely, suppose that $w \in W$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$, $D = \{\beta \in \mathbb{I}\} \bigcup \{\beta \in \mathbb{R} : \beta^2 < k^2\varepsilon_\infty\}$. Then using this vector we can construct the densities of the single-layer potential representation (5)–(7) of an eigenmode $\{\mathrm{E}, \mathrm{H}\} \in U^6$ of the problem (1)–(4), corresponding to the same eigenvalue β. For other side, any eigenmode of the problem (1)–(4), corresponding to an eigenvalue $\beta \in \Lambda_0 \setminus D$ can be represented in the form of single-layer potentials. The densities of this potentials construct an eigenvector $w \in W$ of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β.

Theorem 2 (see [9]). *For each $\beta \in \{\beta \in \mathbb{R} : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has the bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

The results of the theorem 2 generalize the well known results on the dependence of the propagation constants β of the step-index circular dielectric waveguide on the wave number k and dielectric permittivity ε (see, for example, [10]).

3. GENERALIZED NATURAL MODES OF AN INHOMOGENEOUS WAVEGUIDE

In this section we consider the generalized natural modes of an inhomogeneous optical fiber without a sharp boundary. Let the dielectric permittivity ε belongs to the space $C^2(\mathbb{R}^2)$ of twice continuously differentiable in $\mathbb{R}^2$ functions. Denote by ε_+ the maximum of the function ε in the domain Ω_i, let $\varepsilon_+ > \varepsilon_\infty > 0$. A nonzero complex vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is referred to as generalized eigenvector (or eigenmode) of the problem corresponding to an eigenvalue $\beta \in \Lambda$ if the following relations are valid [3]:

$$\mathrm{rot}_\beta \mathrm{E} = i\omega\mu_0 \mathrm{H}, \quad \mathrm{rot}_\beta \mathrm{H} = -i\omega\varepsilon_0\varepsilon \mathrm{E}, \quad x \in \mathbb{R}^2, \tag{11}$$

$$\begin{bmatrix} \mathrm{E} \\ \mathrm{H} \end{bmatrix} = \sum_{l=-\infty}^{\infty} \begin{bmatrix} \mathrm{A}_l \\ \mathrm{B}_l \end{bmatrix} H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{12}$$

Theorem 3 (see [3]). *The imaginary axis $\mathbb{I}$ and the real axis $\mathbb{R}$ of the sheet Λ_0 except the set $G = \left\{\beta \in \mathbb{R} : k^2\varepsilon_\infty < \beta^2 < k^2\varepsilon_+\right\}$ are free of the eigenvalues of the problem* (11), (12). *Surface and complex eigenmodes correspond to real eigenvalues $\beta \in G$ and complex eigenvalues $\beta \in \Lambda_0$, respectively. Leaky eigenmodes correspond to complex eigenvalues β belonging to an "improper" sheet of Λ for which* $\mathrm{Im}\chi_\infty(\beta) < 0$.

If vector $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is an eigenvector of problem (11), (12) corresponding to an eigenvalue $\beta \in \Lambda$, then (see [3])

$$\mathrm{E}(x) = k^2\int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\, \Phi(\beta; x, y)\mathrm{E}(y)dy + \mathrm{grad}_\beta \int_{\Omega_i} \left(\mathrm{E}, \varepsilon^{-1}\mathrm{grad}\varepsilon\right)(y)\Phi(\beta; x, y)dy, \quad x \in \mathbb{R}^2, \tag{13}$$

$$\mathrm{H}(x) = -i\omega\varepsilon_0 \mathrm{rot}_\beta \int_{\Omega_i} (\varepsilon(y) - \varepsilon_\infty)\, \Phi(\beta; x, y)\mathrm{E}(y)dy, \quad x \in \mathbb{R}^2. \tag{14}$$

Using the integral representation (13) for $x \in \Omega_i$ we obtain a nonlinear eigenvalue problem for integral equation on the domain Ω_i. This problem has the operator form

$$A(\beta)\mathrm{F} \equiv (I - B(\beta))\mathrm{F} = 0, \tag{15}$$

where the operator $B(\beta) : (L_2(\Omega_i))^3 \to (L_2(\Omega_i))^3$ satisfies the right side of the integral representation (13) for $x \in \Omega_i$. For any $\beta \in \Lambda$ the operator $B(\beta)$ is compact [3].

It was proved in the paper [3] that the original problem (11), (12) is spectrally equivalent to problem (15). Namely, suppose that $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$ is an eigenmode of problem (11), (12) corresponding to an eigenvalue $\beta \in \Lambda$. Then $\mathrm{F} = \mathrm{E} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to the same eigenvalue β. Suppose that $\mathrm{F} \in [L_2(\Omega_i)]^3$ is an eigenvector of the operator-valued function $A(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$, and also suppose that the same number β is not an eigenvalue of the following problem:

$$\left[\Delta + \left(k^2\varepsilon - \beta^2\right)\right] u = 0, \quad x \in \mathbb{R}^2, \quad u \in C^2(\mathbb{R}^2), \tag{16}$$

$$u = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(\chi_\infty r) \exp(il\varphi), \quad r \geq R_0. \tag{17}$$

Let $\mathrm{E} = B(\beta)\mathrm{F}$ and $\mathrm{H} = (i\omega\mu_0)^{-1}\mathrm{rot}_\beta \mathrm{E}$ for $x \in \mathbb{R}^2$. Then $\{\mathrm{E}, \mathrm{H}\} \in (C^2(\mathbb{R}^2))^6$, and $\{\mathrm{E}, \mathrm{H}\}$ is an eigenvector of the original problem (11), (12) corresponding to the same eigenvalue β.

Theorem 4 (see [3]). *For each $\beta \in \{\beta \in \mathbb{R} : \beta^2 \geq k^2\varepsilon_+\}$ the operator $A(\beta)$ has the bounded inverse operator. The set of all eigenvalues β of the operator-valued function $A(\beta)$ can be only a set of isolated points on Λ. Each eigenvalue β depends continuously on $\omega > 0$, $\varepsilon_+ > 0$, and $\varepsilon_\infty > 0$ and can appear and disappear only at the boundary of Λ, i.e., at $\beta = \pm k\sqrt{\varepsilon_\infty}$ and at infinity.*

4. NUMERICAL METHOD

Describe a projection method for numerical solution of the problem (8). Denote by N the set of integers. We use the representation of the approximate eigenvector of the operator-valued function $A(\beta)$ in the form

$$w_n = (w_n^{(j)})_{j=1}^4, \quad w_n^{(j)}(t) = \sum_{k=-n}^{n} \alpha_k^{(j)} \exp(ikt), \ n \in N, \ j = 1, 2, 3, 4.$$

We look for unknown coefficients $\alpha_k^{(j)}$ by Galerkin method

$$\int_0^{2\pi} (Aw_n)^{(j)}(t) \exp(-ikt)dt = 0, \ k = -n, \ldots, n, \ j = 1, 2, 3, 4.$$

The trigonometric functions $\exp(ikt)$ are the orthogonal eigenfunctions of the singular integral operators $L : C^{0,\alpha} \to C^{1,\alpha}$ and $S : C^{0,\alpha} \to C^{0,\alpha}$, corresponding to the following eigenvalues:

$$\lambda_m^{(L)} = \{\ln 2 \text{ if } m = 0, \ (2|m|)^{-1} \text{ if } m \neq 0\},$$
$$\lambda_m^{(S)} = \{i \text{ if } m = 0, \ i\,\mathrm{sign}(m) \text{ if } m \neq 0\}$$

for the operators L and S respectively. Hence, the action of the main (singular) parts of the integral operators in (8) on the basis functions is expressed in the explicit form.

Denote by W_n^T the set of all trigonometric polynomials of the orders up to n. Denote by $W_n \subset W$ the space of the elements $w_n = (w_n^{(j)})_{j=1}^4$, where $w_n^{(j)} \in W_n^T$. Using the Galerkin method for numerical solution of the problem (8), we get finite-dimensional nonlinear spectral problem

$$A_n(\beta)w_n = 0, \quad A_n : W_n \to W_n. \tag{18}$$

Theorem 5 (see [11]). *If β_0 belongs to the spectrum $\sigma(A)$ of the operator-valued function $A(\beta)$, then there exists some sequence $\{\beta_n\}_{n\in N}$ with $\beta_n \in \sigma(A_n)$, that $\beta_n \to \beta_0$, $n \in N$. If $\{\beta_n\}_{n\in N}$ is a sequence such that $\beta_n \in \sigma(A_n)$ and $\beta_n \to \beta_0 \in \Lambda$, then $\beta_0 \in \sigma(A)$. If $\beta_n \in \sigma(A_n)$, $A_n(\beta_n)w_n = 0$, and $\beta_n \to \beta_0 \in \Lambda$, $w_n \to w_0$, $n \in N$, $\|w_n\| = 1$ then $\beta_0 \in \sigma(A)$ and $A(\beta_0)w_0 = 0$, $\|w_0\| = 1$.*

Figure 2 shows the dispersion curves for the complex modes (on the left) and for the surface guided modes (on the right) of the step-index waveguides of circular and square cross-section. The numerical results obtained by Galerkin method marked by circles and by squares on the left of the Fig. 2. The dispersion curves for circular waveguide are plotted here by solid line, $\widetilde{\beta} = \beta/(k\sqrt{\varepsilon_\infty})$ and $V = kR\sqrt{\varepsilon_+ - \varepsilon_\infty}$. The right side of the Fig. 2 compares the experimental data [12] for surface waves of square waveguide (marked by squares) with our numerical results (solid lines). Here a is a half of the square's side.

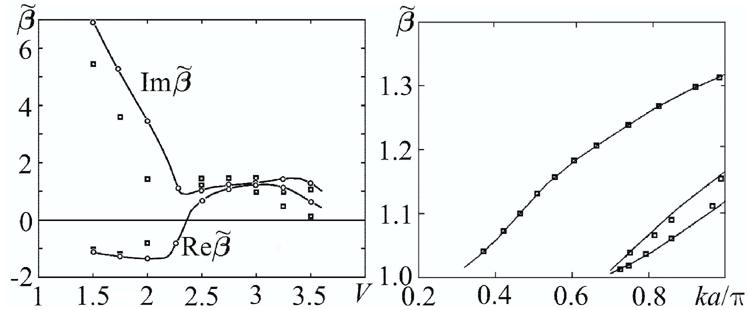

Figure 2: The dispersion curves for the complex modes (on the left) and surface guided modes (on the right) of the step-index waveguides of circular and square cross-section.

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