Explicit algorithms to solve a class of state constrained parabolic optimal control problems

A. Lapin, E. Laitinen, S.Lapin

Abstract

We consider an optimal control problem of a system governed by a linear parabolic equation with the following features: control is distributed, observation is either distributed or final, there are constraints on the state function and on its time derivative. Iterative solution methods are proposed and investigated for the finite difference approximations of these optimal control problems. Due to explicit in time approximation of the state equation and the appropriate choice of the preconditioners in the iterative methods, the implementation of all constructed methods is carried out by explicit formulaes. Computational experiments confirm the theoretical results.

Key words: optimal control, finite difference method, constrained saddle point problem, iterative method

Introduction

State constraints in optimal control of systems governed by partial differential equations play an important role in many real world applications. For instance, in continuous casting process a need to prevent the cracks in a slab and the solidification at a wrong place leads to the bounds on the temperature variable ([1] and bibliography therein). Similar demands arise in the processes of crystal growth [2] and cooling of glass melts [3].

A lot of contributions are known on elliptic optimal control problems with pointwise bounds on the state ([4] – [8]) and on the gradient of state ([9] – [14]). On the other hand, only a few articles deal with parabolic optimal control problems with pointwise bounds on the state [15], [16]. In our best knowledge, numerical analysis of parabolic optimal control problems with pointwise bounds on the time derivative for the state has not yet considered in the literature.

In this paper we consider a parabolic optimal control problem in a parallelepiped with distributed control, distributed or final observation, and pointwise constraints on the control, state and on time derivative of state. We approximate this problem by a finite difference scheme and construct preconditioned Uzawa-type iterative solution methods for the corresponding mesh (finite-dimensional) problems.

The constructed iterative methods have the following two basic properties of the preconditioners: first, they are energy equivalent to the "main" matrix of the problem with the constants of the equivalence which don't depend on mesh parameters, and, secondly, they are easily invertible. The first property of the preconditioners provides the rate of convergence which doesn't depend on the mesh sizes for the problems without state constraints. Similar property is not proved for the state constrained problems but it was observed in the numerical experiments. To get the easily implementable algorithms we construct the saddle point problems with block triangle matrices acting on the direct variables (following the results of [17]). Note, that applying the augmented Lagrangian methods [18], [19] we loose the explicit form of the algorithms despite of the explicit in time approximation of the state equation, namely, we are forced to solve a variational inequality for state mesh function on each iterative step.

This article continues the investigations of [17], [20]-[26] on the iterative solution methods for the constrained saddle point problems with applications to optimal control problems.

1. Formulation of the problem, approximation

Let $\Omega = (0,1)^n$, $Q_T = \Omega \times (0,T]$, $\Sigma = \partial\Omega \times (0,T]$ and homogeneous Dirichlet initial-boundary value problem

$$\frac{\partial y}{\partial t} - \Delta y = u \text{ in } Q_T;\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T];\ y(x,0) = 0, \qquad (1)$$

describes the state of a system. Here $y(x,t)$ and $u(x,t)$ are state and control functions. For any $u \in L_2(Q_T)$ there exists a unique solution of problem (1), such that $y \in L_2(0,T;H_0^1(\Omega))$, $\dfrac{\partial y}{\partial t} \in L_2(Q_T)$, and the following inequality takes place: (cf., e.g. [28], p.370):

$$\|y\|_{L_2(0,T;H_0^1(\Omega))} + \|\frac{\partial y}{\partial t}\|_{L_2(Q_T)} \leqslant c\|u\|_{L_2(Q_T)}. \qquad (2)$$

Let the goal function be defined as

$$J(y,u) = \frac{\gamma_1}{2}\int\limits_{Q_T}(y(x,t)-y_d(x,t))^2dxdt+\frac{\gamma_2}{2}\int\limits_{\Omega}(y(x,T)-z_d(x))^2dx+\frac{\alpha}{2}\int\limits_{Q_T}u^2dxdt,$$

with given functions $y_d \in L_2(Q_T)$, $z_d \in L_2(\Omega)$, and constants $\alpha > 0$, $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$ such that $\gamma_1+\gamma_2 > 0$. Finally, define the sets of point-wise constraints for state, time derivative of state and control:

$$U_{ad} = \{u \in L_2(Q_T) : |u(x,t)| \leqslant u_{\max} \text{ a.e. } (x,t) \in Q_T\},$$
$$Y_{ad} = \{y \in L_2(0,T;H_0^1(\Omega)) : \frac{\partial y}{\partial t} \in L_2(Q_T), \;\; y_{\min} \leqslant y(x,t) \leqslant y_{\max}$$
$$\text{and } dy_{\min} \leqslant \frac{\partial y}{\partial t} \leqslant dy_{\max} \text{ a.e. } Q_T\}.$$

Above $\bar{u} > 0$, $y_{\min}, dy_{\min} \geqslant -\infty$, and $y_{\max}, dy_{\max} \leqslant \infty$. We solve the following optimal control problem:

$$\begin{gathered} \min_{(y,u)\in K} J(y,u), \\ K = \{(y,u) \in Y_{ad} \times U_{ad} : \text{ state equation (1) is satisfied}\}. \end{gathered} \tag{3}$$

Lemma 1. *Problem* (3) *has a unique solution.*

Proof. The sets U_{ad} and Y_{ad} are convex and closed, moreover U_{ad} is bounded. These properties together with linearity of state equation and stability inequality (2) ensure that the set K is convex, closed and bounded. Functional J is continuous and strictly convex on K. Because of all these properties of K and J problem (3) has a unique solution. □

Construct a finite-difference approximation of problem (3) using a uniform in x and t mesh $\omega_x \times \omega_t$ in $\bar{Q}_T$, where ω_x is the uniform mesh of the meshsize h on $\bar{\Omega}$, $\operatorname{card}\omega_x = N_x$, and $\omega_t = \{t_j = j\tau,\ j = 0, 1, \ldots N_t;\ N_t\tau = T\}$. Further we use the notations $y, u, ...$ for the mesh functions and for the vectors of their nodal values as well. By $y_j = y(x, t_j) \in \mathbb{R}^{N_x}$ we denote the values on a time level $t_j = j\tau \in \omega_t$ of a mesh function of x and t, and by $\|.\|_x$ – euclidian norm in the space $\mathbb{R}^{N_x}$.

Let A be the stiffness matrix of mesh Laplasian with homogeneous Dirichlet boundary conditions. It is well-known that the matrix A is symmetric with the spectrum in a segment $[\xi_0, \xi_1]$, where ξ_1 is of order h^{-2}, while $\xi_0 > 0$ is bounded below by a constant which doesn't depend on h.

We approximate state problem (1) by an explicit (forward Euler) finite-difference scheme

$$\frac{y_j - y_{j-1}}{\tau} + Ay_{j-1} = u_j,\ j = 1, 2, \ldots, N_t,\ \ y_0 = 0, \tag{4}$$

in the supposition $\tau \leqslant \dfrac{2}{\xi_1}$ that ensures its stability.

Let the functions y_d and z_d be continuous. Further we use the same notations y_d and z_d for the mesh functions and the vectors of their nodal values. Define a mesh goal function and the sets of the constraints:

$$J_h(y,u) = \frac{\gamma_1}{2}\sum_{j=1}^{N_t} \tau \|y_j - y_{dj}\|_x^2 + \frac{\gamma_2}{2}\|y_{N_t} - z_d\|_x^2 + \frac{\alpha}{2}\sum_{j=1}^{N_t}\tau\|u_j\|_x^2, \qquad (5)$$

$$U_{ad}^h = \{u : |u(x,t)| \leqslant \bar{u}\, \forall x \in \omega_x, \forall t \in \omega_t\},$$
$$Y_{ad}^h = Y_0^h \bigcap Y_1^h,\ Y_0^h = \{y : y_{\min} \leqslant y(x,t) \leqslant y_{\max},\ \forall x \in \omega_x, \forall t \in \omega_t\},$$
$$Y_1^h = \{y : \tau dy_{\min} \leqslant y(x,t) - y(x,t-\tau) \leqslant \tau dy_{\max}\ (y_0 = 0)\ \forall x \in \omega_x, \forall t \in \omega_t\}.$$

The mesh optimal control problem reads as follows:

$$\min_{(y,u)\in K_h} J_h(y,u),$$
$$K_h = \{(y,u) \in Y_{ad}^h \times U_{ad}^h :\ \ (4) \text{ is satisfied}\}. \qquad (6)$$

Lemma 2. *Problem* (6) *has a unique solution.*

Proof. The set K_h is a convex compact and the quadratical function $J_h(y,u)$ is continuous and strictly convex on K_h, whence the statement of the lemma follows. □

Let us rewrite problem (6) in a "vector-matrix" form. To this end we use the following notations: $N = N_t N_x$, $(.,.)$ and $\|.\|$ are the inner product and the norm in $\mathbb{R}^N$, $E \in \mathbb{R}^{N\times N}$ is the unit matrix. Let also matrices $L, M \in \mathbb{R}^{N\times N}$ be defined by the equalities:

$$(Ly)_j = \{\frac{y_j - y_{j-1}}{\tau} + Ay_{j-1} \text{ for } j = 2,\dots,N_t;\ \frac{y_1}{\tau} \text{ for } j = 1\},$$
$$(My)_j = \{0 \text{ for } j = 1,\dots,N_t - 1;\ \frac{1}{\tau}E \text{ for } j = N_t\}.$$

With these notations mesh goal function (5) becomes

$$I(y,u) = \frac{\gamma_1}{2}\|y - y_d\|^2 + \frac{\gamma_2}{2}(M(y - z_d), y - z_d) + \frac{\alpha}{2}\|u\|^2.$$

Introducing block two-diagonal matrix $R \in \mathbb{R}^{N\times N}$, $(Ry)_j = \{y_j - y_{j-1} \text{ for } j = 2,\dots,N_t;\ y_1 \text{ for } j = 1\}$, we replace the constraint $y \in Y_1^h$ in problem (6) by the constraints

$$p = Ry;\ p \in P_{ad}^h = \{\tau dy_{\min} \leqslant p_j \leqslant \tau dy_{\max},\ j = 1, 2, \dots N_t\}.$$

At last, denote by ψ, θ and φ the indicator functions of the sets Y_0^h, P_{ad}^h and U_{ad}^h, respectively.

Finally we obtain the following algebraic form of mesh optimal control problem (6):

$$\min_{Ly=u,\, p=Ry} \{I(y,u) + \psi(y) + \theta(p) + \varphi(u)\}. \tag{7}$$

2. Mesh saddle point problems

Construct Lagrange function for problem (7):

$$\mathcal{L}(y,u,p,\lambda,\mu) = I(y,u) + \psi(y) + \theta(p) + \varphi(u) + (\lambda, Ly - u) + (\mu, Ry - p).$$

A saddle point of this Lagrangian satisfies the following saddle point problem (cf., e.g. [27], p.169):

$$\begin{pmatrix} \gamma_1 E + \gamma_2 M & 0 & 0 & L^T & R^T \\ 0 & \alpha E & 0 & -E & 0 \\ 0 & 0 & 0 & 0 & -E \\ L & -E & 0 & 0 & 0 \\ R & 0 & -E & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \\ \lambda \\ \mu \end{pmatrix} + \begin{pmatrix} \partial\psi(y) \\ \partial\varphi(u) \\ \partial\theta(p) \\ 0 \\ 0 \end{pmatrix} \ni \begin{pmatrix} \gamma_1 y_d + \gamma_2 M z_d \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{8}$$

where $\partial\psi(y), \partial\varphi(u)$ and $\partial\theta(p)$ are the subdifferentials of the corresponding functions. With the notations $z = (y,u,p)^T$, $\eta = (\lambda,\mu)^T$, $f = (\gamma_1 y_d + \gamma_2 M z_d, 0, 0)^T$, $\Psi(z) = \psi(y) + \theta(p) + \varphi(u)$ and

$$\mathcal{A} = \begin{pmatrix} \gamma_1 E + \gamma_2 M & 0 & 0 \\ 0 & \alpha E & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} L & -E & 0 \\ R & 0 & -E \end{pmatrix}$$

problem (8) can be rewritten in a compact form:

$$\begin{pmatrix} \mathcal{A} & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix} + \begin{pmatrix} \partial\Psi(z) \\ 0 \end{pmatrix} \ni \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{9}$$

When investigating the solvability of problem (8) we use the following result:

Proposition 1. *([21]) Let the following assumptions be satisfied:*

$$\begin{gathered} \text{matrix } \mathcal{A} \in \mathbb{R}^{m\times m} \text{ is positive semidefinite} \\ \text{and positive definite on the kernel } \operatorname{Ker} B \text{ of matrix } B, \end{gathered} \tag{10}$$

$$\textit{matrix } B \in \mathbb{R}^{s\times m} \textit{ has a full column rank: } \operatorname{rank} B = s \leqslant m, \tag{11}$$

$$\Psi : \mathbb{R}^m \to \overline{\mathbb{R}} \textit{ is a convex, proper and lower semicontinuous function,} \tag{12}$$

$$\{z \in \mathbb{R}^m : Bz = 0\} \cap \operatorname{int}\operatorname{dom}\Psi \neq \emptyset. \tag{13}$$

Then problem (9) *has a non-empty set of the solutions* $X = \{(z, \eta)\}$, *and* z *is unique.*

Lemma 3. *Let* $\tau \leqslant \dfrac{1}{\xi_1}$. *Then the following inequalities take place:*

$$(Ly, y) \geqslant \frac{\xi_0}{2}(y, y), \quad (Ly, y) \geqslant \frac{1}{2}(My, y) \;\; \forall y. \tag{14}$$

Proof. Set $y_0 = 0$ and using the inequality $\tau \leqslant \dfrac{1}{\xi_1}$ we get

$$(Ly, y) = \sum_{j=1}^{N_t}(\frac{y_j - y_{j-1}}{\tau} + Ay_{j-1}, y_j)_x = \frac{1}{2\tau}\sum_{j=1}^{N_t}(\|y_j\|_x^2 - \|y_{j-1}\|_x^2 + \|y_j - y_{j-1}\|_x^2) +$$
$$+\sum_{j=1}^{N_t}((Ay_{j-1}, y_{j-1})_x + (Ay_{j-1}, y_j - y_{j-1})_x) \geqslant \frac{1}{2\tau}\|y_{N_t}\|_x^2 + \frac{1}{2}\sum_{j=1}^{N_t}((Ay_{j-1}, y_{j-1})_x +$$
$$+\sum_{j=1}^{N_t}(\frac{1}{2\tau}\|y_j - y_{j-1}\|_x^2 - \frac{1}{2}(A(y_j - y_{j-1}), y_j - y_{j-1})_x)) \geqslant \frac{1}{2\tau}\|y_{N_t}\|_x^2 + \frac{1}{2}\sum_{j=1}^{N_t}(Ay_{j-1}, y_{j-1})_x.$$

This estimate together with the inequality $((Ay_{j-1}, y_{j-1})_x \geqslant \xi_0\|y_{j-1}\|_x^2$ yield both inequalities in (14). □

Henceforth we suppose $\tau \leqslant \dfrac{1}{\xi_1}$, so, inequalities (14) are true.

Theorem 1. *Problem* (8) *has a solution* (y, u, p, λ, μ) *with unique* y, u, p, *which coincide with the solution of problem* (7).

Proof. We prove that all assumptions (10) – (12) and (13) of proposition 1 are satisfied for problem (8). To prove (10) we use the definition of the kernel of matrix B, namely, $\operatorname{Ker} B = \{(y, u, p) : Ly = u, Ry = p\}$, and the inequalities $(Ly,y) \geqslant \dfrac{\xi_0}{2}(y,y)$ and $\|Ry\| \leqslant 2\|y\|$. Then for $(y,u,p) \in \mathrm{Ker}B$ we get:

$$(Az,z) \geqslant \alpha\|u\|^2 \geqslant \frac{\alpha}{2}\|u\|^2 + \frac{\alpha}{2}\|Ly\|^2 \geqslant \frac{\alpha}{2}\|u\|^2 + \frac{\alpha\xi_0}{8}\|y\|^2 + \frac{\alpha\xi_0}{32}\|p\|^2.$$

The validity of assumptions (11), (12) follow from the definitions of matrix B and function Ψ. At last, vector $(y_0,u_0,p_0) = (0,0,0)$ satisfies assumption (13). So, all assumptions (10) – (12) and (13) are satisfied for problem (8), and it has a solution (y,u,p,λ,μ) with unique y,u,p due to proposition 1 $\square$

As matrix $\mathcal{A}$ is only positive semidefinite we can't use Udzawa-type methods for solving problem (8). Because of this we make some equivalent transformations of (8) by using one or two last equations of this system to get a saddle point problem with a positive definite matrix. For the technical simplicity we consider only two particular cases of optimal control problem, namely, $\gamma_1 = 1, \gamma_2 = 0$, that corresponds to a problem with distributed observation, and $\gamma_1 = 0, \gamma_2 = 1$, that corresponds to a problem with final observation. Let us emphasize that investigation of the problem with $\gamma_1 > 0$ and $\gamma_2 > 0$ is very similar to that in the case $\gamma_1 = 1, \gamma_2 = 0$, while the case of $\gamma_1 = 0, \gamma_2 = 1$ is a challenging problem.

In the case $\gamma_1 = 1, \gamma_2 = 0$ we add the equality $r(p - Ry) = 0, r > 0$ to third row (inclusion) in system (8) and obtain

$$\begin{pmatrix} E & 0 & 0 & L^T & R^T \\ 0 & \alpha E & 0 & -E & 0 \\ -rR & 0 & rE & 0 & -E \\ L & -E & 0 & 0 & 0 \\ R & 0 & -E & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \\ \lambda \\ \mu \end{pmatrix} + \begin{pmatrix} \partial\psi(y) \\ \partial\varphi(u) \\ \partial\theta(p) \\ 0 \\ 0 \end{pmatrix} \ni \begin{pmatrix} y_d \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

In the case $\gamma_1 = 0, \gamma_2 = 1$ we add the equality $r_1\alpha(Ly - u) = 0, r_1 > 0$ to the first row in (8) and $r_2\alpha(p - Ry) = 0, r_2 > 0$ to the third one. It results in the following saddle point problem:

$$\begin{pmatrix} M + r_1\alpha L & -r_1\alpha E & 0 & L^T & R^T \\ 0 & \alpha E & 0 & -E & 0 \\ -r_2\alpha R & 0 & r_2\alpha E & 0 & -E \\ L & -E & 0 & 0 & 0 \\ R & 0 & -E & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \\ \lambda \\ \mu \end{pmatrix} + \begin{pmatrix} \partial\psi(y) \\ \partial\varphi(u) \\ \partial\theta(p) \\ 0 \\ 0 \end{pmatrix} \ni \begin{pmatrix} Mz_d \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

Below we prove the positive definiteness of the matrices of these saddle point problems

$$\mathcal{A}_{1r} = \begin{pmatrix} E & 0 & 0 \\ 0 & \alpha E & 0 \\ -rR & 0 & rE \end{pmatrix} \text{ and } \mathcal{A}_{2r} = \begin{pmatrix} M + r_1\alpha L & -r_1\alpha E & 0 \\ 0 & \alpha E & 0 \\ -r_2\alpha R & 0 & r_2\alpha E \end{pmatrix} \quad (17)$$

for the appropriate choice of the parameters r and r_1, r_2.

Lemma 4. *For* $0 < r < 1$ *matrix* $\mathcal{A}_{1r}$ *of system* (15) *is positive definite and spectrally equivalent to a block diagonal matrix* $\mathcal{A}_0 = \operatorname{diag}\big(E, \quad \alpha E, \quad rE\big)$*:*

$$(1 - \sqrt{r})(A_0 z, z) \leqslant (A_{1r} z, z) \leqslant (1 + \sqrt{r})(A_0 z, z) \ \forall z = (y, u, p). \quad (18)$$

Proof. It suffice to use the inequality $|(Ry, p)| \leqslant 2\|y\|\|p\| \leqslant r^{-1/2}\|y\|^2 + r^{1/2}\|p\|^2$ when estimating $(\mathcal{A}_{1r} z, z) = \|y\|^2 + \alpha\|u\|^2 + r\|p\|^2 - r(Ry, p)$. □

To investigate the properties of the matrix $\mathcal{A}_{2r}$ we define a domain

$$\Omega(r_1, r_2) = \{0 < r_1 < 2\xi_0, \ 0 < r_2 < r_1\xi_0/2 - r_1^2/4\}\} \quad (19)$$

(recall that ξ_0 is the minimal eigenvalue of the matrix A), and a quadratical form

$$Q(s_1, s_2, s_3) = r_1 s_1^2 + s_2^2 + r_2 s_3^2 - r_1\sqrt{\frac{2}{\xi_0}} s_1 s_2 - 2r_2\sqrt{\frac{2}{\xi_0}} s_1 s_3. \quad (20)$$

Let $\underline{s}$ and $\overline{s}$ be the minimal and maximal eigenvalues of this quadratical form. By direct calculations one can check that $0 < \underline{s} < 1$ if $(r_1, r_2) \in \Omega(r_1, r_2)$. The constants $\underline{s}$, $\overline{s}$ depend only on ξ_0 and the distance between (r_1, r_2) and the boundary of the domain $\Omega(r_1, r_2)$. Choosing (r_1, r_2) "in the middle" of $\Omega(r_1, r_2)$ (for example, $r_1 = \xi_0$, $r_2 = \dfrac{\xi_0^2}{8}$), we obtain $\underline{s}$, $\overline{s}$ which don't depend on the mesh steps τ and h.

Lemma 5. *Let* $(r_1, r_2) \in \Omega(r_1, r_2)$. *Then matrix* $\mathcal{A}_{2r}$ *of system* (16) *is positive definite and energy equivalent to block diagonal matrices* $\mathcal{A}_{00} = \operatorname{diag}\big(M + \alpha L_0, \quad \alpha E, \quad \alpha E\big)$ *and* $\tilde{\mathcal{A}}_0 = \alpha \operatorname{diag}(L_0, E, E)$*,* $L_0 = 0.5(L + L^T)$*:*

$$\underline{s}(\mathcal{A}_{00} z, z) \leqslant (\mathcal{A}_{2r} z, z) \leqslant \overline{s}(\mathcal{A}_{00} z, z) \ \forall z = (y, u, p), \quad (21)$$

$$\underline{s}(\tilde{\mathcal{A}}_0 z, z) \leqslant (\mathcal{A}_{2r} z, z) \leqslant (1 + 2/\alpha)\overline{s}(\tilde{\mathcal{A}}_0 z, z) \ \forall z = (y, u, p). \quad (22)$$

Proof. By using the inequalities $|(Ry,p)| \leqslant 2\|y\|\|p\|$ and $(L_0 y, y) = (Ly, y) \geqslant \frac{\xi_0}{2}\|y\|^2$ we obtain left estimate in (21):

$$(\mathcal{A}_{2r}z, z) = (My, y) + \alpha\big(r_1(L_0y, y) + \|u\|^2 + r_2\|p\|^2 - r_1(y, u) - \\ - r_2(Ry, p)\big) \geqslant (My, y) + \alpha Q(L_0^{1/2}y, u, p) \geqslant (My, y) + \\ + \alpha \underline{s}(\|L_0^{1/2}y\|^2 + \|u\|^2 + \|p\|^2) \geqslant \underline{s}(A_{00}z, z). \quad (23)$$

The right estimate $(\mathcal{A}_{2r}z, z) \leqslant \overline{s}(\mathcal{A}_{00}z, z)$ can be proved similarly.

Now, $(L_0y, y) \geqslant 1/2(My, y)$ due to (14), therefore $\mathcal{A}_{00}$ is spectrally equivalent to the matrix $\tilde{\mathcal{A}}_0 = \alpha\,\mathrm{diag}(L_0, E, E)$:

$$(\tilde{\mathcal{A}}_0 z, z) \leqslant (\mathcal{A}_{00}z, z) \leqslant (1 + 2/\alpha)(\tilde{\mathcal{A}}_0 z, z). \quad (24)$$

The estimates (22) follow from inequalities (21) and (24). □

3. Iterative solution methods for problems (15) and (16)

We will use the following result on the convergence of Uzawa-type method for solving (9):

Proposition 2. *([17]) Let matrix $\mathcal{A}$ be positive definite and the assumptions* (11)– (13) *be satisfied. Let one of the following equivalent condition hold:*

$$(\mathcal{A}z, z) \geqslant \frac{(1+\varepsilon)\rho}{2}(D^{-1}Bz, Bz) \ \ \forall z \in \mathbb{R}^m \quad (25)$$

or

$$(D\eta, \eta) \geqslant \frac{(1+\varepsilon)\rho}{2}(\mathcal{A}_s^{-1}B^T\eta, B^T\eta) \ \ \forall \eta \in \mathbb{R}^s, \quad (26)$$

with $\varepsilon > 0$, $\mathcal{A}_s = 0.5(\mathcal{A} + \mathcal{A}^T)$ and a symmetric and positive definite matrix D. Then Uzawa-type method with preconditioner D

$$\begin{aligned} &\mathcal{A}z^{k+1} + \partial\Psi(z^{k+1}) \ni B^T\eta^k + f, \\ &\frac{1}{\rho}D(\eta^{k+1} - \eta^k) + Bz^{k+1} = 0, \ \rho > 0 \end{aligned} \quad (27)$$

converges for any initial guess η^0: $(z^k, \eta^k) \to (z^*, \eta^*) \in X$ for $k \to \infty$.

Stopping criterion. As a stopping criterion of the iterations (27) we use the smallness of the norm of the residual (cf. [21], [23]). Namely, since we suppose that the inclusion $\mathcal{A}z^k + \partial\Psi(z^k) \ni B^T\eta^{k-1} + f$ is solved exactly, then we control only the norm of the residual vector $r_\eta^k = Bz^k$. The vector of error of k-th iteration $(z^* - z^k, \eta^* - \eta^{k-1})^T$ satisfies the system

$$\begin{pmatrix} \mathcal{A} & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} z^* - z^k \\ \eta^* - \eta^{k-1} \end{pmatrix} + \begin{pmatrix} \partial\Psi(z^*) - \partial\Psi(z^k) \\ 0 \end{pmatrix} \ni \begin{pmatrix} 0 \\ -r_\eta^k \end{pmatrix}.$$

Multiplying this system by $(z^* - z^k, \eta^* - \eta^{k-1})^T$ and using the monotonicity of $\partial\Psi$, we get

$$(\mathcal{A}(z^* - z^k), z^* - z^k) \leqslant (r_\eta^k, \eta^{k-1} - \eta^*) \leqslant \|r_\eta^k\|_{D^{-1}}\|\eta^* - \eta^{k-1}\|_D = o(\|r_\eta^k\|_{D^{-1}}). \tag{28}$$

So, for a positive definite matrix $\mathcal{A}$ inequality (28) gives a posteriori estimate for the iterations error of method (27)

Choice of a preconditioner. The inequalities (25), (26) give the sufficient convergence conditions of iterative method (27) with a wide class of the pairs $(D,\ \rho)$ – a preconditioner and an iterative parameter. Since the problem is finite dimensional, these inequalities are true for any symmetric and positive definite matrix D with an appropriate parameter ρ. So, the main task is constructing the preconditioners for which inequalities (25), (26) are satisfied if $\rho \in (0, \rho_0)$ with ρ_0 independent on mesh steps τ and h. The motivation of this task is that in the case of unconstrained problem ($\partial\Psi = 0$) the optimal preconditioner in method (27) is a matrix D spectrally equivalent to $B\mathcal{A}_s^{-1}B^T$:

$$c_0(B\mathcal{A}_s^{-1}B^T\eta, \eta) \leqslant (D\eta, \eta) \leqslant c_1(B\mathcal{A}_s^{-1}B^T\eta, \eta) \ \ \forall\eta,$$

and the rate of convergence of (27) depends on c_0/c_1.

Thus, further we construct for problems (15) and (16) the easily implementable block-diagonal preconditioners which are spectrally equivalent to $B\mathcal{A}_s^{-1}B^T$ with constants c_0 and c_1 independent on τ and h.

First, we apply method (27) to solving problem (15).

Lemma 6. *For $0 < r < 1$ the matrix $B\mathcal{A}_{1rs}{}^{-1}B^T$, $A_{1rs} = 0.5(A_{1r} + A_{1r}^T)$, is spectrally equivalent to block-diagonal matrix*

$$D = \operatorname{diag}\left((L + \alpha^{-1/2}E)(L^T + \alpha^{-1/2}E),\ \ r^{-1}E\right). \tag{29}$$

In particular,

$$B{\mathcal{A}_{1rs}}^{-1}B^T \leqslant (1-\sqrt{r})^{-1}(\sqrt{1+r}-r)^{-2}D. \qquad (30)$$

Proof. Due to lemma 4 the matrix $B{\mathcal{A}_{1rs}}^{-1}B^T$ is spectrally equivalent to $\tilde{D} = B\mathcal{A}_0^{-1}B^T$:

$$(1+\sqrt{r})^{-1}\tilde{D} \leqslant B{\mathcal{A}_{1rs}}^{-1}B^T \leqslant (1-\sqrt{r})^{-1}\tilde{D}. \qquad (31)$$

It rests to prove that $\tilde{D}$ is spectrally equivalent to D.

Let $\eta = (\lambda, \mu)$. Since $\tilde{D} = \begin{pmatrix} LL^T + \alpha^{-1}E & LR^T \\ RL^T & RR^T + r^{-1}E \end{pmatrix}$, then using the inequality $\|R^T\mu\| \leqslant 2\|\mu\|$ we obtain:

$$(\tilde{D}\eta,\eta) = \|L^T\lambda + R^T\mu\|^2 + \alpha^{-1}\|\lambda\|^2 + r^{-1}\|\mu\|^2 \leqslant (1+\frac{1}{\varepsilon})\|L^T\lambda\|^2 +$$
$$+ (1+\varepsilon)\|R^T\mu\|^2 + \alpha^{-1}\|\lambda\|^2 + r^{-1}\|\mu\|^2 \leqslant (1+\frac{1}{\varepsilon})\|L^T\lambda\|^2 + \alpha^{-1}\|\lambda\|^2 +$$
$$+(1+4r(1+\varepsilon))r^{-1}\|\mu\|^2 \leqslant (1+\frac{1}{\varepsilon})\|L^T\lambda + \alpha^{-1/2}\lambda\|^2 + (1+4r(1+\varepsilon))r^{-1}\|\mu\|^2, \ \varepsilon > 0.$$

For $\varepsilon = \dfrac{\sqrt{1+r}-r}{2r}$ the inequality $(\tilde{D}\eta,\eta) \leqslant (\sqrt{1+r}-r)^{-2}(D\eta,\eta)$ holds. From here and (31) inequality (30) follows.

Further, since $\|L^T\lambda + \alpha^{-1/2}\lambda\|^2 \leqslant 2\|L^T\lambda\|^2 + 2\alpha^{-1}\|\lambda\|^2$ then for any $\delta > 1$ we get

$$(\tilde{D}\eta,\eta) \geqslant (1-\frac{1}{\delta})\|L^T\lambda\|^2 + (1-\delta)\|R^T\mu\|^2 + \alpha^{-1}\|\lambda\|^2 + r^{-1}\|\mu\|^2 \geqslant$$
$$\geqslant \frac{1}{2}(1-\frac{1}{\delta})\|L^T\lambda + \alpha^{-1/2}\lambda\|^2 + (1+4r(1-\delta))r^{-1}\|\mu\|^2.$$

If $\delta \in (1, 1+\frac{1}{4r})$, then $(\tilde{D}\eta,\eta) \geqslant c_0(\delta)(D\eta,\eta)$, $c_0(\delta) > 0$. From here and (31) inequality $B{\mathcal{A}_{1rs}}^{-1}B^T \geqslant (1+\sqrt{r})^{-1}c_0(\delta)D$ follows. □

Method (27) for problem (15) with preconditioner (29) reads as follows:

$$\begin{gathered} y^{k+1} + \partial\psi(y^{k+1}) \ni y_d - L^T\lambda^k - R^T\mu^k, \\ \alpha u^{k+1} + \partial\varphi(u^{k+1}) \ni \lambda^k, \\ rp^{k+1} + \partial\theta(p^{k+1}) \ni rRy^{k+1} + \mu^k, \\ (L+\alpha^{-1/2}E)(L^T+\alpha^{-1/2}E)\frac{\lambda^{k+1}-\lambda^k}{\rho} = Ly^{k+1} - u^{k+1}, \\ \frac{\mu^{k+1}-\mu^k}{r\rho} = Ry^{k+1} - p^{k+1}. \end{gathered} \qquad (32)$$

Theorem 2. *Method* (32) *converges for any initial guess* (λ^0, μ^0) *if*

$$0 < r < 1,\ 0 < \rho < 2(1-\sqrt{r})(\sqrt{1+r}-r)^2. \tag{33}$$

Proof. In virtue of inequality (30) the convergence condition of proposition 2, namely, $(D\eta,\eta) \geqslant \dfrac{(1+\varepsilon)\rho}{2}(\mathcal{A}_{1rs}^{-1}B^T\eta, B^T\eta)$, is true for the parameters r, ρ from (33). The validity of all other assumptions of proposition 2 is proved in theorem 1. □

When implementing method (32) one has to solve the inclusions with respect y^{k+1}, u^{k+1} and p^{k+1}, and a system of linear equations with matrix $(L+\alpha^{-1/2}E)(L^T+\alpha^{-1/2}E)$. As the matrices and the operators of the mentioned inclusions have diagonal form then their solving reduces to pointwise projection. In turn, the matrices $L+\alpha^{-1/2}E$ and $L^T+\alpha^{-1/2}E$ are triangle, so, their solving is also executed by the explicit formulaes.

We can use estimate (28) to control error of the iterative method. In the case under consideration $r_\eta = (r_\lambda, r_\mu) = (Ly-u, Ry-p)$, preconditioner $D = \operatorname{diag}\big((L+\alpha^{-1/2}E)(L^T+\alpha^{-1/2}E),\quad r^{-1}E\big)$ and $(A_{1r}z,z) \geqslant (1-\sqrt{r})(A_0 z, z)$, where $\mathcal{A}_0 = \operatorname{diag}\big(E,\quad \alpha E,\quad rE\big)$. For a fixed r estimate (28) becomes

$$\|y^*-y^k\|^2 + \alpha\|u^*-u^k\|^2 + \|p^*-p^k\|^2 \leqslant \\ \leqslant c\|\eta^*-\eta^{k-1}\|_D(\|(L+\alpha^{-1/2}E)^{-1}(Ly^k-u^k)\| + \|Ry^k-p^k\|), \tag{34}$$

where constant c doesn't depend on mesh steps and α, and $\|\eta-\eta^{k-1}\|_D \to 0$ for $k\to\infty$.

Now, we investigate the iterative solution methods for problem (16). For constructing a preconditioner D we take the matrix $\tilde{\mathcal{A}}_0 = \alpha\operatorname{diag}(L_0, E, E)$, which is energy equivalent to $\mathcal{A}_{2r}$ (cf. (21)).

Lemma 7. *Matrix*

$$\tilde{D} = B\tilde{\mathcal{A}}_0^{-1}B^T = \alpha^{-1}\begin{pmatrix} LL_0^{-1}L^T+E & LL_0^{-1}R^T \\ RL_0^{-1}L^T & RL_0^{-1}R^T+E \end{pmatrix} \tag{35}$$

is spectrally equivalent to block diagonal matrix

$$D = \alpha^{-1}\operatorname{diag}\left(LL_0^{-1}L^T,\quad E\right). \tag{36}$$

Namely, there exist constants $k_0(\xi_0)$ *and* $k_1(\xi_0)$ *such that*

$$k_0(\xi_0)(D\eta,\eta) \leqslant (\tilde{D}\eta,\eta) \leqslant k_1(\xi_0)(D\eta,\eta).$$

Proof. The following inequality is true:

$$LL_0^{-1}L^T \geqslant \frac{\xi_0}{2}E. \tag{37}$$

In fact, it is equivalent to the inequality $L_0^{-1} \geqslant \frac{\xi_0}{2}(L^T L)^{-1}$, which in turn is equivalent to $\|Ly\|^2 \geqslant \frac{\xi_0}{2}(L_0 y, y) = \frac{\xi_0}{2}(Ly, y)$, and this is an obvious consequence of (14). Using (37) and the inequalities $L_0^{-1} \leqslant \frac{2}{\xi_0}E$ and $\|R^T\mu\| \leqslant 2\|\mu\|$ we obtain

$$(\tilde{D}\eta, \eta) = \alpha^{-1}\|L_0^{-1/2}L^T\lambda + L_0^{-1/2}R^T\mu\|^2 + \alpha^{-1}\|\lambda\|^2 + \alpha^{-1}\|\mu\|^2 \leqslant$$
$$\leqslant \alpha^{-1}(2 + \frac{2}{\xi_0})\|L_0^{-1/2}L^T\lambda\|^2 + \alpha^{-1}(1 + \frac{16}{\xi_0})\|\mu\|^2 = k_1(\xi_0)(D\eta, \eta).$$

On the other hand, for any $\delta \in (0, 1)$ we have:

$$\alpha(\tilde{D}\eta, \eta) \geqslant (1 - \delta)\|L_0^{-1/2}L^T\lambda\|^2 + (1 - \frac{1}{\delta})\|L_0^{-1/2}R^T\mu\|^2 + \|\mu\|^2 \geqslant$$
$$\geqslant (1 - \delta)\|L_0^{-1/2}L^T\lambda\|^2 + (1 + (1 - \frac{1}{\delta})\frac{8}{\xi_0})\|\mu\|^2.$$

For a δ, which is close enough to 1 there exists constant $k_0 = k_0(\xi_0) > 0$ such that $1 - \delta \geqslant k_0$ and $(1 + (1 - \frac{1}{\delta})\frac{8}{\xi_0}) \geqslant k_0$, whence $(\tilde{D}\eta, \eta) \geqslant k_0(D\eta, \eta)$. □

Method (27) for problem (16) with preconditioner (36) reads as follows:

$$\begin{gathered}
\alpha u^{k+1} + \partial\varphi(u^{k+1}) \ni \lambda^k, \\
(M + \alpha r_1 L)y^{k+1} + \partial\psi(y^{k+1}) \ni Mz_d - L^T\lambda^k - R^T\mu^k - r_1\alpha u^{k+1}, \\
r_2 p^{k+1} + \partial\theta(p^{k+1}) \ni r_2 R y^{k+1} + \mu^k, \\
LL_0^{-1}L^T\frac{\lambda^{k+1} - \lambda^k}{\alpha\rho} = Ly^{k+1} - u^{k+1}, \\
\frac{\mu^{k+1} - \mu^k}{\alpha\rho} = Ry^{k+1} - p^{k+1}.
\end{gathered} \tag{38}$$

Theorem 3. *Let the vector* (r_1, r_2) *belong to the domain* $\Omega(r_1, r_2)$ *defined in* (19). *Then there exists a constant* ρ_0, *depending only on* ξ_0, *such that method* (38) *converges for any initial guess* (u^0, λ^0, μ^0) *if* $0 < \rho < \rho_0$.

Proof. It suffices to prove inequality of the form (26) in proposition 2.

Using the inequalities $\|L_0^{1/2}L^{-1}u\|^2 \leqslant \dfrac{2}{\xi_0}\|u\|^2$ and $\|Ry\| \leqslant 2\|y\|$, we get

$$(D^{-1}Bz, Bz) = \alpha\|L_0^{1/2}y - L_0^{1/2}L^{-1}u\|^2 + \alpha\|Ry - p\|^2 \leqslant \\ \leqslant \alpha C(\xi_0)\big((L_0y, y) + \|u\|^2 + \|p\|^2\big), \quad (39)$$

where $C(\xi_0)$ is the maximal eigenvalue of the quadratical form

$$(1 + \frac{8}{\xi_0})s_1^2 + 2\sqrt{\frac{2}{\xi_0}}s_1s_2 + \frac{2}{\xi_0}s_2^2 + s_3^2 + \frac{8}{\xi_0}s_1s_3.$$

In what follows we use the quadratical form Q defined in (20). We have:

$$(\mathcal{A}_{2r}z, z) = (My, y) + \alpha r_1(Ly, y) + \alpha r_2\|p\|^2 + \alpha\|u\|^2 - \alpha r_1(u, y) - \alpha r_2(Ry, p) \geqslant \\ \geqslant (My, y) + \alpha Q(L_0^{1/2}y, u, p) \geqslant (My, y) + \alpha\underline{s}\big((L_0y, y) + \|u\|^2 + \|p\|^2\big). \quad (40)$$

The estimates (39) and (40) yield

$$(\mathcal{A}_{2r}z, z) \geqslant \alpha\underline{s}C^{-1}(\xi_0)(D^{-1}Bz, Bz),$$

thus, inequality of the form (26) is true if iterative parameter $\rho \in (0, \rho_0)$, where $\rho_0 = \underline{s} - \dfrac{\rho C(\xi_0)}{2} > 0$. All other assumptions of proposition 2 are satisfied, whence, the statement of the theorem is true. □

When implementing method (38) one has to solve the inclusions with respect y^{k+1}, u^{k+1} and p^{k+1}, and a system of linear equations with matrix $LL_0^{-1}L^T$. Similar to method (32), multivalued operators of the inclusions have diagonal form, matrices of the problems for u^{k+1} и p^{k+1} are diagonal while the matrix of the problem for y^{k+1} is triangle. So, solving of all inclusions reduces to pointwise projection. Solving the system with the matrix $LL_0^{-1}L^T = 2(L^{-1} + L^{-T})^{-1}$ is also executed by the explicit formulaes.

Remark 1. *One can take*

$$D = \operatorname{diag}\big(L(M + \alpha L_0)^{-1}L^T, \quad \alpha^{-1}E\big)$$

as a preconditioner in method (27) *for solving* (16)*. The statement of theorem 3 on the convergence of iterative method for $\rho \in (0, \rho_1)$ with a constant ρ_1, independent on mesh steps is still true. The implementation of the corresponding iterative method again reduces to the explicit calculations.*

Since $r_\eta = (r_\lambda, r_\mu) = (Ly - u, Ry - p)$, $D = \alpha^{-1}\mathrm{diag}\left(LL_0^{-1}L^T,\ \ E\right)$ and $(A_{2r}z, z) \geqslant \underline{s}(\tilde{\mathcal{A}}_0 z, z)$, where $\tilde{\mathcal{A}}_0 = \alpha\,\mathrm{diag}(L_0, E, E)$, then estimate (28) becomes

$$\|y^* - y^k\|_{L_0}^2 + \|u^* - u^k\|^2 + \|p^* - p^k\|^2 \leqslant$$
$$\leqslant c\alpha^{-1/2}\|\eta^* - \eta^{k-1}\|_D(\|Ly^k - u^k\|_{L^{-1}+L^{-T}} + \|Ry^k - p^k\|), \quad (41)$$

where constant c doesn't depend on mesh steps and α, while $\|\eta - \eta^{k-1}\|_D \to 0$ for $k \to \infty$. Above we use the notations $\|.\|_{L_0}$ and $\|.\|_{L^{-1}+L^{-T}}$ for the energy norms of symmetric and positive definite matrices L_0 and $L^{-1} + L^{-T}$, respectively,

4. Numerical examples

We solved problem (3) in the cases $\gamma_1 = 1, \gamma_2 = 0$ and $\gamma_1 = 0, \gamma_2 = 1$, $n = 1$ and $\alpha = 1$. After approximating these problems by finite difference schemes we constructed saddle point problems (15) and (16), and applied for their solution the iterative methods (32) and (38), respectively.

The solution domain was $Q_T = (0, 1) \times (0, T)$ in both cases, the known functions $y_d(x, t) = 2\sin(2\pi x)t$ and $z_d(x) = 2\sin(2\pi)$. For calculations we used operating system Redhat 5 AS and processor 2.4 Ghz Xeon E7330.

We controlled the residuals of the iterations as in (34) and (41). In fact, we multiply both sides of these inequalities by τh to have the mesh analogs of the norms $L_2(Q_T)$ and $H^1(Q_T)$ for the errors. For instance, the estimate (34) for the first problem becomes

$$\|y - y^k\|_{L_2(Q_T)}^2 + \alpha\|u - u^k\|_{L_2(Q_T)}^2 + r\|p - p^k\|_{L_2(Q_T)}^2 \leqslant C(norm1 + norm2)$$

with a small factor $C = (h\tau)^{1/2}\|\eta - \eta^{k-1}\|_D$ and the values $norm1$ and $norm2$ which correspond to the norms of the residuals. The estimate for the second problem is similar with $\|y - y^k\|_{L_2(Q_T)}^2$ replaced by $\|y - y^k\|_{H^1(Q_T)}^2$.

For the first problem ($\gamma_1 = 1.,\ \gamma_2 = 0.$) the values $norm1$ and $norm2$ are shown at Figures 1 and in 2. For the result of Figure 1 we used 0,3 milj nodes and the CPU time was 27 sec. For the result of Figure 2 we used 2,1 milj nodes and the CPU time was 330 sec.

For the second problem ($\gamma_1 = 0.,\ \gamma_2 = 1.$) the values $norm1$ and $norm2$ are shown at Figures 3 and in 4. For the result of Figure 3 we used 0,3 milj nodes and the CPU time was 240 sec. For the result of Figure 4 we used 2,1 milj nodes and the CPU time was 2900 sec.

From the results one can see, that in all cases the rate of the convergence doesn't depend on mesh parameters.

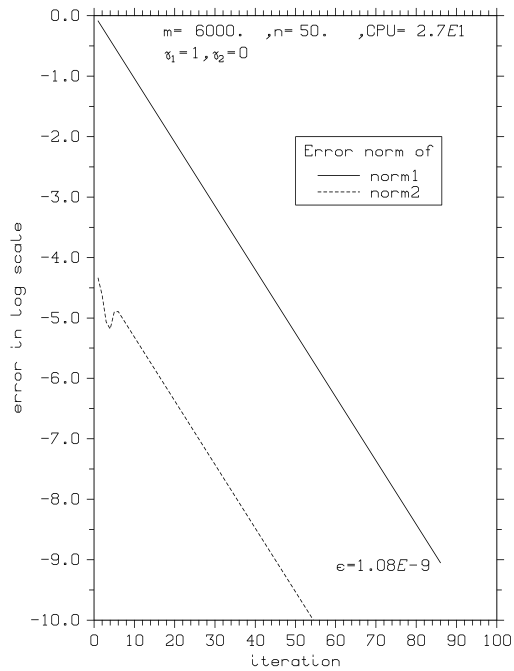

Figure 1:

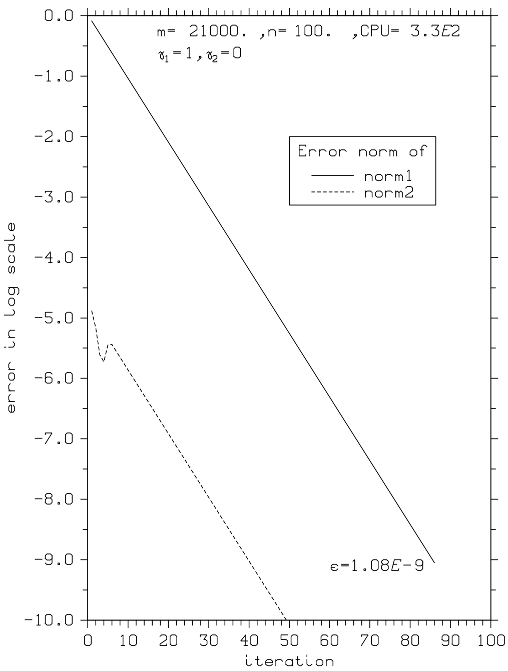

Figure 2:

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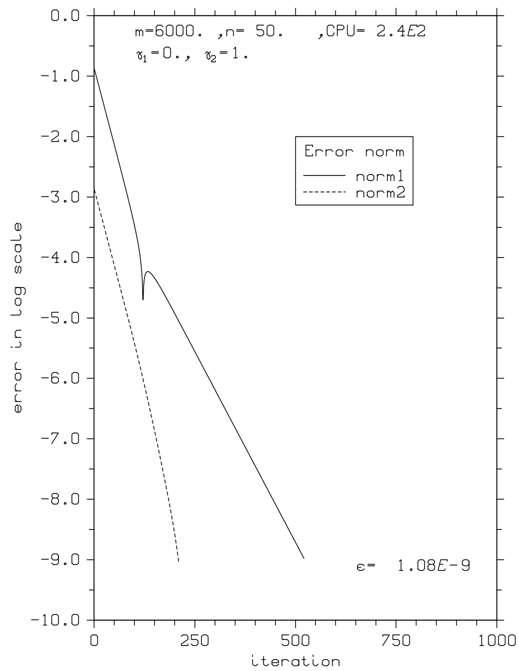

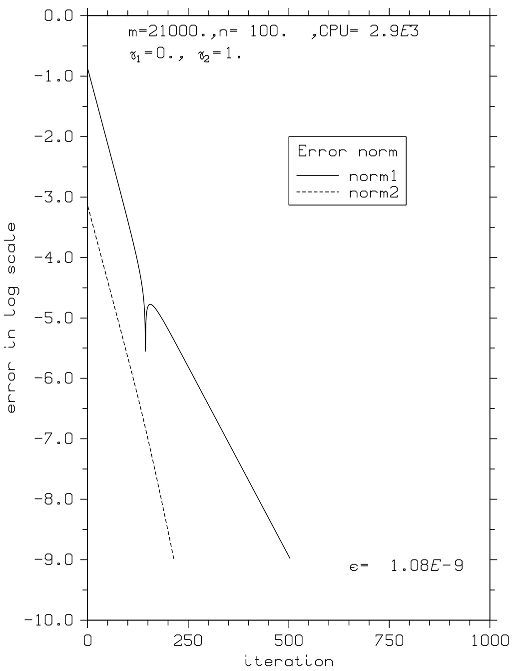

Figure 3:

Figure 4:

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