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On a Class of Multiplicative-Convolution Equations

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Abstract

The aim of this paper is to prove the existence and uniqueness of solution for one class multiplicative-convolution equations in space $\mathcal{A}'_+$, where $\mathcal{A}'_+$ is the space of distributions on $\mathbb{R}$, which are boundary values (in the sense tempered distributions) of functions analytic in upper half-plane of complex plane.

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1 Introduction

In the theory of distributions is known that multiplicative product and convolution of distributions not always exist. Therefore it is important to find the space of distributions where these two operations exist simultaneously.

2 Multiplicative Product and Convolution in $\mathcal{A}'_+$

The Carleman-Fourier transform is defined by [1]:

$$\mathcal{KF}(f)(z) := \begin{cases} \int\limits_0^{+\infty} e^{2\pi itz} f(t)dt, \ \mathrm{Im}z > 0, \\ -\int\limits_{-\infty}^{0} e^{2\pi itz} f(t)dt, \ \mathrm{Im}z < 0. \end{cases}$$

here $f(t)$ is a function of slow growth ($f(t)$ is a continuous on $\mathbb{R}$ function that grows at infinity no faster than a polynomial).

Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions. By $\mathcal{S}'_+(\mathbb{R})$ denote the space of tempered distributions on $\mathbb{R}$ with support in $\mathbb{R}_+ = [0; +\infty]$.

It is well known that any distribution $T \in \mathcal{S}'(\mathbb{R})$ has the structure

$$T = \frac{d^m}{dt^m} f(t),$$

where $\frac{d^m}{dt^m}$ is derivative in the distribution sense, $m \in \mathbb{N}$, $f(t)$ is a function of slow growth. Therefore, if $T \in \mathcal{S}'(\mathbb{R})$, then Carleman-Fourier transform is defined by

$$\mathcal{KF}(T)(z) = (-2\pi iz)^m \mathcal{KF}(f)(z), \ \mathrm{Im}z \neq 0.$$

Let $\mathcal{A}^+(z)$ be the space of analytic functions on $\mathrm{Im}z > 0$ such that $\mathcal{A}^+(z)$ is an image of $\mathcal{S}'_+(\mathbb{R})$ under the Carleman-Fourier transform.

It is known [2] that the space $\mathcal{S}'_+(\mathbb{R})$ with the convolution operation is the convolution algebra. It is also known [3] that the Carleman-Fourier transform is an isomorphism from $\mathcal{S}'_+(\mathbb{R})$ onto $\mathcal{A}^+(z)$. This means that $\mathcal{A}^+(z)$ is the multiplicative algebra with usual unit.

Passing to the limit (in the sense of the weak topology of $\mathcal{S}'$) as $\mathrm{Im}z \to 0$ in $\mathcal{A}^+(z)$, we obtain the space $\mathcal{A}'_+(\mathbb{R}) = \mathcal{A}'_+$ of distributions on $\mathbb{R}$.

The multiplicative product $\forall M, N \in \mathcal{A}'_+$ is defined by the formula

$$\langle MN, \varphi \rangle := \lim_{\varepsilon \to +0} \int\limits_{\mathbb{R}} M^+(x + i\varepsilon) N^+(x + i\varepsilon) \varphi(x) dx, \ \forall \varphi \in \mathcal{S},$$

here $\mathcal{S}$ is the Schwartz space [2].

Obviously, the space $\mathcal{A}'_+$ is the multiplicative algebra with usual unit.

Let $\mathcal{O}'_\alpha$ be the spaces of distributions on $\mathbb{R}$ with the asymptotic behavior $O\left(|t|^\alpha\right)$, $\alpha \geq -1$ [1]. If $M \in \left\{\mathcal{A}'_+ \bigcap \mathcal{O}'_\alpha, \ \alpha \geq -1\right\}$, then appropriate analytic function $M^+(z)$ is determined by

$$M^+(z) = \frac{1}{2\pi i} \left\langle M, \frac{1}{t - z} \right\rangle, \ \mathrm{Im}z > 0. \quad (1)$$

Note also that if a distribution $M \in \mathcal{A}'_+$, then $M^+(z)$ is determined by

$$M^+(z) = \mathcal{KF}\left[\bar{\mathcal{F}}(M)\right](z),\ \mathrm{Im} z > 0, \tag{2}$$

where $\bar{\mathcal{F}}$ is the Fourier cotransform.

Obviously, the multiplicative algebra $\mathcal{A}'_+$ has no zero divisors and does not contain distributions with compact support. For example, the Dirac measure δ does not belong to $\mathcal{A}'_+$.

We can also define a convolution operation in $\mathcal{A}'_+$. Let us consider the following cases.

Case 1. Let Θ'_c be the space of convolutors for $\mathcal{S}'$ [4]. If $U \in \left(\mathcal{A}'_+ \bigcap \Theta'_c\right)$, then $\forall T \in \mathcal{A}'_+$ convolution is defined as

$$\langle U * T, \varphi\rangle := \left\langle T, \check{U} * \varphi\right\rangle,\ \forall\varphi \in \mathcal{S},$$

where $\left\langle \check{U}, \varphi\right\rangle := \langle U, \check{\varphi}\rangle,\ \check{\varphi} := \varphi(-x)$.

Case 2. Let $\mathcal{A}'_c$ be a multiplicative algebra with usual unit and with the asymptotic $O\left(|t|^{-1}\right)$. If $U, T \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$, then convolution is defined as

$$\langle U * T, \varphi\rangle := \left\langle U, \check{T} * \varphi\right\rangle = \left\langle T, \check{U} * \varphi\right\rangle,\ \forall\varphi \in \mathcal{S}.$$

Case 3. If $U \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$ and $T \in \left(\mathcal{A}'_+ \bigcap \mathcal{O}'_\alpha\right)$, then convolution is defined like in case 1.

Thus $\mathcal{A}'_+$ is the multiplicative algebra with unit and $\mathcal{A}'_+$ is the convolution module on convolution algebras $\left(\mathcal{A}'_+ \bigcap \Theta'_c\right)$ and $\left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$.

The Dirac measure δ does not belong to $\mathcal{A}'_+$. But the distribution of Heisenberg-Bogolubov

$$\delta_+ := \frac{1}{2}\delta - \frac{1}{2\pi i} v.p.\frac{1}{x}$$

belongs to $\mathcal{A}'_+$, since δ_+ is the Fourier transform of the Heaviside step function.

δ_+ has the following properties:

1. $\delta_+ \in \left(\mathcal{A}'_+ \cap \mathcal{A}'_c\right)$.
2. $\delta_+ * \delta_+ = \delta_+$.
3. $\delta_+ * T = T$, $\forall T \in \left(\mathcal{A}'_+ \cap \mathcal{O}'_\alpha\right)$.

Thus δ_+ is unit of the convolution algebra $\left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$.

From $\delta_+ \in \mathcal{A}'_+$ it follows that there exists δ_+^α, $\forall\alpha \in \mathbb{R}$. If $\alpha \in \mathbb{N}$, then multiplicative power and differentiation operation are related by the formula

$$\delta_+^\alpha = \frac{1}{(2\pi i)^{\alpha-1}(\alpha-1)!} \cdot \frac{d^{\alpha-1}}{dt^{\alpha-1}}\left(\delta_+\right),\ \forall\alpha \in \mathbb{N}.$$

Generally speaking, we can define the distributions of the form $g(T)$ $\forall T \in \mathcal{A}'_+$, where the condition for analytic function g is

$$g \circ T^+(z) \in \mathcal{A}^+(z),$$

here $T^+(z)$ is the appropriate analytic function for T.

3 The Solution of the Multiplicative-Convolution Equation in $\mathcal{A}'_+$

Consider the multiplicative-convolution equation

$$M\{S * U\} = W, \tag{3}$$

where M, S, W are given distributions from A'_+, U is unknown distribution from $\mathcal{A}'_+$.

To solve the equation (3), we need to perform the following steps.

Step 1. Division in $\mathcal{A}'_+$.

Step 2. Solution of a convolution equation in $\mathcal{A}'_+$.

In step 1 we must solve an equation

$$M \cdot V = W, \tag{4}$$

where V is an unknown distribution from $\mathcal{A}'_+$.

Let $M^+(z)$ be the analytic function appropriate to the distribution M. The equation (4) has an solution (and this solution is unique) if $M^+(z)$ has no zeros in $\mathrm{Im} z > 0$. This solution is defined by the formula:

$$\langle V, \varphi\rangle = \lim_{\varepsilon\to+0} \int\limits_{\mathbb{R}} \frac{W^+(x+i\varepsilon)}{M^+(x+i\varepsilon)}\varphi(x)dx,\ \forall\varphi\in\mathcal{S}.$$

Further, consider the step 2. We have convolution equation in $\mathcal{A}'_+$:

$$S * U = V. \tag{5}$$

It is known from the theory of convolution equations that if the equation (5) has the elementary solution $E \in \left(\mathcal{A}'_+ \bigcap \Theta'_c\right)$, then exists the unique solution of equation (5). This solution is defined by the formula

$$U = E * V \tag{6}$$

and belongs to $\mathcal{A}'_+$.

If the elementary solution $E \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$ and $V \in \mathcal{A}'_+$, then exists the unique solution of equation (5). This solution can also defined by the formula (6). In particular, if $V \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$, then solution $U \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$.

Thus, we have proven the following theorem.

Theorem 3.1 *The equation (3) has the unique solution $U \in \mathcal{A}'_+$ if $M^+(z)$ has no zeros in Im$z > 0$ and if equation (5) has the elementary solution E in $\left(\mathcal{A}'_+ \bigcap \Theta'_c\right)$. Solution U is defined by the formula*

$$U = E * \left\{M^{-1}W\right\}. \tag{7}$$

If $E \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$ and $M^{-1}W \in \mathcal{A}'_+$, then $U \in \mathcal{A}'_+$.

If $M^{-1}W \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$, then $U \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$.

Note that the equation (3) can be interpreted as the boundary condition (in the sense of distributions) for finding analytic function $U^+(z)$ in $\mathrm{Im} z > 0$. The solution of this problem is determined by the formula (1) or (2).

Consider also the convolution-multiplicative equation in $\mathcal{S}'_+$:

$$S * \{MV\} = T \tag{8}$$

In case $\left(\mathcal{A}'_+ \cup \mathcal{O}'_\alpha\right)$, $\alpha \geq -1$, equation (3) is isomorphic to the multiplicative-convolution equation (8) under the Fourier transform. The solution of equation (8) is an image of solution (7) under the Fourier cotransform $\bar{\mathcal{F}}$.

4 Examples

Example 4.1 *Consider the equation*

$$(\tau_\alpha \sigma_+)^{-n} \left\{ (\sigma_+^2 + \gamma \sigma_+) * \tau_\beta U \right\} = \frac{e^{2\pi i k x}}{\pi x} \sin 2\pi k x, \tag{9}$$

where $\alpha, \beta \in \mathbb{R} | \alpha \neq \beta$, $\gamma > 0$, $n \in \mathbb{N}$, $\sigma_+ = -2\pi i \delta_+$, $\delta_+ := \frac{1}{2}\delta - \frac{1}{2\pi i} v.p. \frac{1}{x}$, τ_α *and* τ_β *are shift operators.*

It is clear that $(\tau_\alpha \sigma_+)^{-n} \in \mathcal{A}'_+(\mathbb{R})$. Multiplying both sides of (8) by $(\tau_\alpha \sigma_+)^n$, we obtain

$$(\sigma_+^2 + \gamma \sigma_+) * \tau_\beta U = (\tau_\alpha \sigma_+)^n W,$$

here $W = \frac{e^{2\pi i k x}}{\pi x} \sin 2\pi k x$.

Since $\sigma_+^2 = -2\pi i \delta'_+$ and $\sigma_+ = -2\pi i \delta_+$, we obtain the equation

$$\left(\delta'_+ + \gamma \delta_+\right) * \tau_\beta U = -\frac{1}{2\pi i} (\tau_\alpha \sigma_+)^n W. \tag{10}$$

Operator $\left(\delta'_+ + \gamma \delta_+\right) *$ has the elementary solution $E \in \left(\mathcal{A}'_+ \bigcap \mathcal{A}'_c\right)$:

$$E = \delta_+ * Y(x) e^{-\gamma x},$$

where $Y(x)$ is the Heaviside step function. Hence, exists the unique solution of the equation (10):

$$\tau_\beta U = E * \left[-\frac{1}{2\pi i} (\tau_\alpha \sigma_+)^n W \right].$$

Whence

$$U = \tau_{-\beta} \left\{ E * \left[-\frac{1}{2\pi i} (\tau_\alpha \sigma_+)^n W \right] \right\}. \tag{11}$$

The solution (11) belongs to $\mathcal{A}'_+ \bigcap \mathcal{A}'_c$.

Solution for the appropriate boundary value problem

$$(x-\alpha)^n \left\{ \left(\frac{d}{dx} + \gamma \right) \tau_\beta U \right\} = -\frac{1}{2\pi i} W$$

follows from (1) and (2). This solution has the form

$$U^+(z) = -\tfrac{1}{4\pi^3} \tfrac{e^{2\pi i k(z+\beta)}}{z+\beta} \sin 2\pi k(z+\beta) \tfrac{1}{[z-(\alpha-\beta)]^n \left(z+\beta+\frac{i\gamma}{2\pi}\right)}, \ \mathrm{Im} z > 0. \tag{12}$$

From (12) it follows that the solution (11) can be represented in the form:

$$U = -\frac{1}{4\pi^3} \frac{e^{2\pi i k(x+\beta)}}{x+\beta} \sin 2\pi k(x+\beta) \left(\tau_{\alpha-\beta}\sigma_+\right)^n \frac{1}{\left(x+\beta+\frac{i\gamma}{2\pi}\right)}. \tag{13}$$

Since

$$\left| (\mathrm{Im} z)^n \cdot U^+(z) \right| \le C$$

in the neighborhood of the line $\mathrm{Im} z = 0$, we see that $U \in \left(\mathcal{A}'_+(\mathbb{R}) \bigcap \mathcal{A}'^{(n+1)}_c \right)$ [5].

Consider the following equation

$$e^{-2\pi i t \alpha} \frac{d^n}{dt^n} \left\{ (\gamma + it)\, Y(t) e^{-2\pi i t \beta} V \right\} = (-2\pi i)^{n-2} Y \left[k - |t-k| \right], \tag{14}$$

in $\mathcal{S}'_+$, here $V = \bar{\mathcal{F}} U$.

Equation (14) is isomorphic to the equation (9) under the Fourier transform. The solution of equation (14) is the image of (13) under the Fourier cotransform $\bar{\mathcal{F}}$, i.e.

$$V = \frac{(2\pi i)^{n-1}}{\pi(n-1)!} e^{2\pi i x} Y \left[k - |t-k| \right] * Y(x) e^{-2\pi i x(\alpha-\beta)} x^{n-1} * Y(x) e^{-2\pi i x \beta - \gamma x},$$

$$\forall k, n \in \mathbb{N}, \forall \alpha, \beta \in \mathbb{R}, \gamma > 0.$$

Example 4.2 *Consider the equation*

$$\left(\sigma_+^{-1} - \alpha^2 \sigma_+\right) \left\{ \left[\sigma_+^3 - (2\pi)^3 \beta \sigma_+^2 - (2\pi)^3 i \left(\beta^2 + \gamma^2\right) \sigma_+ \right] * U \right\} = 1 + e^{\frac{\pi i x}{\alpha}} \tag{15}$$

in $\mathcal{A}'_+$*, where* $\alpha > 0$*,* $\beta > 0$*,* $\gamma \in \mathbb{R} \setminus \{0\}$*,* $\sigma_+ = -2\pi i \delta_+$*,* $\delta_+ := \frac{1}{2}\delta - \frac{1}{2\pi i} v.p.\frac{1}{x}$.

The solution for this equation is

$$U = \left(Y(x) e^{-2\pi\beta x} \frac{\sin 2\pi\gamma x}{2\pi\gamma} * \delta_+ \right) * \left[-\frac{1}{2\pi i} (1+e)^{\frac{\pi i x}{\alpha}} x \left(\tau_\alpha \sigma_+\right) \left(\tau_{-\alpha} \sigma_+\right) \right].$$

Appropriate boundary value problem is

$$\left(\sigma_+^{-1} - \alpha^2 \sigma_+\right) \left\{ \left[\left(\frac{d}{dx} + 2\pi\beta \right)^{(2)} + (2\pi\gamma)^2 \right] U \right\} = -\frac{1}{2\pi i} \left(1 + e^{\frac{\pi i x}{\alpha}} \right),$$

and solution for this problem is

$$U^+(z) = \frac{1}{(2\pi)^2} \frac{\gamma}{(z+i\beta)^2 - \gamma^2} \cdot \frac{z \left(1 + e^{\frac{\pi i z}{\alpha}} \right)}{2\pi i \left(z^2 - \alpha^2\right)}, \text{ Im} z > 0.$$

Equation

$$\begin{aligned} \{\delta' + 2\pi i \alpha^2 Y(x)\} * \{Y(x) \left[(2\pi i x + 2\pi\beta)^2 + (2\pi\gamma^2)\right] V\} = \\ = -\tfrac{1}{2\pi i} \left(\delta + \tau_{\frac{1}{2\alpha}} \delta \right) \end{aligned} \tag{16}$$

is isomorphic to the equation (15) under the Fourier transform in $\mathcal{S}'_+$.

The solution for (16) in $\mathcal{S}'_+$ is

$$V = -\frac{Y(x)}{2\pi} e^{-2\pi x\beta} \sin 2\pi\gamma x * Y \left(\frac{1}{4\alpha} - \left| x - \frac{1}{4\alpha} \right| \right) \cos 2\pi\alpha x.$$

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