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SHORT COMMUNICATIONS

A Necessary Condition for the Convergence of Simple Partial Fractions in $L_p(\mathbb{R})$

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1. INTRODUCTION AND MAIN RESULT

Let $z_1, z_2, \dots, z_n$ be complex numbers. By a *simple partial fraction* we mean the function

$$g_n(t) = \sum_{k=1}^{n} \frac{1}{t - z_k}.$$

In the present paper, we assume that $z_k = x_k + iy_k \in \mathbb{C} \setminus \mathbb{R}$, $1 \le k \le n$.

In [1], Protasov studied the convergence of simple partial fractions. In particular, he proved that if the series of functions

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k} \tag{1}$$

converges in $L_p(\mathbb{R})$, then, for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^{1/q+\varepsilon}} < +\infty, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2}$$

Conversely if

$$\sum_{k=1}^{\infty} \frac{1}{|y_k|^{1/q}} < +\infty, \tag{3}$$

then the series (1) converges in $L_p(\mathbb{R})$.

If we assume that the points z_k lie in the angle $\{|z| \le C|y|\}$ with C fixed, then it is easy to see that the sufficient condition (3) is very close to the necessary condition (2).

Nevertheless, conditions (2) and (3) are not exact.

In [2], the author obtained a sufficient condition, which is also a necessary one, assuming that all the poles z_n lie in some angle $\{|z| \le C|y|\}$ with C fixed. The inequality

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|y_k|^{p-1}} < +\infty \tag{4}$$

turns out to be such a condition.

The following statement was proved in [2].

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Theorem 1. *Let* $p > 1$. *If condition* (4) *holds, then the series*

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$. *Conversely if this series converges in* $L_p(\mathbb{R})$, *the sequence* $|y_n|$ *is arranged in increasing order, and* $|z_k| \le C|y_k|$, *then condition* (4) *holds.*

Although the condition for convergence (4) is close to condition (3), it does not follow from it. Indeed, consider the sequence $y_k = k^q \ln^q(k+1)$. For it, the series (4) converges, while the series (3) does not.

The goal in the present paper is to derive the necessary condition for the convergence of the series $g_\infty(t)$ in $L_p(\mathbb{R})$. The following statement holds.

Theorem 2. *Let* $p > 1$. *If the series*

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

converges in $L_p(\mathbb{R})$, *then*

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|z_k^*|^{p-1}} < +\infty, \tag{5}$$

where z_k^* *is the sequence obtained from* z_k *by arranging* $|z_k|$ *in increasing order.*

Proof. Using the Hilbert transform, Protasov showed in [1] that the convergences of the series

$$g_\infty(t) = \sum_{n=1}^{\infty} \frac{1}{t - z_n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left| \operatorname{Im}\left(\frac{1}{t - z_n}\right)\right|$$

in the space $L_p(\mathbb{R})$ are equivalent. Therefore, the convergence of the series g_∞ in $L_p(\mathbb{R})$ is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \frac{|y_n|}{(t - x_n)^2 + y_n^2}$$

in the same space.

This means that, in studying the convergence of the series g_∞, we can assume that all the y_k are positive. Further, our series can be partitioned into two series one of which contains the terms with positive x_k and the other, negative terms. In view of symmetry considerations, it suffices to prove our theorem for the case in which all $x_k \ge 0$.

Thus, we assume that $x_k \ge 0$, $y_k > 0$, $k = 1, 2, \dots$.

Let us prove our theorem for two particular cases:

1) $x_k \le y_k$ for all natural numbers k.

2) $x_k > y_k$ for all natural numbers k.

Let us study the first case. Let us arrange the sequence y_k in increasing order. It is readily seen that

$$\int_{-\infty}^{+\infty} \left(\sum_{k=1}^{\infty} \frac{y_k}{(t - x_k)^2 + y_k^2}\right)^p dt \ge \int_{-\infty}^{+\infty} \sum_{j=1}^{\infty} \frac{y_j}{(t - x_j)^2 + y_j^2} \left(\sum_{k=j}^{\infty} \frac{y_k}{(t - x_k)^2 + y_k^2}\right)^{p-1} dt$$
$$\ge \sum_{j=1}^{\infty} \int_{-y_j + x_j}^{y_j + x_j} \frac{y_j}{(t - x_j)^2 + y_j^2} \left(\sum_{k=j}^{\infty} \frac{y_k}{(t - x_k)^2 + y_k^2}\right)^{p-1} dt$$

$$\geq \sum_{j=1}^{\infty} 2y_j \frac{y_j}{y_j^2 + y_j^2} \biggl(\sum_{k=j}^{\infty} \frac{y_k}{(y_k + x_j + x_k)^2 + y_k^2} \biggr)^{p-1} \geq \sum_{j=1}^{\infty} \biggl(\sum_{k=j+1}^{\infty} \frac{y_k}{(y_k + y_k + y_k)^2 + y_k^2} \biggr)^{p-1}$$

$$= \frac{1}{10^{p-1}} \sum_{j=1}^{\infty} \biggl(\sum_{k=j+1}^{\infty} \frac{1}{y_k} \biggr)^{p-1} \geq \frac{1}{10^{p-1}} \sum_{j=1}^{\infty} \biggl(\sum_{k=j+1}^{2j} \frac{1}{y_{2j}} \biggr)^{p-1} = \frac{1}{10^{p-1}} \sum_{j=1}^{\infty} \frac{j^{p-1}}{y_{2j}^{p-1}},$$

whence we immediately find that the series

$$\sum_{j=1}^{\infty} \frac{j^{p-1}}{y_{2j}^{p-1}}$$

converges and, therefore, in view of the monotonicity of y_n, the series

$$\sum_{j=1}^{\infty} \frac{j^{p-1}}{y_j^{p-1}}$$

must also converge. Since $x_k \leq y_k$, this implies that

$$\sum \left| \frac{k}{z_k^*} \right|^{p-1} < +\infty.$$

The case 1 is proved.

In order to prove the theorem in the second case, note that, for all $t \leq 0$, the following inequality holds:

$$\operatorname{Re}\biggl(\frac{1}{z_k - t} \biggr) = \frac{x_k - t}{(x_k - t)^2 + y_k^2} \geq \frac{x_k - t}{(x_k - t)^2 + (x_k - t)^2} = \frac{1}{2(x_k - t)}.$$

Therefore, since the series g_∞ converges in $L_p(\mathbb{R})$, so must also the series

$$\sum_{k=1}^{\infty} \frac{1}{x_k - t}$$

in $L_p(\mathbb{R}_-)$, i.e.,

$$\int_0^\infty \biggl(\sum_{k=1}^{\infty} \frac{1}{x_k + t} \biggr)^p dt = \int_{-\infty}^0 \biggl(\sum_{k=1}^{\infty} \frac{1}{x_k - t} \biggr)^p dt < +\infty.$$

Let us arrange the sequence x_n in increasing order and find a lower bound for the integral

$$\begin{aligned}
\int_0^\infty \biggl(\sum_{k=1}^{\infty} \frac{1}{x_k + t} \biggr)^p dt &\geq \sum_{j=1}^{\infty} \int_0^\infty \frac{1}{x_j + t} \biggl(\sum_{k=j}^{\infty} \frac{1}{x_k + t} \biggr)^{p-1} dt \\
&\geq \sum_{j=1}^{\infty} \int_0^{x_j} \frac{1}{x_j + t} \biggl(\sum_{k=j}^{\infty} \frac{1}{x_k + t} \biggr)^{p-1} dt \\
&\geq \sum_{j=1}^{\infty} \frac{x_j}{2x_j} \biggl(\sum_{k=j}^{\infty} \frac{1}{x_k + x_j} \biggr)^{p-1} \geq 2^{-p} \sum_{j=1}^{\infty} \biggl(\sum_{k=j}^{\infty} \frac{1}{x_k} \biggr)^{p-1}.
\end{aligned}$$

Hence, using identical arguments to those in case 1, we find that the series

$$\sum_{j=1}^{\infty} \frac{j^{p-1}}{x_j^{p-1}}$$

converges. In view of the fact that $x_j \geq y_j$; we see that

$$\sum_{k=1}^{\infty} \left| \frac{k}{z_k^*} \right|^{p-1} < +\infty.$$

The theorem in case 2 is proved.

The proof in the general case can now be easily obtained by partitioning our series into two series one of which relates to the first case and the other, to the second case. □

2. CONCLUSION

Theorems 1 and 2 can be combined into one statement.

Theorem 3. *A necessary condition for the convergence of the series*

$$g_\infty(t) = \sum_{k=1}^{\infty} \frac{1}{t - z_k}$$

in $L_p(\mathbb{R})$ *is the inequality*

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|z_k^*|^{p-1}} < +\infty.$$

A sufficient condition for the convergence of the series $g_\infty(t)$ *in the same space is the inequality*

$$\sum_{k=1}^{\infty} \frac{k^{p-1}}{|y_k|^{p-1}} < +\infty.$$

Obviously, Theorem 3 cannot be improved without additional assumptions on the sequence x_k. However, under some conditions on the sequence x_k, other criteria for the convergence of the series $g_\infty(t)$ in $L_p(\mathbb{R})$ can be obtained. For example, in the case where the sequence x_k is sufficiently sparse, Danchenko showed in [3] that a necessary and sufficient condition for the convergence of the series $g_\infty(t)$ in $L_p(\mathbb{R})$ is the inequality

$$\sum_{k=1}^{\infty} \frac{1}{y_k^{p-1}} < +\infty.$$

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