# Local and weighted Marcinkiewicz exponents with applications

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**Abstract**

We introduce and study a family of characteristics for non-rectifiable plane curves and arcs in terms of integrals over their complements. Then we apply them to solve some certain versions of the Riemann boundary value problem.

## Introduction

A great body of recent works is dealing with various characteristics of point sets of sophisticated structure: fractals, non-rectifiable curves and so on. The most known are Hausdorff and Minkowskii dimensions (see, for instance, [1, 2]), and a number of new ones: Assouad and Aikawa dimensions and codimensions [3], approximation dimension [4, 5], refined metric dimension [6] and others.

In 2013 [8, 9] the author introduced a family of new metric characteristics for plane sets and called them Marcinkiewicz exponents for closed non-rectifiable Jordan curves. This name is connected with the fact that J. Marcienkiewicz first characterized features of subsets of Euclidean spaces in terms of certain integrals over their complements (see [13]). In the present paper we introduce their weighted and local versions for any compact sets, but mainly we are interested in non-rectifiable curves and arcs on the complex plane. In section 1 we study their properties and relations with known dimensions. In particular, we show that these exponents are characteristics of co-dimensional type.

Then we consider certain applications. We use these characteristics to solve the Riemann boundary value problems (see [10, 11, 12]) in domains with non-rectifiable boundaries. As known, there exists a lot of applications of that problems in mechanics, elasticity theory and others. In section 2 we consider so called continuous formulation of this problem, in section 3 – semi-continuous, and in the final section 4 – the jump problem on an open arc. We obtain new solvability conditions which are sharper than the known ones.

In particular, the improvement of known results is connected with the fact that new features of non-rectifiable curves allow more precise description of their local properties, including a phenomenon of local asymmetry. Let us explain this phenomenon. A curve $\Gamma$ on the complex plane divides small disk $B(t,r) = \{z : |z - t| < r\}$ with center at its interior point $t$ into left and right components $B^+$ and $B^-$. If $\Gamma$ is smooth at point $t$, and $S$ is mapping of symmetry relatively its tangent at point $t$, then the area of set $B^+ \triangle S(B^-)$ is $o(r^2)$ for $r \to 0$, i.e., it decreases faster than the area of $B(t,r)$. We understand this fact as local symmetry. Generally speaking, for a non-rectifiable curve we cannot find a symmetry axis with this property, i.e., non-rectifiable curves are locally asymmetric. In the present paper we introduce left and right (inner and outer) characteristics of plane curves, what allows us to improve solvability conditions for the locally asymmetric curves.

Let us note that the Marcinkiewicz exponents are useful for solving other problems. In paper [7] these characteristics are applied for building of generalized curvilinear integral over non-rectifiable curves. In turn, the obtained construction enables us to solve certain boundary value problems on that curves in other way.

# 1 The Marcinkiewicz exponents

Let $X = (X; d; \mu)$ be a metric measure space equipped with a metric $d$ and a Borel regular outer measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B = B(x; r) = \{y \in X : d(y; x) < r\}, x \in X, r > 0$. We assume that measure $\mu$ is doubling, i.e., there is a constant $C = C(X) > 0$ such that $\mu(B(x; 2r)) \le C\mu(B(x; r))$ for any ball $B \subset X$. Consequently, there exist a constant $C > 0$ and an exponent $s \geq 0$ such that

$$\frac{\mu(B(y;r))}{\mu(B(x;R))} \geq C\left(\frac{r}{R}\right)^s$$

whenever $0 < r \leq R < \operatorname{diam}(X)$, $x \in X$, and $y \in B(x;R)$. The most lower bound of the set of exponents for which the last inequality holds is also denoted by s and called the doubling dimension of $X$. The measure $\mu$ is $Q$-regular, for $Q \geq 1$, if there is a constant $c_Q \geq 1$ such that

$$c_Q^{-1} r^Q \leq \mu(B(x;r)) \leq c_Q r^Q$$

for all $x \in X$ and every $0 < r < \operatorname{diam}(X)$. Clearly, any $Q$-regular measure is doubling, and its doubling dimension is Q. The Lebesgue measure in Euclidean space $\mathbb{R}^n$ is $n$-regular.

Let compact set $E$ be a subset of a fixed open domain $Y \subset X$. We put

$$I_p(E,\mu) := \int\limits_{Y\setminus E} \frac{d\mu}{\operatorname{dist}^p(z,E)}.$$

**Definition 1** *The Marcinkiewicz exponent of set $E$ with respect to measure $\mu$ is the least upper bound of set $\{p : I_p(E,\mu) < \infty\}$. We denote it $\mathfrak{m}(E,\mu)$.*

In paper [9] the Marcinkiewicz exponents are defined for the case where $E$ is a closed curve on the complex plane $\mathbb{C}$, and $\mu$ is restriction of the plane Lebesgue measure $\mathcal{L}$ on either inner or outer domain of this curve.

**Definition 2** *The inner and outer Marcinkiewicz exponents of a closed plane curve $\Gamma$ with respect to measure $\mu$ are $\mathfrak{m}^+(\Gamma,\mu) := \mathfrak{m}(\Gamma,\mu^+)$ and $\mathfrak{m}^-(\Gamma,\mu) := \mathfrak{m}(\Gamma,\mu^-)$, where $\mu^+$ and $\mu^-$ are restrictions of measure $\mu$ on inner and outer domains of $\Gamma$.*

Clearly, in this case $\mathfrak{m}(\Gamma,\mu)$ is the least of inner and outer exponents. We name all these values the Marcinkiewicz exponents in connection with the Marcinkiewicz's idea to characterize sets by means of certain integrals over their complements (see, for instance, [13]).

We introduce also a local version of these values. We put

$$I_p(E,t,r,\mu) := \int\limits_{B(t,r)\setminus E} \frac{d\mu}{\operatorname{dist}^p(z,E)},$$

where $B(t,r)$ is ball of radius $r$ with center $t \in E$.

**Definition 3** *The local Marcinkiewicz exponent of set $E$ with respect to measure $\mu$ is the least upper bound of set $\{p : \lim_{r\to 0} I_p(E,t,r,\mu) < \infty\}$. We denote it $\mathfrak{m}(E,t,\mu)$.*

In what follows we consider mainly the case $d\mu = w(z)d\mathcal{L}$ where $w$ is a non-negative function, and write $\mathfrak{m}(E,w)$ and $\mathfrak{m}(E,t,w)$ instead of $\mathfrak{m}(E,\mu)$ and $\mathfrak{m}(E,t,\mu)$.

If $\Gamma$ is a closed plane curve, then we consider also its inner and outer local Marcinkiewicz exponents $\mathfrak{m}^+(\Gamma,t,\mu) := \mathfrak{m}(\Gamma,t,\mu^+)$ and $\mathfrak{m}^-(\Gamma,t,\mu) := \mathfrak{m}(\Gamma,t,\mu^-)$. Clearly, the local properties of a curve cannot depend on its closeness. Hence, we can introduce analogs of the inner and outer exponents for open arcs, too; see below Definition 6.

Let us describe certain properties of these characteristics. We compare them first with so called Aikawa dimension. H. Aikawa [14] defined it for Euclidean spaces. Then it was defined for arbitrary spaces with doubling measures [3]. According [15], we define Aikawa codimension of a set $E \subset X$ as follows.

**Definition 4** *Let $CA(E,\mu)$ consist of all non-negative values $q$ such that there exists positive constant $c_q$ satisfying estimate*

$$\int\limits_{B(\zeta,r)} \frac{d\mu(z)}{\operatorname{dist}^q(z,E)} \le c_q r^{-q}\mu(B(\zeta,r))$$

*for any $\zeta \in E$ and all sufficiently small $r$. We call $\sup CA(E,\mu)$ Aikawa codimension of set $E$ with respect to measure $\mu$, and denote it $\mathfrak{cda}(E,\mu)$.*

If $q \in CA(E,\mu)$, then integral $I_q(E,\mu)$ is finite. Thus, any set $E \subset X$ satisfies estimate $\mathfrak{cda}(E,\mu) \le \mathfrak{m}(E,\mu)$; if $\Gamma$ is a closed Jordan curve on a plane, then $\mathfrak{cda}(\Gamma,\mu) \le \mathfrak{m}^+(\Gamma,\mu)$ and $\mathfrak{cda}(\Gamma,\mu) \le \mathfrak{m}^-(\Gamma,\mu)$.

We do not define Assouad and Aikawa dimensions; the reader can find their definitions in [3, 15]. According to [15], for $Q$-regular measures $\mu$ both these dimensions of set $E \subset (X;d;\mu)$ are equal to $Q - \mathfrak{cda}(E,\mu)$. Moreover, the Assouad dimension cannot be lesser than the upper Minkowskii dimension $\overline{\mathrm{dm}}(E)$. This value is known also as upper box-counting dimension, Kolmogorov dimension (see, for instance, [1]) and upper metric dimension [16]. One of its definitions is

$$\overline{\mathrm{dm}}(E) := \limsup_{r\to 0} \frac{\log \mathcal{N}_r(E)}{-\log r},$$

where $\mathcal{N}_r(E)$ is the least number of balls of radius $r$ covering $E$. Note that the lower limit of the same fraction was considered in connection with metric dimensions earlier; see [17].

As a conclusion, we obtain

**Lemma 1** *The inequality* $\mathfrak{cda}(E,\mu) \le \mathfrak{m}(E,\mu)$ *is valid for any set* $E$ *in any space* $(X,d,\mu)$*. If* $X = \mathbb{C}$ *and* $\Gamma$ *is a closed Jordan curve on the complex plane, then* $\mathfrak{cda}(\Gamma,\mu) \le \mathfrak{m}^+(\Gamma,\mu)$ *and* $\mathfrak{cda}(\Gamma,\mu) \le \mathfrak{m}^-(\Gamma,\mu)$*.*

*If measure* $\mu$ *is* $Q$*-regular, then* $\mathfrak{m}(E,\mu) \ge Q - \overline{\mathrm{dm}}(E)$*.*

*For* $X = \mathbb{C}$ *with standard metrics and* $\mu = \mathcal{L}$ *we have* $\mathfrak{m}(E,\mathcal{L}) \ge 2 - \overline{\mathrm{dm}}(E)$*. If* $E \subset \mathbb{C}$ *is a continuum, then* $\mathfrak{m}(E,\mathcal{L}) \le 1$*. The Marcinkiewicz exponents of rectifiable curves on the complex plane are equal to 1.*

It remains to show that $\mathfrak{m}(E,\mathcal{L}) \le 1$ for any continuum $E$. Let us call to our mind, that continuum is a connected set containing at least two distinct points. Hence, $\iint\limits_{Q\setminus E} \frac{dxdy}{\mathrm{dist}(x+iy,E)} = \infty$, which concludes the proof of the inequality. The same consideration proves bounds $\mathfrak{m}^+(\Gamma,\mathcal{L}) \le 1$ and $\mathfrak{m}^-(\Gamma,\mathcal{L}) \le 1$ for closed curve $\Gamma$ on the complex plane. As the Minkowskii dimension of any rectifiable plane curve is 1, then the last statement of Lemma is proved, too.

**Lemma 2** *If the set* $E$ *is compact then* $\inf\{\mathfrak{m}(E;t;\mu) : t \in E\} = \mathfrak{m}(E;\mu)$*. For a closed curve* $\Gamma$ *we have* $\inf\{\mathfrak{m}^+(\Gamma;t;\mu) : t \in \Gamma\} = \mathfrak{m}^+(\Gamma;\mu)$ *and* $\inf\{\mathfrak{m}^-(\Gamma;t;\mu) : t \in \Gamma\} = \mathfrak{m}^-(\Gamma;\mu)$*.*

*Proof.* Let us fix a value $p < \mathfrak{m}(E;\mu)$. By definition $I_p(E,\mu) < \infty$. Hence, for any $t \in \Gamma$ and sufficiently small $r > 0$ we have $I_p(E,t,r,\mu) < \infty$, and $\mathfrak{m}(E;t;\mu) \ge p$ for any $t \in E$. Whence, $\inf\{\mathfrak{m}(E;t;\mu) : t \in E\} \ge \mathfrak{m}(E;\mu)$.

Assume that $\inf\{\mathfrak{m}(E;t;\mu) : t \in E\} > \mathfrak{m}(E;\mu)$. Then there exists a value $p$ such that $\inf\{\mathfrak{m}(E;t;\mu) : t \in \Gamma\} > p > \mathfrak{m}(E;\mu)$. Then for any point $t \in E$ there exists radius $r(t) > 0$ such that integral $I_p(E,r,t,\mu)$ converges. The balls $B(t,r(t))$ cover compact set $E$. We select finite covering consisting of balls $B_j = B(t_j,r(t_j))$, $\quad j = 1,2,\ldots,n$, and put $\Delta = Q \bigcap (\bigcup\limits_{j=1}^{n} B_j)$. Clearly, $\int\limits_{\Delta\setminus E} \frac{d\mu}{\mathrm{dist}^p(z,E)} < \infty$. But integral $\int\limits_{Q\setminus\Delta} \frac{d\mu}{\mathrm{dist}^p(z,E)}$ is finite, too. Thus, $I_p(E,\mu) < \infty$, and inequality $p > \mathfrak{m}(E,\mu)$ is impossible. The proof for inner and outer exponents is just the same.

Now we consider certain examples.

**Example 1** *We consider first a curve from the notice [9] (cf. [19], [6] and [20]). We divide the upper side* $\{x+iy : 0 \le x \le 1, y=0\}$ *of square* $S = \{x+iy : 0 \le x \le 1, -1 \le y \le 0\}$ *on the complex plane into segments* $I_n := \{2^{-n} \le x \le 2^{-n+1}, y = 0\}, n = 1, 2, \dots$ *Then we fix* $\alpha \ge 1$ *and* $\beta \ge 1$*, and divide each of the segments* $I_n$ *on* $2^{[n\beta]}$ *equal parts; here* $[\cdot]$ *stands for the entire part. We denote the points of division of segment* $I_n$ *by* $x_{nj}$ *in decreasing order. Let* $p_{nj} := \{x+iy : x_{nj} - C_n \le x \le x_{nj}, 0 \le y \le 2^{-n}\}$*. Here* $C_n = \frac{1}{2}a_n^{\alpha}$*, where* $a_n$ *is a distance between neighboring points of division of segment* $I_n$*, i.e.,* $a_n = 2^{-n-[n\beta]}$*. Then rectangles* $p_{nj}$ *are mutually disjoint. We put* $D_1 := S \bigcup (\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{2^{[n\beta]}} p_{nj})$*, and denote boundary of domain* $D_1$ *by* $\Gamma_1$*. Clearly, this curve consists of infinite number of vertical and horizontal segments condensing to the origin. The summary length of the vertical segments is infinite. As shown in [6],* $\overline{\mathrm{dm}}(\Gamma_1) = \dfrac{2\beta}{\beta+1}$ *for any* $\alpha \ge 1$*. Thus, the upper Minkowskii dimension of* $\Gamma_1$ *does not depend on* $\alpha$*.*

*We consider weight* $w(z) = |z|^{-\gamma}$ *for real* $\gamma \in [0, 1)$*. If point* $t$ *does not coincide with the origin, then its sufficiently small neighborhood contains a rectifiable arc of* $\Gamma_1$*. Hence, we have* $\mathfrak{m}^+(\Gamma_1; t; w) = \mathfrak{m}^-(\Gamma_1; t; w) = 1$ *for* $t \ne 0$*.*

*It remains to consider the case* $t = 0$*. Immediate calculation shows that the integral* $\displaystyle\iint\limits_{B(t,r)\bigcap D} \frac{|z|^{-\gamma}dxdy}{\mathrm{dist}^p(z,\Gamma_1)}$ *converges if and only if* $\alpha(\beta+1)(1-p) > \beta + \gamma - 1$*, i.e.,* $\mathfrak{m}^+(\Gamma_1; 0; w) = 1 - \dfrac{\beta+\gamma-1}{(\beta+1)\alpha}$*. Analogous consideration gives* $\mathfrak{m}^-(\Gamma_1; 0; w) = \dfrac{2-\gamma}{\beta+1}$*. Thus, by virtue of Lemma 2*

$$\mathfrak{m}^+(\Gamma_1; w) = 1 - \frac{\beta+\gamma-1}{\alpha(\beta+1)}, \quad \mathfrak{m}^-(\Gamma_1; w) = \frac{2-\gamma}{\beta+1}.$$

**Example 2** *We put* $S' = \{x+iy : 0 \le x \le 1, 0 \le y \le 1\}$ *and* $D_2 := S' \setminus (\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{j=1}^{2^{[n\beta]}} p_{nj})$*. Curve* $\Gamma_2 := \partial D_2$ *also consists of infinite number of vertical and horizontal segments condensing to the origin. Clearly, the weighted Marcienkiewicz exponents of this curve satisfies all relations from Example 1 if we replace the sign plus in the superscripts by minus and vice versa.*

**Example 3** *Let $\alpha \geq 1$, $\beta > 1$ and $\mu$ is the Lebesgue measure $\mathcal{L}$. We consider domain $D_1$ from Example 1 and domain $D_2$ from Example 2, and put $D_3 := D_2 \bigcup D'$, where $D' := \{z : \overline{z} - 1 \in D_1\}$. Then curve $\Gamma_3 := \partial D_3$ consists of infinite number of vertical and horizontal segments condensing to points 0 and 1. Then $\mathfrak{m}^+(\Gamma_3; 0; \mathcal{L}) = \mathfrak{m}^-(\Gamma_3; 1; \mathcal{L}) = \dfrac{2}{\beta+1}$, $\mathfrak{m}^-(\Gamma_3; 0; \mathcal{L}) = \mathfrak{m}^+(\Gamma_3; 1; \mathcal{L}) = 1 - \dfrac{\beta - 1}{\alpha(\beta+1)}$, and $\mathfrak{m}^\pm(\Gamma_3; t; \mathcal{L}) = 1$ for $t \neq 0, 1$. Thus, $\mathfrak{m}^+(\Gamma_3; \mathcal{L}) = \mathfrak{m}^-(\Gamma_3; \mathcal{L}) = 2 - \overline{\mathrm{dm}}(\Gamma_3)$, but at each point $t \in \Gamma_3$ at least one of exponents $\mathfrak{m}^+(\Gamma_3; t; \mathcal{L})$, $\mathfrak{m}^-(\Gamma_3; t; \mathcal{L})$ is greater.*

In connection with the last example we introduce one more version of the Marcienkiewicz exponent.

**Definition 5** *Let $\Gamma$ be a closed Jordan curve. We call value*

$$\mathfrak{m}^*(\Gamma; \mu) := \inf\{t \in \Gamma : \max\{\mathfrak{m}^+(\Gamma; t; \mu), \mathfrak{m}^-(\Gamma; t; \mu)\}\}$$

*refined Marcienkiewicz exponent.*

Thus, $\mathfrak{m}(\Gamma_3; \mathcal{L}) = 2 - \overline{\mathrm{dm}}(\Gamma_3)$, but $\mathfrak{m}^*(\Gamma_3; \mathcal{L}) > 2 - \overline{\mathrm{dm}}(\Gamma_3)$.

# 2 The Riemann boundary value problem on a closed non-rectifiable curve

The Riemann boundary value problem is well known in complex analysis (see [10, 11, 12]). In the present section we consider it on a closed Jordan curve.

Let a curve $\Gamma$ divide the complex plane $\mathbb{C}$ into domains $D^+$ and $D^- \ni \infty$. We seek a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

The coefficients $G(t)$ and $g(t)$ are given. The boundary values of desired function $\Phi^+(t)$ and $\Phi^-(t)$ are limits of $\Phi(z)$ for $z$ tending to point $t \in \Gamma$ from domains $D^+$ and $D^-$ correspondingly. The simplest case is so called jump problem

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma. \tag{2}$$

The classical results here base on the properties of Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{g(t)dt}{t-z}, \quad z \notin \Gamma,$$

over piecewise-smooth curve $\Gamma$. If $g$ satisfy the Hölder condition

$$\sup\left\{\frac{|g(t') - g(t'')|}{|t' - t''|^{\nu}} : t', t'' \in \Gamma, t' \neq t''\right\} := h_{\nu}(g, \Gamma) < \infty$$

with exponent $\nu \in (0, 1]$, then (see, for instance [10, 11, 12]) the function $\Phi(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ and has boundary values $\Phi^{+}(t)$ and $\Phi^{-}(t)$ satisfying equality (2). Thus, the Cauchy integral with density $g \in H_{\nu}(\Gamma)$ is a solution of jump problem (2). Problem (1) is reducible to the jump problem. Hence, the whole theory of the Riemann boundary value problem on piecewise-smooth curves reduces to application of the cited above result on the boundary behavior of the Cauchy integral. But this integral is not defined for non-rectifiable curves.

Let us describe a way for solving the Riemann problem on non-rectifiable curves offered by B.A. Kats. Assume that $g \in H_{\nu}(\Gamma)$; here and in what follows $H_{\nu}(\Gamma)$ stands for the set of all defined on $\Gamma$ functions satisfying the Hölder condition with exponent $\nu$. We apply to $g$ Whitney extension operator $\mathcal{E}$. As known (see, for instance, [13]), function $u = \mathcal{E}g$ is defined on the whole complex plane, $u|_{\Gamma} = g$, $u \in H_{\nu}(\mathbb{C})$, and $h_{\nu}(u, \mathbb{C}) = h_{\nu}(g, \Gamma)$. In addition, the function $u$ has partial derivatives of all orders in $\mathbb{C} \setminus \Gamma$, and

$$|\nabla u(z)| \leq \frac{h_{\nu}(g, \Gamma)}{\operatorname{dist}^{1-\nu}(z, \Gamma)}. \tag{3}$$

We consider function $\varphi(z) = u(z)\chi^{+}(z)$, where $\chi^{+}(z)$ is a characteristic function of domain $D^{+}$. It equals to 1 for $z \in D^{+}$ and to 0 for $z \in D^{-}$. Clearly, function $\varphi(z)$ has a jump $g$ on $\Gamma$, but it is not holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$. In [21, 22] that function is called a quasi-solution of the jump problem (2). We turn it into a solution of the problem by means of transformation

$$R : \varphi(z) \mapsto \Phi(z) := \varphi(z) - \frac{1}{2\pi i}\iint\limits_{\mathbb{C}}\frac{\partial\varphi}{\partial\overline{\zeta}}\frac{d\zeta d\overline{\zeta}}{\zeta - z}. \tag{4}$$

It is representable as $R = I - T\frac{\partial}{\partial\overline{\zeta}}$, where $I$ is the identity operator, and

$$T : f \mapsto \frac{1}{2\pi i}\iint\limits_{\mathbb{C}}\frac{f(\zeta)d\zeta d\overline{\zeta}}{\zeta - z}.$$

The properties of potential $T$ are well known (see, for instance, [23]). If $f$ has compact support and is integrable with certain exponent $p > 2$, then $Tf$ is continuous in the whole complex plane and satisfies there the Hölder condition with exponent $1 - \frac{2}{p}$, and $\frac{\partial}{\partial \overline{z}} Tf = f(z)$. Thus, transformation $R$ regularizes quasi-solution $\varphi$ if derivative $\frac{\partial \varphi}{\partial \overline{z}}$ is integrable in degree greater than two. Inequality (3) means that the integrability of the first derivatives of $\varphi$ follows from integrability of function $\mathrm{dist}^{-1}(z, \Gamma)$ in appropriate degrees. As known (see, for instance, [19, 21]), it is locally integrable in $\mathbb{C}$ in any power lesser than $2 - \overline{\mathrm{dm}}\,\Gamma$. As a result, there is valid

**Theorem A** *If $g \in H_\nu(\Gamma)$, $\nu > \overline{\mathrm{dm}}(\Gamma)/2$, then the jump problem (2) is solvable.*

This is the first result on solvability of the Riemann boundary value problem on non-rectifiable curves. It was announced in [18] and proved in [19].

But the inner Marcienkievicz exponent is the immediate characteristic of integrability of $\mathrm{dist}^{-1}(z, \Gamma)$ in $D^+$. By virtue of inequality (3) and Definition 2 the derivative $\frac{\partial \varphi}{\partial \overline{z}}$ is integrable with exponent $p > 2$ and, consequently, the problem (2) is solvable for $\nu > (2 - \mathfrak{m}^+(\Gamma; \mathcal{L}))/2$. Example 1 shows that this condition for solvability of the jump problem is sharper than Theorem A. But quasi-solution is not unique. Another quasi-solution with compact support is $\varphi(z) = u(z)(\chi^+(z) - \omega(z))$, where $\omega \in C_0^\infty(\mathbb{C})$ equals to 1 in $\overline{D^+}$. This quasi-solution vanishes in $D^+$, and integrability properties of its derivatives is determined by the outer Marcienkievicz exponent of $\Gamma$. Hence, the problem is solvable for $\nu > (2 - \mathfrak{m}^-(\Gamma; \mathcal{L}))/2$, and Example 2 shows that this result is sharper than Theorem A, too. Thus, there is valid

**Theorem B** *Let $m := \max\{\mathfrak{m}^+(\Gamma; \mathcal{L}); \mathfrak{m}^-(\Gamma; \mathcal{L})\}$ and $g \in H_\nu(\Gamma)$. If $\nu > (2-m)/2$, then the jump problem on a closed curve $\Gamma$ has a solution $\Phi = R\varphi$, where $\varphi$ is one of the described above quasi-solutions. This solution satisfies in domains $D^+$ and $D^-$ the Hölder condition with any exponent lesser than $1 - 2(1-\nu)/m$.*

This result was announced in other terms in [8, 9].

The curve from Example 3 has the same singularities as the curves from Examples 1 and 2, but Theorems B and A give for curve $\Gamma_3$ the same solvability conditions. The following result includes it to the class of curves where Theorem B improves Theorem A.

**Theorem 1** *Let $m^* := \mathfrak{m}^*(\Gamma; \mathcal{L})$ and $g \in H_\nu(\Gamma)$. If $\nu > (2 - m^*)/2$, then the jump problem on a closed curve $\Gamma$ has a solution satisfying the Hölder condition with any exponent lesser than $1 - 2(1-\nu)/m^*$ in domains $D^+$ and $D^-$.*

*Proof.* Let us fix a value $m < m^*$ such that $\nu > (2-m)/2$. By definition of the refined Marcienkiewicz exponent for any $t \in \Gamma$ there exists radius $r = r(t) > 0$ such that either $I_m(\Gamma, t, r, \mathcal{L}^+) < \infty$ or $I_m(\Gamma, t, r, \mathcal{L}^-) < \infty$. The family of balls $\{B(t, r) : t \in \Gamma\}$ covers $\Gamma$. As set $\Gamma$ is compact, since this family contains a finite covering $\{B_j = B(t_j, r(t_j)) : j = 1, 2, \ldots, n\}$. Let $\psi_j \in C_0^\infty(\mathbb{C})$ be a non-negative function with support $\overline{B_j}$, $j = 1, 2, \ldots, n$. Then restriction $\sigma(t)$ of sum $\sum\limits_{j=1}^{n} \psi_j$ on curve $\Gamma$ is positive and $\sigma \in H_1(\Gamma)$. We put $g_j(t) := g(t)\psi_j(t)\sigma^{-1}(t), \quad t \in \Gamma$. Obviously, $g\sigma^{-1} \in H_\nu(\Gamma)$. If $I_m(\Gamma, t_j, r(t_j), \mathcal{L}^+) < \infty$, then we put $\varphi_j := \mathcal{E}(g\sigma^{-1})\psi_j\chi^+$, and if $I_m(\Gamma, t_j, r(t_j), \mathcal{L}^-) < \infty$, then $\varphi_j := \mathcal{E}(g\sigma^{-1})\psi_j(\chi^+ - \omega)$, where $\chi^+$ and $\omega$ are defined above. In the both cases $\varphi_j$ is a quasi-solution of the jump problem with jump $g_j$, its regularization $\Phi_j := R\varphi_j$ solves this problem. Hence, function $\Phi := \sum\limits_{j=1}^{n} \Phi_j$ is solution of the problem (2) satisfying all requirements of the theorem.

We will consider now the uniqueness of obtained solution by means of the E.P. Dolzhenko theorem [24], as it was done in paper [19]. Let $\mathfrak{dh}\, E$ stand for the Hausdorff dimension of compact set $E$ (see, for instance, [1, 3]), which is a subset of domain $G \subset \mathbb{C}$. The Dolzhenko theorem claims that if $\nu > \mathfrak{dh}\, E - 1$, then any holomorphic in $G \setminus E$ function $F \in H_\nu(\overline{G})$ is holomorphic in $G$; otherwise the Hölder space $H_\nu(\overline{G})$ contains a function which is holomorphic in $G \setminus E$, but not in $G$. By virtue of this theorem difference of two solutions of the problem (2) is a constant, if the both solutions satisfy Hölder condition with exponent $\nu > \mathfrak{dh}\, \Gamma - 1$ in $D^+$ and in $D^-$. In this connection we will study the Riemann boundary problem (1) in what follows under additional assumptions

$$\Phi|_{D^+} \in H_\lambda(D^+), \quad \Phi|_{D^-} \in H_\lambda(D^-), \quad \Phi(\infty) = 0. \tag{5}$$

Thus, solution of the jump problem (2) in class (5) is unique for $\lambda > \mathfrak{dh}\, \Gamma - 1$.

Then we apply the standard factorization technique (see [10, 11, 12]) for solving of problem (1). We fix a point $z_0 \in D^+$ and represent non-vanishing coefficient $G(t)$ as $G(t) = (t - z_0)^\varkappa \exp f(t)$, where integer number $\varkappa$ is so called index of the problem. Here $f$ satisfies the Hölder condition with the same exponent as $G$, and we find solution $\Phi_0$ of the jump problem with jump $f$ by means of Theorem 1. Then we evaluate function $X(z)$ equaling to $\exp\Phi_0(z)$ for $z \in D^+$, and to $(z-z_0)^{-\varkappa}\exp\Phi_0(z)$ for $z \in D^-$, and reduce (1) to the jump problem

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \quad t \in \Gamma.$$

The factor $1/X^+$ has intrinsic holomorphic extensions to $D^+$ and to $D^-$, what allows us to build quasi-solutions of this problem and to regularize them under assumptions of Theorem B (cf. [19]). We obtain

**Theorem 2** *Let* $m^* := \mathfrak{m}^*(\Gamma; \mathcal{L})$, $\nu > (2-m^*)/2$ *and* $1-2(1-\nu)/m^* > \lambda > \mathfrak{dh}\,\Gamma - 1$. *If* $G, g \in H_\nu(\Gamma)$ *and* $G(t)$ *does not vanish, then there are valid the following propositions on solvability of the Riemann boundary problem (1) in the class (5):*

*— for* $\varkappa = 0$ *the problem has a unique solution;*

*— for* $\varkappa > 0$ *the problem has a family of solutions depending on* $\varkappa$ *arbitrary complex constants;*

*— for* $\varkappa < 0$ *the problem has a unique solution if* $-\varkappa$ *solvability conditions are fulfilled.*

In other words, under assumptions of Theorem 2 the problem has just the same solvability properties as in the classical case (see [10, 11, 12]).

# 3 The semi-continuous case.

The semi-continuous version of Riemann boundary problem allows violation of equality (1) at a finite set of points of the curve $\Gamma$. We restrict ourselves to the jump problem, i.e., we seek holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ such that it has limit values $\Phi^+(t)$ and $\Phi^-(t)$ from $D^+$ and $D^-$ correspondingly at any point $t \in \Gamma' := \Gamma \setminus \tau$, $\tau := \{t_1, t_2, \ldots, t_m\} \subset \Gamma$, and these values satisfy relation

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma'. \tag{6}$$

For definiteness we have to assume the desired function having prescribed behavior at the points of set $\tau$, for instance (see [10, 11, 12]),

$$\Phi(z) = O(|z-t_j|^{-\gamma}), \quad \gamma = \gamma(\Phi) < 1, \quad j = 1, 2, \ldots, m. \tag{7}$$

The set $\tau$ consists of discontinuity points of both $\Phi$ and $g$. If the curve is of piecewise-smooth, then all discontinuities of solution $\Phi$ are caused by singularities of jump $g$ (see [10, 11, 12]). But in the case under consideration the desired function can loss continuity at a point where the jump is continuous, but local properties of the curve deteriorate; see [25], where this phenomenon is described in terms of the box-counting dimension as "paradoxical discontinuities". The Marcinkiewicz exponents seem to be more convenient for its description. Put

$$w(z) = \prod_{j=1}^{m} |z - t_j|^{-\gamma_j}, \quad 0 \leq \gamma_j < 1, \quad j = 1, 2, \dots, m,$$

$g = wg_0$, $g_0 \in H_\nu(\Gamma)$. We consider local quasi-solutions $\varphi_j(z) := w(z)\xi_j(z)$ where $\xi_j$ equals either $\mathcal{E}(g_0\sigma^{-1})\psi_j\chi^+$ or $w\mathcal{E}(g_0\sigma^{-1})\psi_j(\chi^+ - \omega)$, $\psi_j$, $\sigma$ and $\omega$ are the same functions as in the proof of Theorem 1. The sum $\varphi = \sum\limits_{j=1}^{n} \varphi_j$ is a global quasi-solution. Clearly, the integrability properties of $\dfrac{\partial \varphi}{\partial \overline{z}}$ are determined by derivatives

$$\frac{\partial w\xi_j}{\partial \overline{z}} = w\frac{\partial \xi_j}{\partial \overline{z}} - v_j(z), \quad v_j(z) := w(z)\xi_j(z)\sum_{j=1}^{m} \frac{\gamma_j}{2(\overline{z} - \overline{t}_j)}. \tag{8}$$

The functions $T(v_j)$ is continuous in $\overline{\mathbb{C}} \setminus t_j$, and equals to $O(|z - t_j|^{-\gamma_j})$ at point $t_j$ (see, for example, [23]). Integrability of the first term of (8) is characterized by the Marcinkiewicz exponents with weight $w$. We conclude by means of well known estimates for potential $T$ (see [23]) that function potential $U_j := T\left(w\dfrac{\partial \xi_j}{\partial \overline{z}}\right)$ exists if either $\nu > 1 - \mathfrak{m}^+(\Gamma; t_j; w)$ or $\nu > 1 - \mathfrak{m}^-(\Gamma; t_j; w)$, it satisfies bound $U_j(z) = O(|z - t_j|^{-\gamma'_j})$, $\gamma_j < \gamma'_j < 1$, near point $t_j$, and it is continuous in $\overline{\mathbb{C}} \setminus t_j$ for $\nu > (2 - m^*_j)/2$, $m^*_j := \inf\{\max\{\mathfrak{m}^+(\Gamma; t; \mathcal{L}), \mathfrak{m}^-(\Gamma; t; \mathcal{L})\} : t \in \Gamma \setminus t_j\}$. As a result, we obtain

**Theorem 3** *Let $g = wg_0$, $g_0 \in H_\nu(\Gamma)$, $\nu > 1 - \mathfrak{m}^*(\Gamma; w)$, and $\nu > (2 - m')/2$ where $m' := \inf\{\max\{\mathfrak{m}^+(\Gamma; t; \mathcal{L}), \mathfrak{m}^-(\Gamma; t; \mathcal{L})\} : t \in \Gamma'\}$, then the jump problem (6) has a solution satisfying (7).*

Of course, in general a solution of the semi-continuous jump problem cannot satisfy condition (5), and we have to introduce another uniqueness class.

But we do not discuss this question in the present paper. The same concerns apply to the jump problem on non-rectifiable arc (see next section).

**Example 4** *Let $\Gamma_1$ be the curve from Example 1, $w(z) = |z|^{-\gamma}$, $0 \le \gamma < 1$. We know that the curve is rectifiable outside of any neighborhood of origin, i.e., $\mathfrak{m}^{\pm}(\Gamma_1; t; \mathcal{L}) = \mathfrak{m}^{\pm}(\Gamma_1; t; w) = 1$ for $t \ne 0$, $\mathfrak{m}^{+}(\Gamma_1; 0; w) = 1 - \dfrac{\beta+\gamma-1}{(\beta+1)\alpha}$, and $\mathfrak{m}^{-}(\Gamma_1; 0; w) = \dfrac{2-\gamma}{\beta+1}$. Hence, $m' = 1$, $\mathfrak{m}^{*}(\Gamma; w) = 1 - \dfrac{\beta+\gamma-1}{(\beta+1)\alpha}$. Thus, the jump problem (6) has a solution satisfying (7) under condition $\nu > \max\left\{\dfrac{1}{2}, \dfrac{\beta+\gamma-1}{(\beta+1)\alpha}\right\}$.* [1]

# 4 Non-rectifiable arcs

Let $\Gamma$ be simple directed plane non-rectifiable arc beginning at point $t_1$ and ending at point $t_2$. We put $\Gamma' := \Gamma \setminus \{t_1, t_2\}$, and consider the jump problem (6) on open arc $\Gamma'$. In this section we understand $\Phi^{+}(t)$ and $\Phi^{-}(t)$ as limit values of $\Phi(z)$ at a point $t \in \Gamma'$ from the left and from the right correspondingly. The main difference of the jump problems for closed curves and for open arcs is connected with the fact, that in the case of open arcs we cannot use function $\chi^{+}$ in constructions of quasi-solutions. Here we offer two approaches to overcome this obstacle.

## 4.1 Local analysis

We need an analog of the inner and outer Marcinkiewicz exponents at a point $t \in \Gamma'$.

**Definition 6** *Let $t \in \Gamma'$. We fix small positive radius $r$ such that $\Gamma$ divides disk $B(t, r)$ on left and right components $B^{+}$ and $B^{-}$. The left and right local Marcinkiewicz exponents of arc $\Gamma$ with respect to measure $\mu$ at point $t$ are $\mathfrak{m}^{+}(\Gamma; t; \mu) := \mathfrak{m}(\Gamma; t; \mu^{+})$ and $\mathfrak{m}^{-}(\Gamma; t; \mu) := \mathfrak{m}(\Gamma; t; \mu^{-})$, where $\mu^{+}$ and $\mu^{-}$ are restrictions of measure $\mu$ on components $B^{+}$ and $B^{-}$ correspondingly. We put $\mathfrak{m}^{*}(\Gamma; t; \mu) := \max(\mathfrak{m}^{+}(\Gamma; t; \mu), \mathfrak{m}^{-}(\Gamma; t; \mu))$.*

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[1] The term $\dfrac{1}{2}$ can be omitted, because $\Gamma_1$ is piecewise-smooth outside of any neighborhood of the origin.

We say that a point $t \in \Gamma$ satisfies condition of smooth touch (ST-condition) if there exists a smooth arc $\lambda$ such that the intersection $\lambda \cap \Gamma$ consists of the point $t$. As shown in [26], the set $ST(\Gamma)$ of all points $t \in \Gamma$ satisfying ST-condition is everywhere dense in $\Gamma$.

We study first the case $t_{1,2} \in ST(\Gamma)$. Then there exists a smooth arc $\Lambda$ beginning at $t_1$ and ending at $t_2$, which has not other common points with $\Gamma$. Then union $\Gamma^* := \Gamma \cup \Lambda$ is simple closed curve, $\mathfrak{m}^{\pm}(\Gamma^*; t; \mathcal{L}) = \mathfrak{m}^{\pm}(\Gamma; t; \mathcal{L})$ for $t \in \Gamma'$, and $\mathfrak{m}^{\pm}(\Gamma^*; t; \mathcal{L}) = 1$ for $t \in \Lambda \setminus \{t_1, t_2\}$. Put $\mathfrak{m}^{\pm}(\Gamma); t_{1,2}; \mathcal{L}) := \mathfrak{m}^{\pm}(\Gamma^*; t_{1,2}; \mathcal{L})$, $\mathfrak{m}^{*}(\Gamma; t_{1,2}; \mathcal{L}) := \max(\mathfrak{m}^{+}(\Gamma); t_{1,2}; \mathcal{L}), \mathfrak{m}^{-}(\Gamma); t_{1,2}; \mathcal{L}))$; these values do not depend on the choice of arc $\Lambda$. We extend jump $g \in H_\nu(\Gamma)$ on $\Gamma^*$ (for instance, by the Whitney extension operator $\mathcal{E}$), and find a solution $\Phi^*$ of the jump problem on the closed curve $\Gamma^*$ with the extended jump by means of Theorem 1. Then difference

$$\Phi(z) := \Phi^*(z) - \frac{1}{2\pi i} \int\limits_{\Lambda} \frac{\mathcal{E}g(t)dt}{t - z}$$

is a solution of the jump problem on $\Gamma$, and its behavior at the points $t_{1,2}$ is determined by well known properties of its integral term (see [10, 11, 12]). Thus, there is valid

**Theorem 4** *Let $t_{1,2} \in ST(\Gamma)$, $m^* := \inf\{\mathfrak{m}^*(\Gamma; t; \mathcal{L}) : t \in \Gamma\}$ and $g \in H_\nu(\Gamma)$. If $\nu > (2-m^*)/2$, then the jump problem (6) on arc $\Gamma$ has a solution satisfying estimates*

$$\Phi(z) = (-1)^j \ln|z - t_j| + O(1), \quad z \to t_j, \quad j = 1, 2.$$

Here and in what follows a uniqueness class for solutions of the jump problem can be described in terms of the Hausdorff dimension (see [24]).

Then we consider a situation where the points $t_{1,2}$ do not satisfy the ST-condition. For instance, spiral arc $\{z = r \exp ir^{-\gamma} : 0 < r \leq 1\} \cup \{0\}$, $\gamma > 0$, does not allow smooth touch at the origin. But set $ST(\Gamma)$ is dense in $\Gamma$, and, consequently, $\Gamma$ is representable as union of either finite or countable family of mutually disjoint arcs $\Gamma_j$ with ends from this set. If the family of arcs is countable, then without loss of generality we consider that $t_{1,2}$ are the only limit points of their ends. According Theorem 4, we find solutions $\Phi_j$ of the jump problems on arcs $\Gamma_j$ with jumps $g|_{\Gamma_j}$. Then we regularize series $\sum\limits_{j=1}^{+\infty} \Phi_j$ as it is done in the known proof of Mittag–Leffler Theorem (see, for instance, [27]), and obtain

**Theorem 5** *Let* $m^* := \inf\{\mathfrak{m}^*(\Gamma; t; \mathcal{L}) : t \in \Gamma'\}$. *If* $g$ *satisfies the Hölder condition with exponent* $\nu$ *outside of arbitrarily small neighborhoods of points* $t_{1,2}$, *and* $\nu > (2 - m^*)/2$, *then the jump problem (6) is solvable.*

Note that the last theorem treats the jump problem free of any restrictions on behavior of solutions at the end-points $t_{1,2}$.

## 4.2 Logarithmic kernel

We consider function

$$k_\Gamma(z) := \frac{1}{2\pi i} \ln \frac{z - t_2}{z - t_1}, \quad z \in \mathbb{C} \setminus \Gamma, \tag{9}$$

where branch of the logarithmic function is determined by cut along arc $\Gamma$. In the case $t_j \in ST(\Gamma)$ we have $k_\Gamma(z) = O(\ln |z - t_j|^{-1})$ for $z \to t_j$, $j = 1, 2$. In general the rate of growth of $k_\Gamma(z)$ for $z \to t_{1,2}$ can be arbitrarily high (see Example 5 below). B.A. Kats [26] offered to use this function in constructions of quasi-solutions of the jump problems on arcs. Obviously,

$$k_\Gamma^+(t) - k_\Gamma^-(t) = 1, \quad t \in \Gamma',$$

i.e. product $k_\Gamma \mathcal{E} g$ is a quasi-solution of problem (6) for $g \in H_\nu(\Gamma)$. Its regularization leads in [26] to the following result.

**Theorem C** *Let* $k_\Gamma(z) = O(|z - t_j|^{-\lambda_j})$ *for* $z \to t_j$, $j = 1, 2$. *If* $g \in H_\nu(\Gamma)$, $\nu > \overline{\mathrm{dm}}\,\Gamma/2$, *and* $\lambda_j < (2\nu - \overline{\mathrm{dm}}\,\Gamma)/(2 - \overline{\mathrm{dm}}\,\Gamma)$, *then jump problem (6) has a solution satisfying estimate* $\Phi(z) = g(t_j)k_\Gamma(z) + O(1)$ *for* $z \to t_j$, $j = 1, 2$.

The concept of weighted Marcinkiewicz exponents implies the following sharpening of this theorem.

**Theorem 6** *Let* $k_\Gamma \in L_{loc}(\mathbb{C})$. *If* $g \in H_\nu(\Gamma)$, $\nu > (2 - \mathfrak{m}(\Gamma, |k_\Gamma|))/2$, *then jump problem (6) has a solution satisfying estimate* $\Phi(z) = g(t_j)k_\Gamma(z) + O(1)$ *for* $z \to t_j$, $j = 1, 2$.

The local exponents allow us to obtain the following complement of this result. Let us fix two points $\tilde{t}_{1,2} \in ST(\Gamma)$. They divide $\Gamma$ into three arcs: arc $\Gamma_1$ beginning at $t_1$ and ending at $\tilde{t}_1$, arc $\Gamma_2$ beginning at $\tilde{t}_1$ and ending at $\tilde{t}_2$, and arc $\Gamma_3$ beginning at $\tilde{t}_2$ and ending at $t_2$. We consider the jump problems on these arcs. The problem for arc $\Gamma_2$ is solved in Theorem 4. Theorem 6 describes solutions for arcs $\Gamma_1$ and $\Gamma_2$. As points $\tilde{t}_{1,2}$ are arbitrarily close to $t_{1,2}$, then their solvability is determined by local exponents $\mathfrak{m}(\Gamma; t_{1,2}; |k_\Gamma|)$. We obtain

**Theorem 7** *Let* $k_\Gamma \in L_{loc}(\mathbb{C})$, $m^* := \inf\{\mathfrak{m}^*(\Gamma; t; \mathcal{L}) : t \in \Gamma'\}$, $m_{end} := \min\{\mathfrak{m}(\Gamma; t_j; |k_\Gamma|) : j = 1, 2\}$. *If* $g \in H_\nu(\Gamma)$, $\nu > (2-m^*)/2$, *and* $\nu > 1-m_{end}$, *then jump problem (6) has a solution. If, in addition,* $\nu > (2-m_{end})/2$, *then it has a solution satisfying estimate* $\Phi(z) = g(t_j)k_\Gamma(z) + O(1)$ *for* $z \to t_j$, $j = 1, 2$.

**Example 5** *We fix a positive value* $q$, *and consider infinite sequence of points* $z_1 = 1$, $z_{2+4k} = -z_{4(k+1)} = \frac{1+i}{(k+1)^q}$, $z_{3+4k} = -z_{5+4k} = \frac{1-i}{(k+1)^q}$ *for* $k = 0, 1, 2, \ldots$ . *We connect consecutive points* $z_n$ *and* $z_{n+1}$ *by segment* $S_n$, *and build arc* $\Gamma = \{0\} \bigcup \left( \bigcup\limits_{n=1}^{+\infty} S_n \right)$ *beginning at the origin and ending at point* $z_1 = 1$. *Clearly, the length of arc* $\Gamma$ *is finite length for* $q > 1$, *and infinite for* $q \leq 1$; *all points of* $\Gamma$ *excluding origin satisfy the* $ST-$*condition. Immediate calculations show that* $k_\Gamma(z) = (2\pi i)^{-1} \ln |z - 1| + O(1)$ *for* $z \to 1$, *and* $k_\Gamma(z) = O(|z|^{-1/q})$ *for* $z \to 0$. *Hence,* $k_\Gamma \in L_{loc}(\mathbb{C})$ *for* $q > 1/2$. *As the arc is locally rectifiable everywhere excluding origin, then* $\mathfrak{m}^\pm(\Gamma; t; \mathcal{L}) = 1$ *for* $t \in \Gamma'$, $\mathfrak{m}(\Gamma; 1; \mathcal{L}) = \mathfrak{m}(\Gamma; 1; |k_\Gamma|) = 1$. *For* $q > 1$ *we have* $\mathfrak{m}(\Gamma; 0; \mathcal{L}) = 1$, *too. Easy evaluation shows that for* $q < 1$ *we have* $\mathfrak{m}(\Gamma; 0; \mathcal{L}) = 2q/(q + 1)$, *and* $\mathfrak{m}(\Gamma; 0; |k_\Gamma|) = (2q - 1)/(q + 1)$ *for* $1/2 < q < 1$.

*By virtue of Theorem 5 the jump problem on this arc has a solution free of restrictions at its ends for* $\nu > 1/2$, *but as the arc is piecewise-smooth outside of any neighborhood of origin, then that solution exists for any* $\nu > 0$. *For* $q > 1/2$, $\nu > 1.5(1 + q)^{-1}$ *it has a solution satisfying estimates* $\Phi(z) = g(t_j)k_\Gamma(z) + O(1)$ *for* $z \to t_j$, $j = 1, 2$.

Probably, we can apply the Marcinkiewicz exponents in the boundary value problems for various versions of generalized analytic functions on non-rectifiable curves.

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