The Journal of Symbolic Logic

Additional services for *The Journal of Symbolic Logic:*

THE COMPLEMENTS OF LOWER CONES OF DEGREES AND THE DEGREE SPECTRA OF STRUCTURES

URI ANDREWS, MINGZHONG CAI, ISKANDER SH. KALIMULLIN, STEFFEN LEMPP, JOSEPH S. MILLER and ANTONIO MONTALBÁN

Link to this article: http://journals.cambridge.org/abstract_S0022481215000596

How to cite this article:
URI ANDREWS, MINGZHONG CAI, ISKANDER SH. KALIMULLIN, STEFFEN LEMPP, JOSEPH S. MILLER and ANTONIO MONTALBÁN THE COMPLEMENTS OF LOWER CONES OF DEGREES AND THE DEGREE SPECTRA OF STRUCTURES. The Journal of Symbolic Logic, Available on CJO 2016 doi:10.1017/jsl.2015.59

Request Permissions : Click here

THE JOURNAL OF SYMBOLIC LOGIC, Page 1 of 10

THE COMPLEMENTS OF LOWER CONES OF DEGREES AND THE DEGREE SPECTRA OF STRUCTURES

URI ANDREWS, MINGZHONG CAI, ISKANDER SH. KALIMULLIN, STEFFEN LEMPP, JOSEPH S. MILLER, AND ANTONIO MONTALBÁN

Abstract. We study Turing degrees $\mathbf{a}$ for which there is a countable structure $\mathcal{A}$ whose degree spectrum is the collection $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$. In particular, for degrees $\mathbf{a}$ from the interval $[\mathbf{0}', \mathbf{0}'']$, such a structure exists if $\mathbf{a}' = \mathbf{0}''$, and there are no such structures if $\mathbf{a}'' > \mathbf{0}'''$.

§1. Introduction. The *degree spectrum* of a countable structure $\mathcal{A}$ in a finite language is the set of all Turing degrees of isomorphic copies of $\mathcal{A}$ on the universe ω:

$$\mathbf{Sp}\,(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{A} \cong \mathcal{B} \ \& \ \dom\,(\mathcal{B}) = \omega\}.$$

Knight [7] showed that each degree spectrum is either the singleton $\{\mathbf{0}\}$ (the trivial case), or closed upward. Therefore, the degree spectrum of a noncomputable structure $\mathcal{A}$ can be alternatively defined as the collection of degrees that compute copies of $\mathcal{A}$.

Slaman [12] and Wehner [13] have proved independently that there exist noncomputable structures that are computable in all nonzero degrees, i.e., the collection $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{0}\}$ is a degree spectrum of a structure. Wehner's solution used the following theorem:

THEOREM 1.1 (Wehner [13]). *There is a family $\mathcal{G}$ of sets such that $\mathcal{G}$ is uniformly X-c.e. if and only if X is not computable.*

To get a structure with the desired degree spectrum, it is enough now to use a coding $\mathcal{G} \mapsto \Gamma_\mathcal{G}$ of families of sets into undirected graphs such that $\mathcal{G}$ is uniformly X-c.e. if and only if $\Gamma_\mathcal{G}$ has an X-computable copy. For example, we can let $\Gamma_\mathcal{G}$ be the disconnected union of infinitely repeated graphs Γ_A, for each $A \in \mathcal{G}$, where each Γ_A consists of a single vertex connected with independent $(n+3)$-cycles for each $n \in A$.

Note that this coding can be carried over to produce partial orderings, lattices, rings, integral domains, commutative semigroups, and 2-step nilpotent groups with the degree spectrum $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{0}\}$ (see [2]). For linear orderings [10], equivalence structures, and abelian p-groups [6], we only know that degree spectra can be equal to $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{0}\}$ within the Δ_2^0-degrees.

Using a coding of families into structures, Kalimullin [3,4] generalized the result of Slaman and Wehner to the complements of lower cones of some nonzero degrees: if a degree $\mathbf{a}$ is low or c.e. then the collection $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$ is the degree spectrum

of a structure. In the next section, we will give an alternative proof for the c.e. case and, after that, we will be able to combine both cases, finding an appropriate family for a degree $\mathbf{a}$ that is low over a c.e. degree $\mathbf{g} \le \mathbf{a}$.

Kalimullin [5] has also proved that there is a degree $\mathbf{a} \le \mathbf{0}''$ such that the collection $\{\mathbf{x} : \mathbf{x} \nleq \mathbf{a}\}$ is not the degree spectrum of any structure. In the last two sections, we will see that there are large classes of degrees $\mathbf{a}$ with this property, and that we can even take $\mathbf{a} = \mathbf{0}''$.

§2. Families that are uniformly c.e. in the complement of a cone.

THEOREM 2.1 (Kalimullin [4]).   *For each c.e. set A, there is a family $\mathcal{G}_A$ of c.e. sets such that $\mathcal{G}_A$ is uniformly X-c.e. if and only if $X \nleq_T A$.*

PROOF (Montalbán). Without loss of generality, we may assume that the set $A \subseteq \omega$ is infinite. Fix a computable bijection $f : \omega \to A$ and let g be the true stage function for A defined by

$$g(0) = 0, \quad g(n + 1) = (\mu s > g(n))[(\forall t > s)[f(t) > f(s)]].$$

Then $g \equiv_T A$ and the set of strings $C_g = \{\sigma \in \omega^{<\omega} : \sigma \nsubseteq g\}$ is c.e.

Consider the family of c.e. sets

$$\mathcal{G}_A = \left\{ \{n\} \oplus (C_g \cup g_F) : n \in \omega \ \& \ F \text{ is finite} \ \& \ F \neq W_n^g \right\},$$

where $g_F = \{g \restriction x : x \in F\} = \{\tau \subset g : |\tau| \in F\} \subseteq \omega^{<\omega} - C_g$, and

$$\{n\} \oplus X = \{2n\} \cup \{2x + 1 : x \in X\}.$$

The main idea of this definition is to diagonalize against all A-c.e. sets W_n^g, $n \in \omega$, along the co-c.e. path g. This ensures that the family is not A-c.e.; if it were A-c.e., then knowing g we could effectively enumerate a set not equal to W_n^g, for each n, which contradicts the Recursion Theorem. On the other hand, if a string has not yet been enumerated into C_g, then it might be an initial segment of g, so we can use it to estimate W_n^g. Based on this estimation, we code a finite part of given set $X \nleq_T A$ into F to eventually ensure the diagonalization $F \neq W_n^g$. We explain both parts of the proof more formally below.

First, we show that $\mathcal{G}_A$ is not uniformly A-c.e. Suppose otherwise. Since $g \equiv_T A$, there is a computable function a such that

$$\mathcal{G}_A = \left\{ W_{a(k)}^A : k \in \omega \right\}.$$

Define computable functions b and c such that, for every $n \in \omega$,

$$2n \in W_{a(b(n))}^g \text{ and } W_{a(b(n))}^g = \{n\} \oplus W_{c(n)}^g \in \mathcal{G}_A.$$

Now, define a computable function d such that, for every $n \in \omega$,

$$W_{d(n)}^g = \left\{ x : g \restriction x \in W_{c(n)}^g \right\}.$$

Then $W_{d(n)}^g \neq W_n^g$ for every $n \in \omega$, since $W_{c(n)}^g = C_g \cup g_F$ with $F = W_{d(n)}^g$. This contradicts the Recursion Theorem, so $\mathcal{G}_A$ is not uniformly A-c.e.

Next, let $X \nleq_T A$. To prove that $\mathcal{G}_A$ is uniformly X-c.e., we fix an infinite X-computable set U that does not contain any infinite A-c.e. subsets. (For example,

a set consisting of codes of initial segments of X will suffice.) Let $U = \{u(0) < u(1) < u(2) < \cdots\}$ for an X-computable function u. We may assume that $u(s) > s$ for every s.

Fix integers $n, m, s \in \omega$. Below we X-computably define the sequence $H_\sigma = H_\sigma^{n,m,s}$ by induction on $\sigma \in \omega^{<\omega}$, beginning with the empty string $\sigma = \lambda$:

$$H_\lambda = D_m;$$

$$H_{\sigma * i} = \begin{cases} H_\sigma \cup \{u(|\sigma|)\}, & \text{if } H_\sigma = W_n^\sigma \text{ and } s < |\sigma|, \\ H_\sigma & \text{otherwise,} \end{cases}$$

where D_m is the m-th finite set, and W_n^σ is the set containing all numbers enumerated by the W_n-operator with the use at most σ before the end of the stage $t = |\sigma|$. Let $H^{n,m,s} = \bigcup_{\sigma \subset g} H_\sigma^{n,m,s}$.

Suppose that $D_m \neq W_n^g$, and the stage s is large enough that $D_m \neq W_n^\sigma$, for every $\sigma \subset g$ with $|\sigma| > s$. Then the definition ensures that $H^{n,m,s} = D_m$. Note also that for arbitrary $n, m, s \in \omega$, the set $H^{n,m,s}$ is finite, and therefore $H^{n,m,s} \neq W_n^g$; otherwise $H^{n,m,s} - D_m = W_n^g - D_m$ would be an infinite A-c.e. subset of U. Thus,

$$\{H^{n,m,s} : m, s \in \omega\} = \{F : F \text{ is finite \& } F \neq W_n^g\}$$

for every $n \in \omega$. We also have

$$\{\tau \subset g : (\exists \sigma \subseteq \tau)[|\tau| \in H_\sigma^{n,m,s}]\} = \{\tau \subset g : |\tau| \in H^{n,m,s}\} = g_{H^{n,m,s}},$$

since $u \in H_{\sigma * i}^{n,m,s} - H_\sigma^{n,m,s}$ implies $u > |\sigma|$; and

$$C_g \cup \{\tau : (\exists \sigma \subseteq \tau)[|\tau| \in H_\sigma^{n,m,s}]\} = C_g \cup \{\tau \subset g : (\exists \sigma \subseteq \tau)[|\tau| \in H_\sigma^{n,m,s}]\} = C_g \cup g_{H^{n,m,s}}.$$

Therefore, the family

$$\mathcal{G}_A = \{\{n\} \oplus (C_g \cup \{\tau : (\exists \sigma \subseteq \tau)[|\tau| \in H_\sigma^{n,m,s}]\}) : n, m, s \in \omega\}$$

is uniformly X-c.e.   ⊣

COROLLARY 2.2 (Kalimullin [4]). *For every c.e. degree* $\mathbf{a}$ (*in particular, for* $\mathbf{a} = \mathbf{0}'$), *there exists a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \nleq \mathbf{a}\}$.

THEOREM 2.3. *Let G be a c.e. set. Then for each set A with $G \leq_T A$ and $A' \leq_T G'$, there is a family $\mathcal{G}_A$ of c.e. sets such that $\mathcal{G}_A$ is uniformly X-c.e. if and only if $X \nleq_T A$.*

PROOF. As in the previous proof, we take a function $g \equiv_T G$ such that the set $C_g = \{\sigma \in \omega^{<\omega} : \sigma \nsubseteq g\}$ is c.e.

Consider the family of c.e. sets

$$\mathcal{G}_A = \{\{n\} \oplus (C_g \cup g_F) : n \in \omega \text{ \& } F \text{ is finite \& } F \neq W_n^A\},$$

where $g_F = \{g \upharpoonright x : x \in F\} \subseteq \omega^{<\omega} - C_g$.

Again, $\mathcal{G}_A$ is not uniformly A-c.e. Indeed, if $\mathcal{G}_A = \{W_{a(n)}^A : n \in \omega\}$ for a computable function a, then we have computable functions b and c such that $W_{a(b(n))}^A = \{n\} \oplus W_{c(n)}^A \in \mathcal{G}_A$, so that using $g \leq_T A$ we have a computable function d such that, for all $n \in \omega$,

$$W_{d(n)}^A = \left\{x : g \upharpoonright x \in W_{c(n)}^A\right\} \neq W_n^A.$$

This contradicts the Recursion Theorem.

Let $X \not\leq_T A$. We again fix an infinite X-computable set $U = \{0 < u(0) < u(1) < u(2) < \cdots \}$ that does not contain any infinite A-c.e. subsets. Since $A' \leq_T g'$ there is a Turing functional Φ such that

$$W_n^A(z) = \lim_s \Phi^g(n, z, s)$$

for every $n, z \in \omega$. For finite uses $\sigma \in \omega^{<\omega}$, we introduce the following approximation of W_n^A:

$$V_n^\sigma = \{z : (\exists s < |\sigma|)[\Phi^\sigma(n, z, s) \downarrow = 1 \ \& \ (\forall t < |\sigma|)[t > s \implies \Phi^\sigma(n, z, t) \uparrow]]\},$$

where $\Phi^\sigma(n, z, s)$ is the value (if any) computed before the stage $|\sigma|$. It is clear that $W_n^A = \lim_{\sigma \subset g} V_n^\sigma$.

The definition of the sequences H_σ, $\sigma \in \omega^{<\omega}$, is similar to one from the previous proof. In contrast with the previous proof, we have only $W_n^A = \lim_{\sigma \subset g} V_n^\sigma$ instead of $W_n^g = \bigcup_{\sigma \subset g} W_n^\sigma$. This forces us to add to the list of parameters $n, m, s \in \omega$ one more parameter $x \in \omega$ that acts as a witness of $D_m \neq W_n^A$. Also we need to use standard length functions ℓ_σ to estimate and to control the agreement between H_σ and W_n^A.

Below is the formal inductive X-computable definition of the sequence $H_\sigma = H_\sigma^{n,m,x,s}$, for $\sigma \in \omega^{<\omega}$:

$$H_\lambda = D_m;$$

$$H_{\sigma*i} = \begin{cases} H_\sigma \cup \{u(|\sigma|)\}, & \text{if } x \notin D_m \triangle V_n^\sigma, \ x < |\sigma|, \ s < |\sigma|, \text{ and} \\ & \qquad (\forall \rho \subset \sigma)[\ell_\sigma > \ell_\rho], \\ H_\sigma & \text{otherwise,} \end{cases}$$

where $D_m \triangle V_n^\sigma = (D_m - V_n^\sigma) \cup (V_n^\sigma - D_m)$, and

$$\ell_\sigma = \max \{y \leq |\sigma| : (\forall z < y)[z \in H_\sigma \iff z \in V_n^\sigma]\}.$$

Let $H^{n,m,x,s} = \bigcup_{\sigma \subset g} H_\sigma^{n,m,x,s}$. We must prove that

$$\{H^{n,m,x,s} : m, x, s \in \omega\} = \{F : F \text{ is finite } \& \ F \neq W_n^A\}$$

for every $n \in \omega$. If $D_m \neq W_n^A$, then take an x such that $x \in D_m \triangle W_n^A$. Take s large enough that $x \in D_m \triangle V_n^\sigma$ for every $\sigma \subset g$ with $|\sigma| > s$. The definition ensures that $H^{n,m,x,s} = D_m$.

It remains to prove that $H^{n,m,x,s}$ is finite and $H^{n,m,x,s} \neq W_n^A$, for every n, m, x, s. Suppose that $H^{n,m,x,s}$ is infinite. Then $\lim_{\sigma \subset g} \ell_\sigma = \infty$, and so

$$H^{n,m,x,s} = \lim_{\sigma \subset g} V_n^\sigma = W_n^A$$

is an infinite A-c.e. subset of $U \cup D_m =^* U$, contradicting our assumption about U. Therefore, $H^{n,m,x,s}$ is finite.

Suppose that $H^{n,m,x,s} = W_n^A$. Then $\lim_{\sigma \subset g} \ell_\sigma = \infty$. Since $H^{n,m,x,s}$ is finite, by definition we should have $x \in D_m \triangle W_n^A = \lim_{\sigma \subset g}(D_m \triangle V_n^\sigma)$ and $H^{n,m,x,s} = D_m \cup \{u_1, \ldots, u_k\}$ with $u_i = u(|\sigma_i|) \geq |\sigma_i| > x$ for each $i = 1, \ldots, k$. Then $x \in H^{n,m,x,s} \triangle W_n^A$, which is a contradiction.

We have proved that $\{H^{n,m,x,s} : m, x, s \in \omega\} = \{F : F \text{ is finite } \& \ F \neq W_n^A\}$ for every $n \in \omega$. As in the previous proof, this implies that the family

$$\mathcal{G}_A = \{\{n\} \oplus (C_g \cup \{\tau : (\exists \sigma \subseteq \tau)[|\tau| \in H_\sigma^{n,m,x,s}]\}) : n, m, x, s \in \omega\}$$

is uniformly X-c.e. $\dashv$

COROLLARY 2.4. *Suppose that for a degree* $\mathbf{a}$ *there exists a c.e. degree* $\mathbf{g} \leq \mathbf{a}$ *such that* $\mathbf{a}' \leq \mathbf{g}'$. *Then there exists a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.

COROLLARY 2.5 (Kalimullin [3]). *If* $\mathbf{a}' \leq \mathbf{0}'$, *then there exists a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.

COROLLARY 2.6. *If* $\mathbf{0}' \leq \mathbf{a}$ *and* $\mathbf{a}' \leq \mathbf{0}''$, *then there exists a structure* $\mathcal{A}$ *such that* $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.

§3. Impossible degree spectra: a highness property. A structure $\mathcal{B}$ has the *c.e. extension property* if each existential type $Th_\exists(\mathcal{B}, \vec{b})$ is c.e. As we will explain later, if the degree spectrum of $\mathcal{B}$ is the complement of a lower cone, then $\mathcal{B}$ must have this property.

THEOREM 3.1. *Let A and B be sets such that* B'' *has A-c.e. degree and* $B''' \leq_T A'$. *Then each B-computable structure* $\mathcal{B}$ *with the c.e. extension property has an A-computable copy* $\mathcal{A} \cong \mathcal{B}$.

PROOF. Let $\mathcal{B}$ be a B-computable structure. Without loss of generality we can assume that $\mathcal{B}$ has a relational signature and the universe of $\mathcal{B}$ is ω. Since $\mathcal{B}$ has the c.e. extension property, for each tuple of integers $\vec{b}$ the set $Th_\exists(\mathcal{B}, \vec{b})$ of all existential facts about $\vec{b}$ in $\mathcal{B}$ is c.e. Note that the relation

$$R(\vec{b}, e) \iff W_e = Th_\exists(\mathcal{B}, \vec{b})$$

is $\Pi_2^B \subseteq \Delta_3^B \subseteq \Delta_1^{B''}$, so that we can fix a B''-computable function $e(\vec{b})$ such that

$$W_{e(\vec{b})} = Th_\exists(\mathcal{B}, \vec{b}).$$

If $\mathcal{C}$ is a finite substructure of $\mathcal{B}$, then we will write $e(\mathcal{C})$ for $e(\vec{b})$, where $\vec{b}$ is the sequence of elements of $\mathcal{C}$ written in increasing order (as integers).

It is easy to see by a direct back-and-forth strategy that the theorem follows immediately if $e(\vec{b})$ is A-computable. In the general case we have $e \leq_T B'' \leq_T B''' \leq_T A'$, but we also know that B'' has A-c.e. degree.

Now instead of each single back-and-forth step, we can use a straightforward technical algorithm based on the Limit Lemma:

LEMMA 3.2. *Assume that the following data is given*:
(1) *a finite structure* $\mathcal{C}$;
(2) *a monomorphism* $f : \mathcal{C} \to \mathcal{B}$;
(3) *a value* $e(f(\mathcal{C}))$ *such that* $Th_\exists(\mathcal{B}, f(\mathcal{C})) = W_{e(f(\mathcal{C}))}$;
(4) *a finite structure* $\mathcal{D} \supseteq \mathcal{C}$ *that is embeddable into* $\mathcal{B}$ *extending* f *(we can check this fact using* $e(f(\mathcal{C}))$);
(5) *an element* $b \in \mathcal{B}$.

Then there is an A-computable algorithm that generates, uniformly in the given data, an infinite sequence of finite structures

$$\mathcal{D} = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots$$

such that:
(1) $\mathcal{E} = \bigcup_s \mathcal{E}_s$ *is finite*;
(2) *there is a monomorphism* $g : \mathcal{E} \to \mathcal{B}$ *extending* f *such that* $b \in g(\mathcal{E})$.

Fix an A-c.e. set $X \equiv_T B''$. Our construction of an A-computably copy $\mathcal{A} \cong \mathcal{B}$ uses the Robinson low guessing method [11], which helps tame the process of guessing facts that are computable from a low c.e. set (in this case, relative to A). To this end, we will define a Turing operator Γ^X. Using the Recursion Theorem and the fact that $X' \leq_T A'$, we can fix a Turing operator Δ such that $\Delta^{A'} = \operatorname{dom} \Gamma^X$. We will use Γ and Δ to certify A-computable guesses about computations from $X \equiv_T B''$.

The construction itself splits into simultaneously working modules M_n, $n \in \omega$; each of them produces a finite structure $\mathcal{A}_n$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$. We will let $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

The module M_n executes the procedure from Lemma 3.2 with the input

(1) $\mathcal{C} = \mathcal{A}_{n-1}$ (we assume $\mathcal{A}_{-1} = \emptyset$);
(2) the monomorphism $f_{n-1} \colon \mathcal{A}_{n-1} \to \mathcal{B}$ provided by M_{n-1};
(3) the value $e(f_{n-1}(\mathcal{A}_{n-1}))$ provided by M_{n-1};
(4) the finite structure $\mathcal{D} \supseteq \mathcal{C}$ equal to the union of all $\mathcal{A}_m$, $m \in \omega$, as they appear at the moment the procedure starts;
(5) the element $b = n$ from $\mathcal{B}$

The output structure $\mathcal{A}_n = \mathcal{E}$ should satisfy the properties:

(1) $\mathcal{A}_n$ is finite;
(2) there is a monomorphism $f_n \colon \mathcal{E} \to \mathcal{B}$ extending f_{n-1} such that $b \in f_n(\mathcal{A}_n)$.

While the procedure is executing, we are trying to calculate both a monomorphism $f_n \colon \mathcal{A}_n \to \mathcal{B}$ extending f_{n-1} such that $b \in f_n(\mathcal{A}_n)$, and the value $e(f_n(\mathcal{A}_n))$. Precise calculation of this data requires the B''-oracle, so we can pass to M_{n+1} only a limit approximation based on the current A-computable enumeration of $X \equiv_T B''$. Before passing to M_{n+1}, we certify the current approximation: Define $\Gamma^X(n) \downarrow$ with the same use as the current calculation of f_n and $e(f_n(\mathcal{A}_n))$, then wait for $\Delta^{A'}(n) = 1$ for the current A-computable enumeration of A'. If our approximation later becomes incorrect via inconsistency of the Γ-use, wait until we see $\Delta^{A'}(n) = 0$ and then restart the procedures in all M_m for $m \geq n$.

Since $\Delta^{A'} = \operatorname{dom} \Gamma^X$, there are only finitely many restarts for each M_n, $n \in \omega$. Each module M_n successfully defines a finite structure $\mathcal{A}_n$ with embedding $f_n \colon \mathcal{A}_n \to \mathcal{B}$. Since $n \in f_n(\mathcal{A}_n)$, the function $f = \bigcup_n f_n$ is an isomorphism between the A-computable structure $\mathcal{A} = \bigcup_n \mathcal{A}_n$ and the structure $\mathcal{B}$. $\dashv$

COROLLARY 3.3. *If $\mathbf{a} \geq \mathbf{0}'$ and $\mathbf{a}' \geq \mathbf{0}'''$ (in particular, if $\mathbf{a} = \mathbf{0}^{(n)}$ for $n \geq 2$), then there is no countable structure $\mathcal{A}$ such that $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.*

PROOF. Suppose for a contradiction that $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$. Then each existential type of $\mathcal{A}$ is c.e. relative to every element of the co-null class

$$\{X \in 2^\omega : X \not\leq_T \mathbf{a}\}.$$

By [9], this is possible only if the existential type is c.e. Thus, $\mathcal{A}$ has the c.e. extension property. By assumption, $\mathcal{A}$ has no $\mathbf{a}$-computable copy.

Consider first the case when $\mathbf{0}'' \not\leq \mathbf{a}$. By a modification of Friedberg's Jump Inversion Theorem (see [8, §13.3, Corollary X(b)]), there are degrees $\mathbf{a}_0, \mathbf{a}_1 \geq \mathbf{0}'$ such that

$$\mathbf{0}'' = \mathbf{a}_0' = \mathbf{a}_1' = \mathbf{a}_0 \cup \mathbf{a}_1.$$

Applying the same result to each of the degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, we get degrees $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}_3$ such that

$$\mathbf{0}'' = \mathbf{b}_0'' = \mathbf{b}_1'' = \mathbf{b}_2'' = \mathbf{b}_3'' = \mathbf{b}_0 \cup \mathbf{b}_1 \cup \mathbf{b}_2 \cup \mathbf{b}_3.$$

Since $\mathbf{0}'' \not\leq \mathbf{a}$, there exists a degree $\mathbf{b} \in \{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ such that $\mathbf{b} \not\leq \mathbf{a}$. Then $\mathcal{A}$ has a $\mathbf{b}$-computable copy, but $\mathbf{b}'' = \mathbf{0}''$ has c.e. degree relative to $\mathbf{a} \geq \mathbf{0}'$ and $\mathbf{b}''' = \mathbf{0}''' \leq \mathbf{a}'$. This contradicts Theorem 3.1.

Suppose now that $\mathbf{0}'' \leq \mathbf{a}$. Relativizing the low simple set construction, one can find an $\mathbf{a}$-c.e. degree $\mathbf{c} > \mathbf{a}$ such that $\mathbf{c}' = \mathbf{a}'$. By a double jump inversion argument, as above, there exists a degree $\mathbf{b} \not\leq \mathbf{a}$ such that $\mathbf{b}'' = \mathbf{c}$. Then $\mathcal{A}$ has a $\mathbf{b}$-computable copy but $\mathbf{b}'' = \mathbf{c}$ has $\mathbf{a}$-c.e. degree and $\mathbf{b}''' = \mathbf{c}' \leq \mathbf{a}'$. Again, this contradicts Theorem 3.1.   ⊣

§4. **Impossible degree spectra: a nonlowness property.** Just for convenience, we define $\mathcal{C}(\mathbf{x})$ as the class of degrees $\mathbf{a} \geq \mathbf{x}$ such that for every function $f \leq_T \mathbf{x}'$, there is a $g \leq_T \mathbf{a}$ that is not dominated by f.

THEOREM 4.1. *If* $\mathbf{a} \in \mathcal{C}(\mathbf{0}')$, *then there is no countable structure* $\mathcal{A}$ *such that* $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.

PROOF. We use the following forcing lemma. The proof follows from the proof of a similar result for non-GL_2 degrees in [1, Theorem 2.8] with the observation that the function from the proof that we need to escape is computable from $\mathbf{0}''$.

LEMMA 4.2. *Let* $\mathcal{P}$ *be a forcing notion computable from* $\mathbf{0}'$ *(i.e., both membership and the partial order are computable from* $\mathbf{0}'$*) and let* $\langle D_n \rangle$ *be a class of dense sets such that there is a* $\mathbf{0}''$*-computable density function* $d(p, n) = q$ *which gives a forcing condition* q *that extends* p *and is in* D_n. *Then every* $\mathbf{a} \in \mathcal{C}(\mathbf{0}')$ *computes a generic sequence in* $\mathcal{P}$ *with respect to the dense sets* $\langle D_n \rangle$.

We may assume that $\mathbf{a}$ is not above $\mathbf{0}''$ (because the case $\mathbf{a} \geq_T \mathbf{0}''$ is covered in Corollary 3.3). As above, we can perform double jump inversion with cone avoidance to get a degree $\mathbf{b}$ that is not below $\mathbf{a}$ and such that $\mathbf{b}'' = \mathbf{0}''$.

For a contradiction, assume that $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$ is the degree spectrum of a countable structure $\mathcal{A}$, and let $\mathcal{M}$ be a model of $\mathcal{A}$ computable in $\mathbf{b}$. As always, we assume that models are defined on ω. Again, the goal is to show that $\mathbf{a}$ can compute a model $\mathcal{N}$ isomorphic to $\mathcal{M}$ using the fact that $\mathcal{M}$ has the c.e. extension property.

The key idea is that using $\mathbf{b}'' = \mathbf{0}''$ we could compute the correct c.e. indices for the existential types of finite approximations to $\mathcal{N}$, along with associated embeddings into $\mathcal{M}$. This allows us to give a very robust construction of a structure $\mathcal{N}$ isomorphic to $\mathcal{M}$. The plan is to use this robustness and Lemma 4.2 to show that $\mathbf{a}$ can approximate such a construction well enough to also compute an isomorphic copy of $\mathcal{M}$. We need to define an appropriate forcing notion.

First, we define an auxiliary $\mathbf{0}'$-computable ordering $\mathcal{Q}$; our forcing notion $\mathcal{P}$ will be defined based on $\mathcal{Q}$. Let $\mathcal{Q}$ be all 5-tuples (s, τ, σ, f, e) where s is a natural number, τ is an initial segment of the standard $\mathbf{0}'$-computable enumeration of $\emptyset''$ at stage s, σ is an initial segment of the model $\mathcal{N}$ we want to build, coded as a binary string, τ verifies that f is a partial isomorphism from $\mathcal{N}$ to the given model $\mathcal{M}$ and that e is the c.e. index of the existential type of the finitely many elements we have decided in the range of f.

Given two tuples $p = (s, \tau, \sigma, f, e)$ and $q = (s', \tau', \sigma', f', e')$, q extends p ($q \leq_{\mathcal{Q}}$ p) if the following holds:

(1) $s \leq s'$ and $\sigma' \supseteq \sigma$, i.e., the model is extended consistently.
(2) either at stage s', (computable in $\mathbf{0}'$) we see that the approximation to $\emptyset''$ disagrees with τ (in which case we believe p is wrong); or
   (a) σ', after coded as an existential formula over the free variables in the (assumed) type W_e, is in W_e (so σ' extends σ while respecting W_e);
   (b) τ' verifies that $W_{e'}$ *extends* W_e in the sense that for every formula in $W_{e'}$, after adding existential quantifiers for the last appropriate number of free variables (depending on the numbers of free variables in W_e and $W_{e'}$), is in W_e; and
   (c) f' extends f.
   (i.e., we believe p is correct and we extend it to q appropriately for each component.)

It is easy to see that $\mathbf{0}'' = \mathbf{b}''$ computes a sequence in $\mathcal{Q}$ such that the limit of the f's is total and onto, and furthermore, such that the τ's are always correct initial segments of $\emptyset''$. Our plan is to use the degree $\mathbf{a}$ to approximate such a construction. Note that in an approximation, we might figure out that some guess $p \in \mathcal{Q}$ is wrong (i.e., discover that $\emptyset''$ disagrees with τ). In this case, we should ignore the restraints (f, e) that p puts on extensions, but we still want to obey the restriction placed by previous guesses that have not been proved wrong. (Note that previous guesses are not automatically respected because our order on $\mathcal{Q}$ is not actually transitive.) For this reason, we want to keep track of all guesses, so we define a forcing notion $\mathcal{P}$ by taking "chains of conditions in $\mathcal{Q}$".

A forcing condition in $\mathcal{P}$ is a finite sequence $\bar{p}$ of conditions $\langle p_0, p_1, \ldots, p_n \rangle$ such that each p_i is in $\mathcal{Q}$ and p_i extends p_j whenever $i > j$. Given two conditions $\bar{p}$ and $\bar{q}$, we say $\bar{q}$ extends $\bar{p}$ if $\bar{p}$ is an initial segment of $\bar{q}$ (as sequences). It is easy to see that the forcing notion $\mathcal{P}$ is computable from $\mathbf{0}'$.

Consider the dense sets

$$D_n = \{\bar{p} : \text{in the last condition } p_m \text{ of } \bar{p}, \tau_m \text{ is a correct initial segment of } \emptyset''$$
$$\text{and } n \text{ is in both the domain and range of } f_m\}.$$

We show that $\mathbf{0}''$ computes a density function f such that $f(\bar{p}, n)$ extends a given $\bar{p}$ and is in D_n. First of all, given $\bar{p} = \langle p_0, p_1, \ldots, p_m \rangle$, $\mathbf{0}''$ can figure out whether the τ part of these conditions are correct or not, and find a stage s_0 when we detect all incorrect ones in the $\mathbf{0}'$-computable enumeration. Now for the last remaining correct p_i, σ_m is, by definition of the partial order, in W_{e_i}. This means that the current model can still be consistently mapped to the given model $\mathcal{M}$ by a partial isomorphism extending the correct f_i in p_i. So we can extend the sequence by adding a new condition (s, τ, σ, f, e) where σ contains n in the universe; f is an extension of the partial isomorphism as above by adding n into both the domain and range; e is the correct existential type of the current elements in the domain (and range) of f; all is witnessed by a correct initial segment τ of $\emptyset''$; and s is a stage number greater than s_0 and large enough to enumerate τ in the $\mathbf{0}'$-computable enumeration.

Now applying Lemma 4.2, we get a generic sequence G computable in $\mathbf{a}$ meeting all the dense sets D_n's. Let $(p_0, p_1, p_2, \ldots)$ be the sequence of $\mathcal{Q}$-conditions in $\bar{p} \in G$.

This gives an **a**-computable model $\mathcal{N}$ by projecting the elements of G onto their σ coordinates, i.e., $\mathcal{N} = \cup_i \sigma_i$. In addition, when we meet a dense set D_n, we guarantee that the partial isomorphism f is correct and preserved in the construction by any forcing extension, and we make sure that n is in the domain and range of f. So the union of these correct partial isomorphisms is an isomorphism from $\mathcal{N}$ to $\mathcal{M}$. This gives the desired contradiction. ⊣

As a final remark, recall that a degree $\mathbf{a} \geq \mathbf{0}'$ is $\overline{GL_2}$ over $\mathbf{0}'$ if $\mathbf{a}'' > (\mathbf{a} \vee \mathbf{0}'')'$. It is well-known that all degrees $\overline{GL_2}$ over $\mathbf{0}'$ are in the class $\mathcal{C}(\mathbf{0}')$, and restricted to the degrees in $[\mathbf{0}', \mathbf{0}'']$ they are exactly the degrees that are non-low$_2$ over $\mathbf{0}'$, i.e., $\mathbf{a}'' > \mathbf{0}'''$. Recall that if a degree $\mathbf{d}$ is low over $\mathbf{0}'$, then $\{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$ is a degree spectrum, so in the interval $[\mathbf{0}', \mathbf{0}'']$, we only have a very small gap left, namely the degrees that are low$_2$ but not low over $\mathbf{0}'$.

COROLLARY 4.3. *If $\mathbf{0}' \leq \mathbf{a} \leq \mathbf{0}''$ and $\mathbf{a}'' > \mathbf{0}'''$, then there is no countable structure $\mathcal{A}$ such that $\mathbf{Sp}(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \not\leq \mathbf{a}\}$.*

REMARK 4.4. It follows from Theorem 3.1 that every low$_2$ algebraic structure with the c.e. extension property has copies in the degrees $\mathbf{a}$ such that $\mathbf{a} \geq \mathbf{0}'$ and $\mathbf{a}' \geq \mathbf{0}'''$, i.e., in the degrees of functions dominating all $\mathbf{0}'$-computable functions. Moreover, by the proof of Theorem 4.1, every low$_2$ algebraic structure with the c.e. extension property has copies in the degrees $\mathbf{a} \in \mathcal{C}(\mathbf{0}')$, i.e., in the degrees $\mathbf{a} \geq \mathbf{0}'$ such that there is no $\mathbf{0}'$-computable function dominating all **a**-computable functions. Note also that by Corollary 2.2, not every low algebraic structure with the c.e. extension property has a copy in a c.e. degree $\mathbf{a} < \mathbf{0}'$, even if $\mathbf{a}' = \mathbf{0}''$.

REFERENCES

[1] MINZHONG CAI and RICHARD SHORE, *Domination, forcing, array nonrecursiveness and relative recursive enumerability*, this JOURNAL, vol. 77 (2012), pp. 33–48.
[2] DENIS R. HIRSCHFELDT, BAKHADYR KHOUSSAINOV, RICHARD A. SHORE, and ARKADII M. SLINKO, *Degree spectra and computable dimensions in algebraic structures*. **Annals of Pure and Applied Logic**, vol. 115 (2002), no. 1–3, pp. 71–113.
[3] ISKANDER SH. KALIMULLIN, *Spectra of degrees of some structures*. **Algebra Logika**, vol. 46 (2007), no. 6, pp. 729–744.
[4] ———, *Almost computably enumerable families of sets*. **Sbornik. Mathematics**, vol. 199 (2008), no. 10, pp. 1451–1458.
[5] ———, *Restrictions on the degree spectra of algebraic structures*. **Siberian Mathematical Journal**, vol. 49 (2008), no. 6, pp. 1034–1043.
[6] ISKANDER KALIMULLIN, BAKHADYR KHOUSSAINOV, and ALEXANDER MELNIKOV, *Limitwise monotonic sequences and degree spectra of structures*. **Proceedings of the American Mathematical Society**, vol. 141 (2013), pp. 3275–3289.

[7] JULIA F. KNIGHT, *Degrees coded in jumps of orderings*, this JOURNAL, vol. 51 (1986), no. 4, pp. 1034–1042.

[8] ROGERS, H. JR., **Theory of Recursive Functions and Effective Computability**, McGraw Hill, New York, 1968.

[9] KAREL DE LEEUW, EDWARD F. MOORE, CLAUDE E. SHANNON, and NORMAN SHAPIRO, *Computability by probabilistic machines*, **Automata Studies: Annals of Mathematics Studies**, pp. 183–212, no. 34, Princeton University Press, Princeton, N. J., 1956.

[10] RUSSELL MILLER, *The Δ_2^0-spectrum of a linear order*, this JOURNAL, vol. 66 (2001), no. 2, pp. 470–486.

[11] ROBERT W. ROBINSON, *Interpolation and embedding in the recursively enumerable degrees*. **Annals of Mathematics**, vol. 93 (1971), pp. 285–314.

[12] THEODORE A. SLAMAN, *Relative to any nonrecursive set*. **Proceedings of the American Mathematical Society**, vol. 126 (1998), no. 7, pp. 2117–2122.

[13] STEPHAN WEHNER, *Enumerations, countable structures and Turing degrees*. **Proceedings of the American Mathematical Society**, vol. 126 (1998), no. 7, pp. 2131–2139.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
480 LINCOLN DR.
MADISON, WI 53706, USA
E-mail: andrews@math.wisc.edu, lempp@math.wisc.edu, jmiller@math.wisc.edu
URL: www.math.wisc.edu/∼andrews
URL: www.math.wisc.edu/∼lempp
URL: www.math.wisc.edu/∼jmiller

DEPARTMENT OF MATHEMATICS
DARTMOUTH COLLEGE
HANOVER, NH 03755, USA
E-mail: mingzhong.cai@dartmouth.edu
URL: math.dartmouth.edu/∼cai

INSTITUTE OF MATHEMATICS AND MECHANICS
KAZAN FEDERAL UNIVERSITY
KREMLEVSKAYA ST. 18
420008 KAZAN, RUSSIA
E-mail: iskander.kalimullin@kpfu.ru
URL: kpfu.ru/main?p_id=10721

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA-BERKELEY
EVANS HALL #3840
BERKELEY, CA 94720, USA
E-mail: antonio@math.berkeley.edu
URL: www.math.berkeley.edu/∼antonio