Available online at www.sciencedirect.com

ScienceDirect

Advances in Space Research 62 (2018) 65–70

ADVANCES IN SPACE RESEARCH (a COSPAR publication)

www.elsevier.com/locate/asr

Problem of stability of multidimensional solutions of the BK class equations in space plasma

Vasily Yu. Belashov [a,*], Elena S. Belashova [b], Oleg A. Kharshiladze [c]

[a] Kazan Federal University, Kazan, Russia
[b] Kazan National Research Technical University named after A.N. Tupolev – KAI, Kazan, Russia
[c] Iv. Javaxishvilis Tbilisi State University, Tbilisi, Georgia

Received 22 November 2017; received in revised form 31 March 2018; accepted 3 April 2018
Available online 12 April 2018

Abstract

The problem of stability of the multidimensional solutions of the BK class equations describing the nonlinear waves which are forming on the low-frequency branch of oscillations in plasma for cases when $\beta \equiv 4\pi nT/B^2 \ll 1$ and $\beta > 1$ is studied. In first case, for $\omega < \omega_B = eB/Mc, k\lambda_D \ll 1$ the FMS waves are excited, and their dynamics under conditions $k_x^2 \gg k_\perp^2,\ v_x \ll c_A$ near the cone of $\theta = \arctan\ (M/m)^{1/2}$, is described by the equation of the BK class known as the GKP equation for magnetic field $h = B_\sim/B$ with due account of the high order dispersive correction defined by values of plasma parameters and angle $\theta = (\mathbf{B}, \mathbf{k})$. In another case, the dynamics of the finite-amplitude Alfvén waves propagating near-to-parallel to $\mathbf{B}$ is described by the equation of the same class known as the 3-DNLS equation for $h = (B_y + iB_z)/2B|1-\beta|$. To study the stability of multidimensional solutions in both cases the method of investigation of the Hamiltonian bounding with deformation conserving momentum by solving the variation problem is used. As a result, we have obtained the conditions of existence of the 2D and 3D soliton solutions in the BK system for cases of the GKP and 3-DNLS equation (i.e. for the FMS and Alfvén waves, respectively) in dependence on the equations' coefficients, i.e. on the parameters of both plasma and wave.

Keywords: Nonlinear waves; Stability; BK class equations; GKP equation; 3-DNLS equation; FMS waves; Alfvén waves; Space plasma

1. Introduction. Basic equations

In this paper we study the formation, structure, stability and dynamics of the multidimensional solitons forming on the low-frequency branch of oscillations in a plasma for cases $\beta \equiv 4\pi nT/B^2 \ll 1$ and $\beta > 1$. These oscillations are described by the Belashov-Karpman (BK) class of equations

$$\partial_t u + \mathrm{A}(t,u)u = f, \quad f = \kappa \int_{-\infty}^{x} \Delta_\perp u dx,$$
$$\Delta_\perp = \partial_y^2 + \partial_z^2 \tag{1}$$

which with operator

$$\mathrm{A}(t,u) = \alpha u \partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3) \tag{2}$$

turns into the generalized Kadomtsev-Petviashvili (GKP) equation and in case when $\beta \equiv 4\pi nT/B^2 \ll 1$ for $\omega < \omega_B = eB/Mc,\ k\lambda_D \ll 1$ describes propagation of the fast magnetosonic (FMS) waves in magnetized plasma with $k_x^2 \gg k_\perp^2,\ v_x \ll c_A$ near the cone of $\theta = arctan\ (M/m)^{1/2}$ (Belashov, 1994). In this case function u has a sense the dimensionless amplitude of the magnetic field of the wave,

* Corresponding author at: Institute of Physics, Kazan Federal University, 18 Kremlyovskaya, Kazan 420008, Russia.
E-mail address: vybelashov@yahoo.com (V.Yu. Belashov).

https://doi.org/10.1016/j.asr.2018.04.001
0273-1177/

$h = B_{\sim}/B$, the coefficients at the terms describing nonlinearity, dissipation and dispersion effects, respectively, are defined by values of plasma parameters and angle $\theta = (\mathbf{B}, \mathbf{k})$. In opposite case, when operator

$$\mathrm{A}(t,u) = 3s|p|^2 u^2 \partial_x - \partial_x^2 (i\lambda + \nu) \tag{3}$$

Eq. (1) turns into the 3-dimensional derivative nonlinear Schrödinger (3-DNLS) equation and in case when $\beta > 1$ describes the dynamics of the finite-amplitude Alfvén waves propagating near-to-parallel to $\mathbf{B}$ for $u = h = (B_y + iB_z)/2B|1 - \beta|$, $\mathbf{h} = \mathbf{B}_\perp/B_0$ where $p = (1 + ie)$, and e is "an eccentricity" of the polarization ellipse of the Alfvén wave (Belashov and Belashova, 2015a,b). The upper and lower signs of $\lambda = \pm 1$ correspond to the right and left circularly polarized wave, respectively; sign of nonlinearity is accounted by coefficient $s = \mathrm{sgn}\,(1 - p) = \pm 1$ in nonlinear term; $\kappa = -r_A/2$, $r_A = v_A/\omega_{0i}$.

The sets of Eqs. (1), (2) and (1), (3) are not completely integrable ones, and a problem of existence of multidimensional stable soliton solutions requires especial investigation. Let us consider the problem of stability of possible multidimensional solutions for two particular cases of the BK system mentioned above which correspond two branches of oscillations in space plasma. Let us assume that dissipation is absent in the medium, i.e. $\nu = 0$ in Eqs. (2) and (3). At first, for the whole diapason of the dispersion coefficients' change we will give the estimations and formulate the sufficient conditions of stability of the GKP equation solutions in the two-dimensional (2D) and three-dimensional (3D) geometry on the basis of transformational properties of the Hamiltonian. Further, we will consider the same problem for the 3-DNLS equation in the 3D geometry.

2. Case of the GKP equation

To study the solutions stability, performing some coordinate transformation, rewrite Eqs. (1) and (2) in form

$$\partial_x(\partial_t u + 6u\partial_x u - \varepsilon\partial_x^3 u - \lambda\partial_x^5 u) = \Delta_\perp u, \tag{4}$$

where $\varepsilon = \beta|\gamma|^{-1/2}, \lambda = \mathrm{sgn}\gamma$. Note that (4) is now the Hamiltonian equation. Rewriting it into the form

$$\partial_t u = \partial_x(\delta \mathrm{H}/\delta u) = -\int_{-\infty}^{\infty} \delta'(x - x')(\delta H/\delta u)dx', \tag{5}$$

with Hamiltonian

$$\mathrm{H} = \int \left[-\frac{\varepsilon}{2}(\partial_x u)^2 + \frac{\lambda}{2}(\partial_x^2 u)^2 + \frac{1}{2}(\nabla_\perp \partial_x v)^2 - u^3 \right] d\mathbf{r} \tag{6}$$

and $\partial_x^2 v = u$, we obtain the Hamiltonian equation where the continuum of values, $u \in \mathrm{M}$, plays the role of the point coordinates in the phase space M, the matrix $\omega(x,x') = \delta'(x - x')$ is skew-symmetric and, because of the inversibility of the operator ∂_x on the decreasing functions for $|x| \to \infty$, is a non-degenerate one on u. Thus the Hamiltonian structure can be represented by the Poisson bracket (Belashov and Vladimirov, 2005)

$$\{S,R\} = \int_{-\infty}^{\infty} (\delta S/\delta u)\partial_x(\delta R/\delta u)\,dx, \quad S,R \in \mathrm{M},$$

with $S;R \in \mathrm{M}$, which satisfies the Jacobi's identity since ω does not depend on the point u in the space M.

The problem of the stability of the soliton-like solutions of (5) was studied before in (Belashov, 1991) on the basis of an analysis of transformational properties of the Hamiltonian (6) in the 2D and 3D geometry ($\partial_z = 0$ and $\partial_{yz} \neq 0$, respectively) for $\lambda = \pm 1$, $\varepsilon \gtrless 0$ (corresponding to different types of the medium).

The stationary solutions of Eq. (4) are defined from the variation problem,

$$\delta(\mathrm{H} + \mathrm{v}P_x) = 0 \tag{7}$$

where $P_x = \frac{1}{2}\int u^2 d\,\mathbf{r}$ is the momentum projection onto the x axis, v is a Lagrange's factor, which illustrates the fact that all finite solutions of Eq. (4) are the stationary points of the Hamiltonian for fixed P_x.

Consider now the problem of stability. In a dynamic system, according to the Lyapunov's theorem, the stationary points corresponding to the maximum or minimum of the Hamiltonian H are absolutely stable. If an extremum is local then the locally stable solutions are possible. The unstable states correspond to the monotonous dependence of H on its variables, i.e., those cases when the stationary point is the saddle point. According to that, all we need is to prove that the Hamiltonian H is limited from below for the fixed P_x.Similar to what was done for the classic KP equation in (Kuznetsov and Turitsyn, 1982), we consider the scale transformations in the real vector space R,

$$u(x,\mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{(1-d)/2}u(x/\zeta, \mathbf{r}_\perp/\eta) \tag{8}$$

(where d is the dimension of the problem, and $\zeta, \eta \in \mathrm{R}$) which conserves the momentum P_x. The Hamiltonian as a function of the parameters ζ, η now takes the form

$$\mathrm{H}(\zeta,\eta) = a\zeta^{-2} + b\zeta^2\eta^{-2} - c\zeta^{-1/2}\eta^{(1-d)/2} + e\zeta^{-4}, \tag{9}$$

where

$$a = -(\varepsilon/2)\int (\partial_x u)^2 d\mathbf{r}, \quad b = (1/2)\int (\nabla_\perp \partial_x v)^2 d\mathbf{r},$$

$$c = \int u^3 d\mathbf{r},$$

$$e = (\lambda/2)\int (\partial_x^2 u)^2 d\mathbf{r}.$$

The necessary conditions for the existence of the Hamiltonian's extremum are given by

$$\partial_\zeta \mathrm{H} = 0 \text{ and } \partial_\eta \mathrm{H} = 0, \tag{10}$$

The latter enables us to obtain the extremum's coordinates, (ζ_i, η_j), if it exists. Holding the inequalities

$$\left| \begin{matrix} \partial_\zeta^2 \mathrm{H}(\zeta_i, \eta_j) & \partial_{\zeta\eta}^2 \mathrm{H}(\zeta_i, \eta_j) \\ \partial_{\eta\zeta}^2 \mathrm{H}(\zeta_i, \eta_j) & \partial_\eta^2 \mathrm{H}(\zeta_i, \eta_j) \end{matrix} \right| > 0, \qquad (11)$$
$$\partial_\zeta^2 \mathrm{H}(\zeta_i, \eta_j) > 0$$

guarantees that the corresponding quadratic form is the positively definite one and therefore these inequalities give the sufficient condition of the existence of the (local) minimum at the point (ζ_i, η_j).Consider Eq. (2) for $d = 2$, i.e., with $\partial_z = 0$. In this case Eqs. (10) form the following set (Belashov, 1991; Belashov and Vladimirov, 2005):

$$G \equiv (c^4/32b)\,t^4 - (at + 2e)^3 = 0,$$
$$t = \zeta^2, \qquad (12)$$
$$\eta = \left[(4b/c)^2 \zeta^5\right]^{1/3}.$$

Analysis of (12) shows (see Appendix A) that it has one positive root, $t \in \mathrm{R}$, for every quadruple of the functions a, b, c, $e \in \mathrm{R}$ in the case $e > 0$ and any a; two positive roots, $t_{1,2} \in \mathrm{R}$, for $e < 0$ and $a > 0$; and in the case $e < 0$, $a \le 0$ we have $t \notin \mathrm{R}$.Inequalities (11) for $d = 2$, taking into account expressions (12), lead to

$$G - (C_{11}a^3t^3 + C_{12}a^2et^2 + C_{13}ae^2t + C_{14}e^3) < 0, \qquad (13.1)$$

$$G - (C_{21}a^3t^3 + C_{22}a^2et^2 + C_{23}ae^2t + C_{24}e^3) < 0, \qquad (13.2)$$

where $C_{nm} > 0$ are constants. It follows that conditions (11) are fulfilled on the set $S_t \subset \mathrm{R}$ of solutions of the set (12) for $e > 0$ and $a \ge 0$, and, consequently, the Hamiltonian H (ζ, η) is bounded from below. Solving (13.1) and (13.2) in the R-space for $e > 0$ and $a < 0$, we obtain for $S_t^{(13.1)} \cap S_t^{(13.2)} = A_t \subset \mathrm{R}$ that $\sup A_t = (3C_{11})^{-1} \times [2C_1 \cos(\varphi_1/3) - C_{12}]a^{-1}e$, and $\inf A_t = 0$ ($t = 0$ is not the root of the set (12) and we therefore discard it). Here,

$$C_1 = = (C_{12}^2 - 3C_{11}C_{13})^{1/2} \quad \text{and}$$
$$\varphi_1 = \mathrm{Arccos}\left\{(2C_1^3)^{-1}[C_{12}(C_{12}^2 - 3C_1^2) - 27C_{11}^2C_{14}]\right\}.$$

Taking into account (A4) (see Appendix A; note that $S_t \cap A_t \neq \varnothing$), we conclude that for $e > 0$ and $a < 0$, the sufficient condition of the existence of the local minimum of H (ζ, η) is the relation $S_t \subseteq A_t$, i.e.,

$$(a/c)(b/e)^{1/4} \geqslant (6C_{11})^{-1}[C_1\cos(\varphi_1/3) - C_{12}/2]. \qquad (14)$$

Analogously, considering inequalities (13) for the case $e < 0$ and $a > 0$, we obtain that

$$\inf B_t^{(1)} = (3C_{21})^{-1}[2C_2\cosh(\varphi_2/3) - C_{22}]a^{-1}e, \ \sup B_t^{(1)}$$
$$= (3C_{11})^{-1}[2C_1 \times \cos(\varphi_1/3 + 4\pi/3) - C_{12}]a^{-1}e, \inf B_t^{(2)}$$
$$= (3C_{11})^{-1}[2C_1\cos(\varphi_1/3 + 2\pi/3) - C_{12}]a^{-1}e,$$

where

$$B_t^{(1)} \cup B_t^{(2)} = S_t^{(13.1)} \cap S_t^{(13.2)} = A_t \subset \mathrm{R}; \ \ C_2 = (C_{22}^2 - 3C_{21}C_{23})^{1/2};$$
$$\varphi_2 = \mathrm{Arcosh}\left\{(2C_2^3)^{-1}[C_{22}(C_{22}^2 - 3C_2^2) - 27C_{21}^2C_{24}]\right\}.$$

Taking into account equalities (A8) we obtain that $B_t^{(1)} \subset S_t \Rightarrow B_t^{(1)} \cap S_t = B_t^{(1)}$, and $B_t^{(2)} \cap S_t = \varnothing$. Then in (A5), by changing $a^4b/c^4e \le -2^4 \dot{s} 3^{-3} \dot{s} Q^{-1}$ ($Q > 1$) and using inequalities (A7), we obtain $Q = -2^8 \cdot 3^{-3}(T + 2)/T^2$ ($T = \inf B_t^{(1)}ae^{-1}$). This corresponds to the sufficient condition of the existence of the local minimum of the Hamiltonian $\mathrm{H}(\zeta, \eta)$, namely $\inf S_t = \inf B_t^{(1)}$, which can now be rewritten as

$$a^4b/c^4e \leqslant 2^{-4}T^2/(T + 2). \qquad (15)$$

Fig. 1 shows the change of the Hamiltonian $\mathrm{H}(\zeta, \eta)$ for the test values of the integrals a; b; c, and e for $d = 2$, $\lambda = \pm 1$, $\varepsilon \gtrless 0$.Consider now Eq. (5) for $d = 3$ ($\partial_{yz} \neq 0$). In this case, for every quadruple a; b; c; $e \in \mathrm{R}$, with $a \neq 0$, we immediately obtain from (10)

$$\zeta_i = (16ab)^{-1}(3c^2 \pm \sqrt{9c^4 - 512ab^2e}),$$
$$\eta_j = (2b/c)\,\zeta_i^{5/2}, \qquad (16)$$
$$i = 1, 2; \quad j = 1, 2, 3, 4.$$

We note here that $(\zeta_i, \eta_j) \notin \mathrm{R}$ for $\zeta_i < 0$, and therefore we consider below only the roots $\zeta_i > 0$ (we map out equality $\zeta_i = 0$, taking into account $e \neq 0$, otherwise (5) degenerates into the standard KP equation).Inequalities (11) taking into account (16) are now given by

$$a\zeta^2 - (c^2/2b)\zeta + 10e/3 > 0, \qquad (17.1)$$

$$a\zeta^2 + (c^2/48b)\zeta + 10e/3 > 0. \qquad (17.2)$$

In the case $e > 0$ and $a > 0$, the condition $\zeta_i \in \mathrm{R}$, i.e.,

$$c^4 \geqslant (512/9)ab^2e \qquad (18)$$

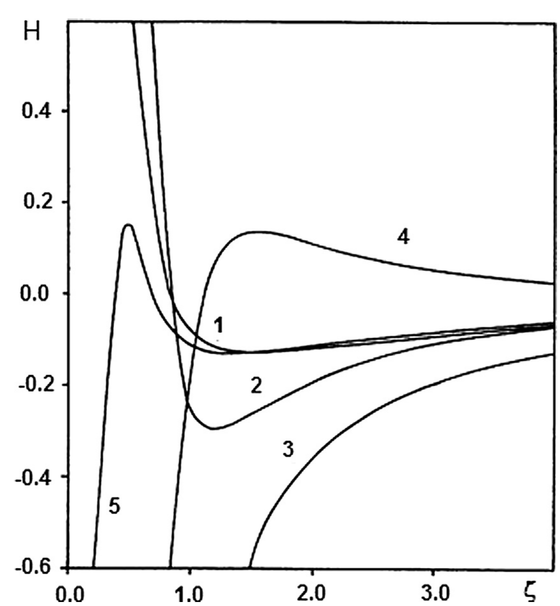

Fig. 1. Change of the Hamiltonian $\mathrm{H}(\zeta, \eta)$ in the 2D case ($d = 2$) along lines $\eta = [(4b/c)^2\zeta^5]^{1/3}$ for the test values of the integrals: 1 – $a = 0.5$, $b = 0.5$, $c = 1$, $e = 0.02$; 2 – $a = -0.5$, $b = 0.5$, $c = 0.5$, $e = 0.5$; 3 – $a = -0.5$, $b = 0.5$, $c = 1$, $e = -0.02$; 4 – $a = 1$, $b = 1$, $c = 0.5$, $e = -1$; 5 – $a = 0.5$, $b = 0.5$, $c = 1$, $e = -0.02$.

gives $\zeta_{1,2} > 0$. Elementary analysis then shows that $S^{(16)}_{\zeta_1} \cap S^{(17.1)}_{\zeta_1} = \varnothing$ and, if strict inequality (18) holds $S^{(16)}_{\zeta_2} \subset S^{(17)}_{\zeta_2}$. Thus, for the existence of the local minimum of $\mathrm{H}(\zeta,\eta)$ for $e > 0$ and $a > 0$, it is sufficient to have

$$ab^2e/c^4 < 9/512. \tag{19}$$

When $e > 0$ and $a < 0$, for each quadruple of $a, b, c, e \in \mathrm{R}$ we have from the first equality of (13) $\zeta_1 < 0$ and, therefore, $S^{(16)}_{\eta_{1,2}} \cap \mathrm{R} = \varnothing$. For $S^{(16)}_{\zeta_2}$, elementary analysis of inequalities (17) gives us $S^{(16)}_{\zeta_2} \subset S^{(17)}_{\zeta_2}$. Thus for any $e > 0$ and $a < 0$ the function $\mathrm{H}(\zeta,\eta)$ is limited from below.

Analogous consideration in the case $e < 0$ shows that for $a < 0$, when condition (18) is satisfied for every quadruple a, b, c, $e \in \mathrm{R}$, we have $\zeta_{1,2} < 0$ and, therefore, $S^{(16)}_{\eta_{1,2,3,4}} \cap \mathrm{R} = \varnothing$; for $a > 0$ we have $\zeta_2 < 0 \Rightarrow S^{(16)}_{\eta_{3,4}} \cap \mathrm{R} = \varnothing$, $\zeta_1 > 0$ but $S^{(16)}_{\zeta_1} \cap S^{(17.1)}_{\zeta_1} = \varnothing$. For $a = 0$ and $e \gtrless 0$ (i.e., $e \neq 0$), instead of (16) we have

$$\zeta_i = 16be/3c^2, \quad \eta_j = (2b/c)\zeta_i^{5/2},$$
$$i = 1, \quad j = 1,2$$

for every triplet of functions $b, c, e \in \mathrm{R}$. For $e < 0$ it immediately follows that $S_{\eta_j} \cap \mathrm{R} = \varnothing$. For $e > 0$, it is not difficult to show that $S_\zeta \subset S^{(14)}_\zeta$.

Fig. 2 shows the change of the Hamiltonian $H(\zeta,\eta)$ for the test values of the integrals a, b, c, e for $d = 3$, $\lambda = \pm 1$, $\varepsilon \gtrless 0$. To sum up the above results, we conclude the following. In the 2D case the Hamiltonian (6) of the Eq. (5) is limited from below at the fixed projection of the momentum P_x for the integral values $e > 0$ and $a \geq 0$ (i.e. when $\lambda = 1$, $\varepsilon \leq 0$ in expression (6)) and has the local minima for $e > 0$ and $a < 0$ ($\lambda = 1$, $\varepsilon > 0$) and $e < 0$ and $a > 0$ ($\lambda = -1$, $\varepsilon < 0$) when the conditions (14) and (15), respectively, are satisfied.

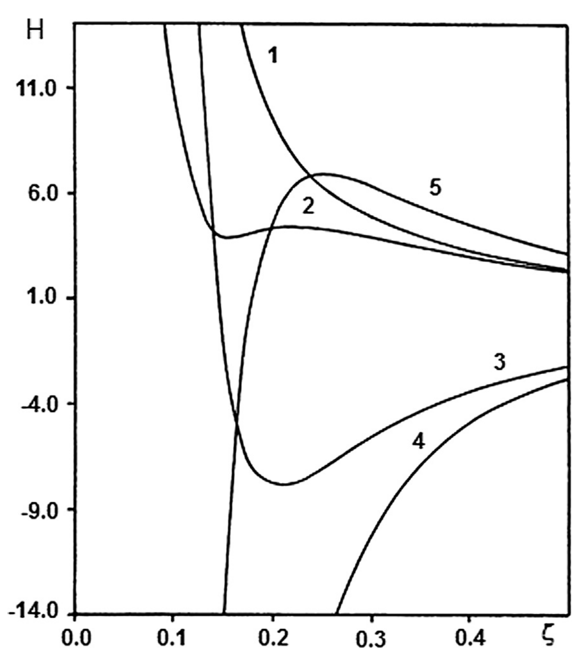

Fig. 2. Change of the Hamiltonian $\mathrm{H}(\zeta,\eta)$ for $d = 3$ along the lines $\eta = (2b/c)\zeta^{5/2}$ for the test values of the integrals: 1 – $a = 1, b = 1, c = 1, e = 0.025$; 2 – $a = 1, b = 1, c = 1, e = 0.017$; 3 – $a = -0.5, b = 1, c = 0.5, e = 0.02$; 4 – $a = -0.5, b = 1, c = 0.5, e = -0.02$; 5 – $a = 1, b = 1, c = 0.5, e = -0.02$.

In the 3D case H has a local minimum for $e > 0$ and $a \geq 0$ [i.e. when $\lambda = 1$, $\varepsilon \leq 0$ in (6)] if the condition (19) is satisfied, and it is limited from below for $e > 0$ and $a < 0$ ($\lambda = 1$, $\varepsilon > 0$). Note that the class of scale transformations (8) of course does not include all possible deformations of the Hamiltonian H but the estimations obtained above justify that it is limited for the cases considered when, according to the Lyapunov's theorem, absolutely and locally stable soliton solutions should exist. Analysis of the boundedness of H on the numerical solutions of (4) for $d = 2$ and $d = 3$ obtained in (Karpman and Belashov, 1991a,b; Belashov, 1997) was presented in (Belashov and Vladimirov, 2005) and it has confirmed the results presented above. This is a noteworthy fact that the GKP equation accounting, unlike the usual KP equation, the next order dispersive correction has the stable 3D solutions.

3. Case of the 3-DNLS equation

Consider now a case of the 3-DNLS equation in the BK system, i.e. Eqs. (1) and (3), using the same approach as in previous section for the GKP equation [see also (Belashov, 1999)]. We rewrite 3-DNLS Eqs. (1) and (3) by performing the formal change $u \to h$ into the Hamiltonian form

$$\partial_t h = \partial_x(\delta\mathrm{H}/\delta h), \tag{20}$$

with the Hamiltonian (Belashov, 2014)

$$\mathrm{H} = \int_{-\infty}^{\infty}\left[\frac{1}{2}|h|^4 + \lambda s h h^* \partial_x\varphi + \frac{1}{2}\kappa(\nabla_\perp\partial_x w)^2\right]d\mathbf{r},$$
$$\partial_x^2 w = h, \quad \varphi = \arg(h), \tag{21}$$

which has a sense of energy of the system, and solve the variation problem (7) where $P_x = \frac{1}{2}\int |h|^2 d\,\mathbf{r}$ is the momentum projection onto the x axis, that illustrates the fact that all finite solutions of Eq. (20) are the stationary points of the Hamiltonian for fixed P_x. It is needed now to prove the Hamiltonian's boundedness (from below) for fixed P_x. Consider the scale transformation $h\ (x, \mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{-1}h\ (x/\zeta, \mathbf{r}_\perp/\eta)$ $(\zeta, \eta \in \mathrm{C})$ conserving P_x, in complex vector space C. The Hamiltonian as a function of ζ, η is given by

$$\mathrm{H}(\zeta,\eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} + c\zeta^2\eta^{-2}, \tag{22}$$

where $a = (1/2)\int |h|^4 d\mathbf{r}$, $b = \lambda s \int h h^* \partial_x\varphi d\mathbf{r}$, $c = (\sigma/2)\int (\nabla_\perp\partial_x w)^2 d\mathbf{r}$. The necessary conditions for the existence of the extremum (10) immediately allows us to obtain the extremum's coordinates

$$\zeta = -a/c, \quad \eta = \{-(a/b)[1 + (a^2/c^{2)})]\}, \tag{23}$$

where $b < 0$ if $\eta \in R \subset C$ because $a > 0$, $c > 0$ by definition, and $b > 0$ if $\eta \in C$. The sufficient conditions for the existence of the local minimum of H at the point (ζ_i, η_i) are given by inequalities (11) and we therefore obtain for $b < 0$

$$a/c < d = (2\sqrt{2})^{-1}\sqrt{13 + \sqrt{185}}. \tag{24}$$

Thus it follows from (11) and (22)–(24) that the Hamiltonian H of Eq. (20) is limited from below, i.e.

$$H > -3bd/(1 + 2d^2) \tag{25}$$

where $b < 0$ if condition (24) holds. In this case the 3D solutions of 3-DNLS equation are stable. The solutions are unstable in the opposite case, $ac^{-1} \geqslant d,\ b < 0$. Condition $b < 0$ corresponds to the right circularly polarized wave with $\beta = 4\pi nT/B^2 > 1$, i.e. when $\lambda = 1,\ s = -1$ in Eqs. (1) and (3), and to the left circularly polarized wave when $\lambda = -1, s = 1$. It is necessary to note that the sign change $\lambda = 1 \to -1,\ s = -1 \to 1$ is equivalent to the change $t \to -t,\ \kappa \to -\kappa$ and for negative κ the Hamiltonian becomes negative in the area "occupied" by the 3D wave weakly limited in the $\mathbf{k}_\perp$ -direction; in this case condition (25) is not satisfied. The change of the sign of b to positive [when $\lambda = 1,\ s = 1$ or $\lambda = -1,\ s = -1$ in Eqs. (1) and (3)] is equivalent to the analytical extension of solution from real values of $y,\ z$ to the pure imaginary ones: $y \to -iy,\ z \to -iz$ and, therefore, equivalent to the change of sign of κ in the basic equations. In this case instead of inequality (25) the opposite inequality will take place. From the physical point of view this means that if the opposite inequality is satisfied, the right polarized wave with the positive nonlinearity and the left polarized wave with the negative nonlinearity are stable. Note that in the particular case $\kappa = 0$ in Eqs. (1) and (2) (1D approximation), instead of inequality (25) and the opposite one, it is easy to obtain the conditions $H > 0$ and $H < 0$, respectively, that is completely in agreement with the results obtained in (Dawson and Fontán, 1988) for the 1-DNLS equation.

Thus the analysis of the transformation properties of the Hamiltonian of the 3-DNLS equation allows us to determine the ranges of the respective coefficients as well as H which has the sense of the energy of the system, corresponding to the stable and unstable 3D solutions. So, we have proved the possibility of existence in the 3-DNLS model of absolutely stable 3D solutions.

4. Conclusion

So, we have investigated analytically the problem of stability of multidimensional solitons and nonlinear wave packets in the framework of model of the BK class of equations. Under the assumption of negligible dissipative effects, these solutions coincide with those of the GKP class equations in the form (1) and (2) with $\nu = 0$ and the 3-DNLS Eqs. (1) and (3). In the first section we have presented analytical estimates and formulate the sufficient conditions for the stability of solutions of GKP equation in the 2D and 3D cases, based on the transformational properties of the system's Hamiltonian for the whole range of the dispersive coefficients. Then an analogous problem for the 3-DNLS equation in the 3D geometry has been studied. Despite the fact that the considered classes of the Hamiltonian's deformations for both equations do not include all possible deformations of H, the obtained results clearly demonstrate the stability of the solutions if some (found and formulated) conditions are satisfied and can at least be considered as the necessary conditions of the stability of the multidimensional solutions.

The application of our analysis to the problem of the FMS waves beam's propagation in magnetized plasma enables us to prove (Belashov and Belashova, 2016), for example, that the 3D beam propagating at θ angle to magnetic field doesn't focuse and becomes stationary and stable in the cone of $\theta < \arctan\ (M/m)^{1/2}$ when inequality

$$(m/M - \cot^2\theta)^2[\cot^4\theta(1 + \cot^2\theta)]^{-1} > 4/3$$

is satisfied. Let us note also that obtained results give us the possibility to interpret correctly some our numerical and theoretical results on the dynamics of the internal gravity waves' solitons, induced by the pulse-type sources, which propagate at heights of the ionosphere F region (Belashov and Belashova, 2015b) from the point of view of such solitons stability.

Note also, that our analytical results presented above are well confirmed by the results of our numerical experiments on study of structure and stability of multidimensional solitons in the model of the 3-DNLS equations (Belashov and Vladimirov, 2005; Belashov, 2014; Belashov and Belashova, 2015a). So, we have obtained that for a solitary wave propagating in a plasma, on a level with wave spreading and wave collapse (in other terminology, self-contraction), the formation of the 3D solitons can be observed. These results are well applicable and useful in studies of dynamics of the Alfvén waves propagating in space plasma.

Obtained results and our approach can be also useful for other nonlinear systems describing the wave processes in a laser and space plasmas, for example in the problems which have been considered for the generalized nonlinear Schrödinger (NLS) equation (Kharshiladze et al., 2017) and other nonlinear systems in a plasma and in near-Earth environment [see, for example, (Popel et al., 1995, Kopnin et al., 2004; Belashov, 2017)].

Acknowledgments

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University. This work was also supported by the Shota Rustaveli National Science Foundation (SRNF), grant no. FR17_252.

Appendix A

Performing the transform $t \to t' + 8a^3b/c^4$ in Eq. (12), we obtain the reduced equation

$$t'^4 + pt'^2 + qt' + r = 0. \quad \text{(A1)}$$

The cubic resolvent kernel $z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0$, using the change $z \to x - 2p/3$, can be reduced to the equation

$$x^3 + p'x + q' = 0, \quad \text{(A2)}$$

where $p' = 2^{10}be^3/c^4$; $q' = -2^{14}a^2b^2e^4/c^8$, with the discriminant

$$D = 2^{26}b^3c^{-12}e^8(2^4 3^{-3}e + a^4bc^{-4}). \quad \text{(A3)}$$

In the case $e > 0$ and, $a \gtrless 0$ we have $D > 0$, therefore Eq. (A2) as well as the resolvent kernel in the real vector space R for each quadruple of the values of functions $a, b, c, e \in$ R have one root. Thus, using the Descartes' rule of signs, we can conclude, that (12) for $e > 0$ and $a \gtrless 0$ has one positive root $t \in$ R (note that $t \leq 0$ does not satisfy (12), in this case $\zeta \notin$ R). It follows from the analysis of Eq. (12) that in the space R for $S_t \subset$ R the equalities

$$\inf_{a>0} S_t = \sup_{a<0} S_t = 4(be^3)^{1/4}/c, \quad \inf_{a<0} S_t = 0. \quad \text{(A4)}$$

take place.

Consider now the case when $e < 0$, $a \gtrless 0$. It follows from Eq. (12) that for $a \leq 0$ this equation foes not have roots $t > 0$ in the space R, therefore, we limit ourselves by an analysis of (12) for $a > 0$. When

$$F = a^4b/c^4e < -2^4 3^{-3} \quad \text{(A5)}$$

we have $D > 0$ from (A3). It then follows that (A2) and the resolvent kernel in the space R for each quadruple of the functions $a, b, c, e \in$ R have one root, and Eq. (12), taking into account the rule of signs, has two positive roots $t_{1,2} \in$ R.Let us estimate boundaries of the set $S_t \subset$ R. With the two changes $t \to t + h$ and $t \to -t + h$ in (12), we obtain, respectively, the sets

$$c^4 > 8ia^{4-i}bh^{-i}(ah + 2e)^{i-1}, \quad \text{(A6)}$$

$$(-1)^i c^4 > (-1)^i 8ia^{4-i}bh^{-i}(ah + 2e)^{i-1}, \quad \text{(A7)}$$

$I = 1, 2, 3, 4.$

Solving inequalities (A6) and (A7) with the condition (A5), we obtain

$$\begin{aligned}
&\sup S_t = 2^5 \cdot 3^{-3}\left[\sqrt{10}\cos(\psi_1/3 + 2\pi/3) - 4\right] a^{-1}e,\\
&\psi_1 = \text{Arccos}\left[-11/\left(2^5 \cdot 5\sqrt{10}\right)\right],\\
&\inf S_t = \min\left\{\max(h'_1, h'_2), 2^4 3^{-3}\left(\sqrt{10} - 8\right) a^{-1}e\right\},\\
&h'_1 = 8\left[\sqrt{-2FF'}\cos(\psi_2/3) + F\right]a^{-1}e,\\
&h'_2 = 8\left[\sqrt{-2FF'}\cos(\psi_2/3 + 4\pi/3) + F\right]a^{-1}e,\\
&\psi_2 = \text{Arccos}\left\{(2^7F^2 + 3 \cdot 2^3F + 3)/\left(2^3F'\sqrt{-2FF'}\right)\right\},\\
&FF' = 1 - 2F.
\end{aligned} \quad \text{(A8)}$$

If condition (A5) is not satisfied, we have $D \leq 0$. In this case, a simple analysis shows that (A2) and the resolvent kernel for each quadruple of the functions $a, b, c, e \in$ R $(e < 0, a > 0)$ have one positive and two negative roots. Therefore, Eqs. (A1) and (12) in the real vector space R have no roots.

References

Belashov, V.Yu., 1991. Stability of two- and three-dimensional solitons in weakly dispersive media. Sov. Phys. Doklady 36, 626–630.

Belashov, V.Yu., 1994. Nonlinear effects for FMS waves propagating in magnetized plasma. Plasma Phys. Control. Fusion 36, 1661–1669.

Belashov, V.Yu., 1997. The KP Equation and its Generalizations. Theory and applications. NEISRI FEB RAS, p. 162.

Belashov, V.Yu., 1999. The problem of stability for three-dimensional Alfvén waves propagating in magnetized plasma. Doklady Phys. 44, 327–329.

Belashov, V.Yu., Vladimirov, S.V., 2005. Solitary Waves in Dispersive Complex Media. Theory, Simulation, Applications. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, p. 304.

Belashov, V.Yu., 2014. Dynamics of multidimensional nonlinear wave structures of the solitons and vortex types in dispersive complex media. Theory, simulation, applications. In:17th Int. Congress on Plasma Physics – ICPP 2014, Lisbon, Portugal, Sept. 15–19, 2014, Book of Abstracts. BAP.P.35.

Belashov, V.Yu., Belashova, E.S., 2015a. Nonlinear dynamics of the 3D Alfvén waves in plasma of ionosphere and magnetosphere. J. Atm. Solar-Terr. Phys. 136, 150–154.

Belashov, V.Yu., Belashova, E.S., 2015b. Dynamics of IGW and traveling ionospheric disturbances in regions with sharp gradients of the ionospheric parameters. Adv. Space Res. 56, 333–340.

Belashov, V.Yu., Belashova, E.S., 2016. Nonlinear dynamics of 3D beams of fast magnetosonic waves propagating in the ionospheric and magnetospheric plasma. Geomag. Aeronomy 56, 716–723.

Belashov, V.Yu., 2017. Multidimensional nonlinear ion-acoustic waves in a weakly relativistic plasma. J. Astrophys. Aerospace Techn. 5, 57.

Dawson, S.P., Fonátn, C.F., 1988. Soliton decay of nonlinear Alfven waves: numerical studies. Phys. Fluids 31, 83–89.

Karpman, V.I., Belashov, V.Yu., 1991a. Dynamics of two-dimensional solitons in weakly dispersive media. Phys. Lett. 154A, 131–139.

Karpman, V.I., Belashov, V.Yu., 1991b. Evolution of three-dimensional nonlinear pulses in weakly dispersive media. Phys. Lett. 154A, 140–144.

Kharshiladze, O., et al., 2017. Modeling of nonelastic interactions of optical solitons. VIII Annual Meeting of the Georgian Mechanical Union, Tbilisi, Georgia, Sept. 29-30, 2017. Tbilisi University Press, 40–41.

Kopnin, S.I. et al., 2004. Localized structures of nanosize charged dust grains in Earth's middle atmosphere. Planet. Space Sci. 52, 1187–1194.

Kuznetsov, E.A., Turitsyn, S.K., 1982. Two- and three-dimensional solitons in weakly dispersive media. Soviet Phys. JETP 55, 844–852.

Popel, S.I. et al., 1995. Theory of modulational interactions in plasmas in the presence of an external magnetic field. Phys. Rep. C 259, 327–405.