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STATES ON SYMMETRIC LOGICS: EXTENSIONS

Airat Bikchentaev* — Mirko Navara**

Dedicated to Professor Anatolij Dvurečenskij

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. We continue the study of *symmetric logics*, i.e., collections of subsets generalizing Boolean algebras and closed under the symmetric difference. We contribute to several open questions. One of them is whether there is a non-Boolean symmetric logic such that all states on it are $\triangle$-subadditive.

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1. Motivation

Orthomodular posets and, in particular, orthomodular lattices appear as algebraic structures of events in quantum mechanics, cf. [6, 7, 15, 16]. The natural requirement that the event system must allow "sufficiently many" states leads (in its stronger form) to orthomodular posets which can be represented as collections of subsets of a set generalizing σ-algebras [6]. In such collections, the set-theoretical symmetric difference can be introduced as an additional operation [13] which cannot be derived from the lattice-theoretical operations and orthocomplementation [8]. Thus we arrive at the notion of a symmetric logic.

During the study of symmetric logics, many questions remained open (cf. [1,2]). Here we answer some of them. We introduce necessary additional constructions with symmetric logics in Section 3. In Section 4, we clarify under which conditions a symmetric logic becomes a Boolean algebra.

2. Basic notions

2.1. Concrete logics

Let Ω be a non-empty set. By 2^Ω we denote the set of all subsets of Ω. For $n \in \mathbb{N}$, we define $\Omega_n = \{1, 2, \dots, n\}$.

Let us recall [6] that a collection $\mathcal{E} \subseteq 2^\Omega$ of subsets of Ω is called a *concrete (quantum) logic* if the following conditions hold true:

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(C1) $\Omega \in \mathcal{E}$,

(C2) $A \in \mathcal{E} \implies A^c := \Omega \setminus A \in \mathcal{E}$,

(C3) $A, B \in \mathcal{E},\ A \cap B = \varnothing \implies A \cup B \in \mathcal{E}$.

A concrete logic $\mathcal{E}$ is called a *σ-class* [6] if it satisfies the following strengthening of (C3):

(C3') $\{A_n \mid n \in \mathbb{N}\} \subseteq \mathcal{E},\ A_m \cap A_n = \varnothing$ whenever $m \neq n \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{E}$.

A family $\mathcal{E} \subseteq 2^\Omega$ is a concrete logic if and only if it satisfies (C1) and the following condition:

(C4) $A, B \in \mathcal{E},\ A \subseteq B \implies B \setminus A \in \mathcal{E}$.

2.2. Symmetric logics

The set 2^Ω is a group with respect to the symmetric difference operation: $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Notice that

$$\begin{aligned} A^c \triangle B &= (A \triangle B)^c\,, \\ A^c \triangle B^c &= A \triangle B\,. \end{aligned}$$

The principal notion of this paper is the following [12: Definition 3.2]: A *symmetric logic* is a concrete quantum logic $\mathcal{E}$ satisfying:

(S) $A, B \in \mathcal{E} \implies A \triangle B \in \mathcal{E}$.

A family $\mathcal{E} \subseteq 2^\Omega$ is a symmetric logic if and only if it satisfies (C1) and (S) [1: Proposition 1]. Symmetric logics were investigated e.g. in [1–4, 8, 9, 12, 13].

***Example* 2.1.** Let $n \in \mathbb{N}$ and $\Omega_{2n} = \{1, 2, \ldots, 2n\}$. Then the family

$$\mathcal{E}_{2n}^{\text{even}} = \{A \subseteq \Omega_{2n} \mid \operatorname{card} A \text{ is even}\}$$

is a symmetric logic on Ω_{2n}.

***Example* 2.2.** Let $\mathcal{E} \subset 2^\Omega$ be a concrete quantum logic and $T \in \mathcal{E}$, $T \neq \varnothing$. Then the family $\mathcal{E}_T = \{A \in \mathcal{E} \mid A \subseteq T\}$ is a concrete logic with the greatest element T. Moreover, if $\mathcal{E}$ is a symmetric logic, then $\mathcal{E}_T$ is also a symmetric logic.

2.3. States

We say that a mapping $m\colon \mathcal{E} \to [0,1]$ is a *state* (or a *probability measure*) on a concrete logic $\mathcal{E}$ if $m(\Omega) = 1$ and $m(A \cup B) = m(A) + m(B)$ whenever $A, B \in \mathcal{E},\ A \cap B = \varnothing$. Let us denote by $P(\mathcal{E})$ the set of all states on a concrete logic $\mathcal{E}$. For each $a \in \Omega$, we define the state m_a *concentrated* in a by

$$m_a(C) = \begin{cases} 1 & \text{if } a \in C\,, \\ 0 & \text{if } a \notin C \end{cases}$$

for all $C \in \mathcal{E}$. Recall that a state $m \in P(\mathcal{E})$ is called *subadditive* [15: p. 829] if for each $A, B \in \mathcal{E}$ there exists a set $C \in \mathcal{E}$ such that $C \supseteq A \cup B$ and $m(C) \leq m(A) + m(B)$.

If $\mathcal{E}$ is a Boolean algebra then any state $m \in P(\mathcal{E})$ is subadditive. There exists a concrete quantum logic which is not a Boolean algebra and all of its states are subadditive. This result was established in [14] with substantial help of the techniques developed in [10] and [11] (see also [15: p. 831]).

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From now on, we suppose that $\mathcal{E}$ is a symmetric logic. A state $m \in P(\mathcal{E})$ is called $\triangle$-*subadditive* [3] if

$$m(A \triangle B) \leq m(A) + m(B) \qquad \text{for any pair} \quad A, B \in \mathcal{E}\,.$$

The set of all $\triangle$-subadditive states is convex. Every subadditive state $m \in P(\mathcal{E})$ is $\triangle$-subadditive (hint: $C \supseteq A \cup B \supseteq A \triangle B$), but the reverse implication does not hold in general. In [2], the following situations were demonstrated:

1) a $\triangle$-subadditive state which is not subadditive;
2) a two-valued state which is not $\triangle$-subadditive.

3. Auxiliary constructions

3.1. Symmetric logics generated by mappings

Let X, Y be sets and $F\colon X \to Y$ be a mapping. We extend it to a mapping $F\colon 2^X \to 2^Y$ by $F(A) := \{F(x) \mid x \in A\}$. Then $F(A \cup B) = F(A) \cup F(B)$ for all $A, B \subseteq X$. The following conditions are equivalent (see [5: Chap. 1, §6, Exercise]):

a) F is an injection;
b) $F^{-1}(F(A)) = A$ for all $A \subseteq X$;
c) $F(A \cap B) = F(A) \cap F(B)$ for all $A, B \subseteq X$;
d) $F(A) \cap F(B) = \varnothing$ for all $A, B \subseteq X$ with $A \cap B = \varnothing$;
e) $F(A \setminus B) = F(A) \setminus F(B)$ for all $A, B \subseteq X$ with $B \subseteq A$.

In particular, $F(A^c) = F(X) \setminus F(A) \subseteq Y \setminus F(A) = F(A)^c$. Thus, if F is a bijection, then $F(A^c) = F(A)^c$.

We use the notation $F(\mathcal{E}) = \{F(A) \mid A \in \mathcal{E}\}$.

Proposition 3.1. *Let $F\colon X \to Y$ be a bijection. If $\mathcal{E} \subseteq 2^X$ is a concrete logic (a σ-class, a symmetric logic, resp.), then $F(\mathcal{E}) \subseteq 2^Y$ is a concrete logic (a σ-class, a symmetric logic, resp.).*

3.2. Restrictions of symmetric logics

For symmetric logics, a strengthening of Example 2.2 works; the element T need not be taken from $\mathcal{E}$:

Proposition 3.2. *Let $\mathcal{E} \subseteq 2^\Omega$ be a symmetric logic and $T \subseteq \Omega$, $T \neq \varnothing$. Then the family*

$$\mathcal{E}|_T = \{A \cap T \mid A \in \mathcal{E}\} \subseteq 2^T$$

is a symmetric logic with the greatest element T.

Proof. We shall verify conditions (C1) and (S). We have $T = \Omega \cap T \in \mathcal{E}|_T$ and, for all $A, B \in \mathcal{E}$,

$$(A \cap T) \triangle (B \cap T) = (A \triangle B) \cap T \in \mathcal{E}|_T$$

by distributivity of the intersection with respect to the union. $\square$

If $\mathcal{E}$ is a concrete logic which is not a symmetric logic, then the reduction $\mathcal{E}|_T$ need not be a concrete quantum logic, even if $T \in \mathcal{E}$, as shown by the following example:

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***Example* 3.3.** Let $\Omega = \{1,2,3,4,5\}$, $T = \{2,3,5\}$, $A = \{1,2\}$, $B = \{1,3\}$, and

$$\mathcal{E} = \{\varnothing, \Omega, T, T^c, A, A^c, B, B^c\}.$$

Then $\mathcal{E}$ is a concrete quantum logic, but $\mathcal{E}|_T = \{\varnothing, T, \{2\}, \{3,5\}, \{3\}, \{2,5\}\}$ is not a concrete quantum logic because $\{2\} \cup \{3\} \notin \mathcal{E}|_T$.

***Example* 3.4.** In Example 2.1, let us take $m \in P(\mathcal{E}_{2n}^{\text{even}})$ and $T \in \mathcal{E}_{2n}^{\text{even}}$, $m(T) > 0$. Then formula

$$\overline{m}(A \cap T) := \frac{m(A) + m(T) - m(A \triangle T)}{2\, m(T)} \qquad \text{for} \quad A \subseteq T$$

defines a signed measure $\overline{m}$ on 2^T [3: Theorem 2.1]. Moreover, m is $\triangle$-subadditive iff $\overline{m}$ is a state [3: Theorem 2.3].

4. When all states are $\triangle$-subadditive

All states on Boolean algebras are subadditive and hence $\triangle$-subadditive.

Problem 4.1. ([2: Problem 7.1]) Let $\mathcal{E}$ be a symmetric logic such that any state $m \in P(\mathcal{E})$ is $\triangle$-subadditive. Is it true that $\mathcal{E}$ is a Boolean algebra?

In this section, we shall prove that Problem 4.1 has a positive solution in the finite case (Theorem 4.3), but not in the infinite case (Proposition 4.7). For the former, we shall use the following lemma (proved also for infinite sets for a possible future use):

Lemma 4.2. *Let Ω be a finite or infinite set with* $\operatorname{card}\Omega \geq 2$. *Let $\mathcal{E} \subseteq 2^\Omega$ be a symmetric logic such that each state on $\mathcal{E}$ is $\triangle$-subadditive and let $T \subseteq \Omega$, $T \neq \varnothing$. Then each state on $\mathcal{E}|_T$ is $\triangle$-subadditive.*

Proof. Consider $T \subseteq \Omega$, $T \neq \varnothing$, and a symmetric logic $\mathcal{E} \subseteq 2^\Omega$ such that each state on $\mathcal{E}$ is $\triangle$-subadditive. By Proposition 3.2, the family $\mathcal{E}|_T$ is a symmetric logic with the greatest element T. Let us show that every $m \in P(\mathcal{E}|_T)$ is $\triangle$-subadditive. Suppose the contrary: there exist $m \in P(\mathcal{E}|_T)$ and $A, B \in \mathcal{E}|_T$ such that $m(A \triangle B) > m(A) + m(B)$. We extend the measure m to the function $\widetilde{m}$ on $\mathcal{E}$ by the formula $\widetilde{m}(U) := m(U \cap T)$ for all $U \in \mathcal{E}$. Then

a) $\widetilde{m} \in P(\mathcal{E})$ and b) $\widetilde{m}$ is not $\triangle$-subadditive.

For the proof of a), put $U, V \in \mathcal{E}$ with $U \cap V = \varnothing$. Then $(U \cap T) \cap (V \cap T) = \varnothing$ and

$$\begin{aligned}\widetilde{m}(U \cup V) &= m((U \cup V) \cap T) = m((U \cap T) \cup (V \cap T)) \\ &= m(U \cap T) + m(V \cap T) = \widetilde{m}(U) + \widetilde{m}(V)\end{aligned}$$

by distributivity of the intersection with respect to the union and additivity of m. Thus $\widetilde{m} \in P(\mathcal{E})$.

For the proof of b), put $U, V \in \mathcal{E}$ such that $A = U \cap T$, $B = V \cap T$. Then

$$\begin{aligned}\widetilde{m}(U \triangle V) &= m((U \triangle V) \cap T) = m((U \cap T) \triangle (V \cap T)) = m(A \triangle B) \\ &> m(A) + m(B) = m(U \cap T) + m(V \cap T) = \widetilde{m}(U) + \widetilde{m}(V)\end{aligned}$$

by distributivity of the intersection with respect to the symmetric difference. We have a contradiction. ☐

Theorem 4.3. *Let $\mathcal{E}$ be a finite symmetric logic such that each state on $\mathcal{E}$ is $\triangle$-subadditive. Then $\mathcal{E}$ is a Boolean algebra.*

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P r o o f. Suppose that $\mathcal{E}$ is a finite symmetric logic of subsets of Ω. Without loss of generality, we assume that $\mathcal{E}$ satisfies

$$\forall a, b \in \Omega : \ \big[a \neq b \implies \exists A \in \mathcal{E} : \ (a \in A \ \ \& \ \ b \notin A)\big].$$

This means that each two points $a, b \in \Omega$ can be separated by an element of $\mathcal{E}$. Such a representation can be always found by the identification of points which cannot be separated. As $\mathcal{E}$ is finite, so is Ω.

We use induction on $n = \operatorname{card}\Omega$. We assume that $\Omega = \Omega_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. If $n = 1$, then $\Omega_1 = \{1\}$ and $\mathcal{E} = \{\varnothing, \Omega_1\}$ is a Boolean algebra. If $n = 2$ and $\mathcal{E}$ separates points, then $\mathcal{E} = \{\varnothing, \{1\}, \{2\}, \Omega_2\}$, which is a Boolean algebra.

The induction conjecture: for $n > 2$, suppose that every symmetric logic $\mathcal{E} \subseteq 2^{\Omega_k}$ for all $k \in \{1, \ldots, n\}$, such that each state on $\mathcal{E}$ is $\triangle$-subadditive, is a Boolean algebra.

Consider a symmetric logic $\mathcal{E} \subseteq 2^{\Omega_{n+1}}$ such that every state on $P(\mathcal{E})$ is $\triangle$-subadditive. Let us prove that $\mathcal{E}$ is also a Boolean algebra.

Now we show that $\{j\} \in \mathcal{E}$ for all $j \in \Omega_{n+1}$. Let us suppose the contrary: let $\{j\} \notin \mathcal{E}$ for some $j \in \Omega_{n+1}$. Let $\{A_k\}_{k=1}^{l} \subseteq \mathcal{E}$ be such that $\bigcap\limits_{k=1}^{l} A_k = \{j\}$. For $i \in \Omega_{n+1}$, consider $T_i = \Omega_{n+1} \setminus \{i\}$ and the symmetric logic $\mathcal{E}|_{T_i}$. We have $A_k \cap T_i \in \mathcal{E}|_{T_i}$ for all $k \in \Omega_l$ and for all $i \in T_j$. By Proposition 3.2, the family $\mathcal{E}|_{T_i}$ is a symmetric logic with the greatest element T_i. By Lemma 4.2 every $m \in P(\mathcal{E}|_{T_i})$ is $\triangle$-subadditive, and by the induction conjecture $\mathcal{E}|_{T_i}$ is a Boolean algebra. Hence $\left(\bigcap\limits_{k=1}^{l} A_k\right) \cap T_i \in \mathcal{E}|_{T_i}$ for all $i \in T_j$. We have $\left(\bigcap\limits_{k=1}^{l} A_k\right) \cup \{i\} = \{i, j\} \in \mathcal{E}$ for all $i \in T_j$. By taking symmetric differences of such elements of $\mathcal{E}$ we prove that every set $A \subseteq \Omega_{n+1}$ with even cardinality lies in $\mathcal{E}$. With the notation of Example 2.1, $\mathcal{E}_{n+1}^{\mathrm{even}} \subseteq \mathcal{E}$.

Case I. Let $n + 1$ be odd. Then every set $A \subseteq \Omega_{n+1}$ with odd cardinality lies in $\mathcal{E}$ as a complement of some set of even cardinality. Thus $\mathcal{E} = 2^{\Omega_{n+1}}$.

Case II. Let $n + 1$ be even, $n + 1 = 2t$ for some $t \in \mathbb{N}$, $t \geq 2$. It was proved earlier that $\mathcal{E}_{n+1}^{\mathrm{even}} \subseteq \mathcal{E}$. There exist non-$\triangle$-subadditive measures on $\mathcal{E}_{n+1}^{\mathrm{even}}$. For example, put

$$\begin{aligned} m(\{1\}) &= \frac{2}{n+1}, \\ m(\{2\}) &= -\frac{1}{n+1}, \\ m(\{3\}) &= m(\{4\}) = \cdots = m(\{n+1\}) = \frac{1}{n+1} \end{aligned}$$

and define $\widetilde{m} \in P(\mathcal{E}_{n+1}^{\mathrm{even}})$ by the formula $\widetilde{m}(A) := \sum\limits_{j \in A} m(\{j\})$ for $A \in \mathcal{E}_{n+1}^{\mathrm{even}}$. Then

$$\frac{3}{n+1} = \widetilde{m}(\{1,3\}) = \widetilde{m}(\{1,2\} \triangle \{2,3\}) > \widetilde{m}(\{1,2\}) + \widetilde{m}(\{2,3\}) = \frac{1}{n+1}.$$

Thus $\mathcal{E}_{n+1}^{\mathrm{even}} \subsetneq \mathcal{E}$ and there exists $A \in \mathcal{E}$ with $\operatorname{card} A = 2u - 1$ for some $u \leq t$. Without loss of generality we assume that $A = \Omega_{2u-1}$. If $u \geq 2$, then $A \triangle \{2, 3, \ldots, 2u - 1\} = \{1\} \in \mathcal{E}$. If $u = 1$, then $A = \{1\} \in \mathcal{E}$. In both cases, $\{1\} \in \mathcal{E}$ and $\{1\} \triangle \{1, j\} = \{j\} \in \mathcal{E}$ for all $j \in T_1$. Thus $\mathcal{E} = 2^{\Omega_{n+1}}$. □

The following example will be used to show that Theorem 4.3 cannot be extended to infinite symmetric logics:

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***Example* 4.4.** Let Ω be an uncountable set. We define

$$\begin{aligned} \mathcal{B} &:= \{A \subseteq \Omega \mid \operatorname{card} A \text{ is finite or } \operatorname{card}(\Omega \setminus A) \text{ is finite}\}, \\ \mathcal{E}_\Omega^{\mathrm{even}} &:= \{A \subseteq \Omega \mid \operatorname{card} A \text{ is even or } \operatorname{card}(\Omega \setminus A) \text{ is even}\} \subseteq \mathcal{B}. \end{aligned}$$

Then $\mathcal{B}$ is an algebra (=field) of subsets of Ω and $\mathcal{E}_\Omega^{\mathrm{even}}$ is a symmetric logic.

Remark 4.5. Example 4.4 can be described also as a *kernel logic* in the sense of [9, 11]. We may define a measure $\mu\colon \mathcal{B} \to \mathbb{Z}_2$ with values in the two-element cyclic group $\mathbb{Z}_2$ so that μ attains 0 at Ω and 1 at all singletons. Then $\operatorname{Ker}\mu = \{A \in \mathcal{B} \mid \mu(A) = 0\} = \mathcal{E}_\Omega^{\mathrm{even}}$.

We shall show that Example 4.4 gives counterexamples to a conjecture formulated in [2]. For this, we shall use the following property:

Proposition 4.6. *The symmetric logic $\mathcal{E}_\Omega^{\mathrm{even}}$ from* Example 4.4 *contains each union of two disjoint sets from $\mathcal{B} \setminus \mathcal{E}_\Omega^{\mathrm{even}}$.*

Proof. Let $A, B \in \mathcal{B} \setminus \mathcal{E}_\Omega^{\mathrm{even}}$, $A \cap B = \varnothing$. If A, B are finite, they have odd cardinalities and $A \cup B$ has an even cardinality. If A is infinite, then $\Omega \setminus A$ and $B \subseteq \Omega \setminus A$ have odd cardinalities and $\Omega \setminus (A \cup B) = (\Omega \setminus A) \setminus B$ has an even cardinality. The sets A, B cannot be both infinite. □

Proposition 4.7. *Each state on $\mathcal{E}_\Omega^{\mathrm{even}}$ from* Example 4.4 *is $\triangle$-subadditive.*

Proof. Let $A, B \in \mathcal{E}_\Omega^{\mathrm{even}}$ and let m be a state on $\mathcal{E}_\Omega^{\mathrm{even}}$. We use the notation $K_{A,B} = \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$. Notice that $K_{A,B}$ is a partition of unity, i.e., it consists of mutually disjoint sets whose union is Ω.

We distinguish 3 cases:

Case I. Assume that $A \cap B \in \mathcal{E}_\Omega^{\mathrm{even}}$. Then $A \setminus B, B \setminus A \in \mathcal{E}_\Omega^{\mathrm{even}}$ and

$$\begin{aligned} A \setminus B &\subseteq A, & B \setminus A &\subseteq B, \\ m(A \setminus B) &\le m(A), & m(B \setminus A) &\le m(B). \end{aligned}$$

Thus

$$m(A \triangle B) = m(A \setminus B) + m(B \setminus A) \le m(A) + m(B).$$

Case II. Assume that $A \cap B \notin \mathcal{E}_\Omega^{\mathrm{even}}$ and $\operatorname{card}(A \cap B) \ge 3$. Then $A \cap B$ can be expressed as a union of 3 disjoint sets, say R, S, T, from $\mathcal{B} \setminus \mathcal{E}_\Omega^{\mathrm{even}}$. Also elements of $K_{A,B}$ do not belong to $\mathcal{E}_\Omega^{\mathrm{even}}$. According to Proposition 4.6, $\mathcal{E}_\Omega^{\mathrm{even}}$ contains disjoint sets $R \cup (A \setminus B)$, $S \cup (B \setminus A)$, $T \cup (A^c \cap B^c)$ (which form a partition of unity) and also $R \cup S$. Then

$$\begin{aligned} A \setminus B \subseteq R \cup (A \setminus B) &\subseteq A, & B \setminus A \subseteq S \cup (B \setminus A) &\subseteq B, \\ m(R \cup (A \setminus B)) &\le m(A), & m(S \cup (B \setminus A)) &\le m(B). \end{aligned}$$

Thus

$$\begin{aligned} m(A \triangle B) &= m((A \setminus B) \cup (B \setminus A)) \\ &\le m((A \setminus B) \cup (B \setminus A)) + m(R \cup S) \\ &= m((A \setminus B) \cup (B \setminus A) \cup (R \cup S)) \\ &= m((R \cup (A \setminus B)) \cup (S \cup (B \setminus A))) \\ &= m(R \cup (A \setminus B)) + m(S \cup (B \setminus A)) \\ &\le m(A) + m(B). \end{aligned}$$

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Case III. Assume that Cases I and II do not apply. Then $\operatorname{card}(A \cap B) = 1$. The partition of unity $K_{A,B}$ contains exactly one infinite set, say C. In C, we can find uncountably many mutually disjoint nonempty sets from $\mathcal{E}_\Omega^{\mathrm{even}}$. Among them, we may find a $U \subset C$ which has measure 0, $m(U) = 0$. (Otherwise, we have uncountably many disjoint sets of non-zero measure; we get a contradiction because we can choose a finite subfamily whose union has measure greater than 1.) We define new sets $A_0 := A \cup U \in \mathcal{E}_\Omega^{\mathrm{even}}$, $B_0 := B \cup U \in \mathcal{E}_\Omega^{\mathrm{even}}$. (It is possible that $A_0 = A$ or $B_0 = B$.) As U has measure 0, the measures remain unchanged, in particular,

$$\begin{aligned} m(A_0) &= m(A)\,, \\ m(B_0) &= m(B)\,, \\ m(A_0 \triangle B_0) &= m(A \triangle B)\,. \end{aligned}$$

(The latter equality holds because $A_0 \triangle B_0$ is either $A \triangle B$ or $(A \triangle B) \setminus U$.) The important difference is that

$$\operatorname{card}(A_0 \cap B_0) = \operatorname{card}((A \cap B) \cup U) = \operatorname{card}(A \cap B) + \operatorname{card}(U) \geq 3\,,$$

thus Case II applies to A_0, B_0 (in place of A, B). This proves the desired inequality

$$m(A \triangle B) = m(A_0 \triangle B_0) \leq m(A_0) + m(B_0) = m(A) + m(B)\,.$$

$\square$

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* *N. I. Lobachevskii Institute*
of Mathematics and Mechanics
Kazan Federal University
Kremlevskaya 18, 420008
Kazan
RUSSIAN FEDERATION
E-mail: Airat.Bikchentaev@kpfu.ru

** *Center for Machine Perception*
Department of Cybernetics
Faculty of Electrical Engineering
Czech Technical University in Prague
Technická 2
166 27 Prague 6
CZECH REPUBLIC
E-mail: navara@cmp.felk.cvut.cz
URL: http://cmp.felk.cvut.cz/~navara

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