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Optimization Problems in a Theory of Supercritical Fluid Extraction of Oil

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Abstract—In terms of shrinking core model we discuss optimization problems, which occur in supercritical fluid extraction of oil from polydisperse packed bed of ground plant material. We determine the optimal way of packing the particles into the extraction vessel and solve a problem of minimization of extraction time.

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Introduction. Supercritical fluid extraction (SFE) of oil from vegetable raw materials has recently become one of the basic technological processes in food, biofuel, and pharmaceutical industry. During SFE supercritical fluid (SF) (usually carbon dioxide or water) is pumped through a reactor unit filled with the crushed seeds of oil-bearing crops. Moderate temperatures, absence of harmful solvent residues (typical for traditional methods), and an easy way to separate the solvent from the extract [1] make SFE attractive for practical use. Various mathematical models [2] have been proposed to predict the extraction process. The so-called shrinking core (SC) model [3–7] can be considered physically as most feasible, and has been employed in computational simulations [4, 5]. At the same time, analytical approaches are also shown [5–7] to be useful in theoretical study of SFE processes. The paper generalizes these results in several aspects. A complete analytical solution to the SFE problem is obtained for the uniform distribution of polydisperse ground material in the extraction vessel (Item 2). In Item 3 we determine optimum packing of ground particles along the vessel with respect to their size. The problem of minimization of the full extraction time is solved in Item 4. Possible applications of the obtained theoretical results in SFE technology are discussed in conclusion (Item 5).

1. Mathematical model of the extraction process. Let us introduce the time t and spatial coordinate z varying from 0 to 1 along the vessel, from its inlet to outlet. Let $F(a)$ be the overall particle-size distribution function (ODF) of ground plant material with the distribution density $f(a)$. By definition, $dF = f\,da$ is the volumetric fraction of particles with dimensionless size from a to $a + da$. These functions depend on conditions and time of plant material milling. The model of the process is presented in terms of solute concentration $c(t, z)$ in the fluid phase and fraction $s(t, z, a)$ of oil extracted from particles of size a at the moment t in the cross-section z; $0 \le s, c \le 1$.

Master equations for c and s take the following dimensionless form [7, 8]

$$\frac{\partial c}{\partial z} = \frac{\partial}{\partial t}\int_0^\infty s\, f(a)\, da, \quad \frac{\partial s}{\partial t} = \frac{d(s)}{a^2}\,(1 - c)\,. \tag{1}$$

The first of them represents the mass balance of oil in the fluid phase outside particles, and the second one describes the oil mass transfer from particles of size a in the packed bed to the pore space. The driving force of this flux is the difference between the equilibrium oil concentration, assumed to be unity, and the current local concentration c in the fluid phase. Cumulative diffusion coefficient $d(s)$ depends

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on the particle shape of packed bed particles, and for the most important case of spherical (isometrical) particles it is

$$d(s) = \frac{0.5\,(1-s)^{1/3}}{1-(1-s)^{1/3}}.$$

For flat (1D) and cylindrical (2D) particles coefficient $d(s)$ is determined by

$$d_{1D}(s) = \frac{1}{2s}, \quad d_{2D}(s) = \frac{-1}{\ln(1-s)}.$$

The following initial and boundary conditions complete equations (1)

$$s\,(0,z,a) = 0, \quad c(t,0) = 0. \tag{2}$$

The so-called overall extraction curve (OEC) $Y(t)$ (the fraction of available oil extracted from the reactor at the moment t) is of the principal practical interest and is a focus of simulations. Along with OEC it is convenient to consider the zonal oil fraction $y(t,z)$ extracted from the packed bed interval $[0,z]$ at the time t

$$y\,(t,z) = \int_0^t c\,(t,z)\,dt.$$

Obviously, $Y(t) = y(t,1)$.

Integration of Eqs. (1) with respect to the time t and initial condition (2) directly leads to the problem for $y(t,z)$

$$\frac{\partial y}{\partial z} = \int_0^\infty s\,f(a)da, \tag{3}$$

$$t - y = a^2\varphi\,(s)\,. \tag{4}$$

The function $\varphi(s)$ is defined on the interval $0 < s < 1$ and is determined by the integral

$$\varphi\,(s) = \int_0^s \frac{ds}{d(s)}.$$

It increases monotonically from zero at $s = 0$ to unity at $s = 1$. The latter condition $\varphi(1) = 1$ can be always satisfied by appropriate normalization of particle size a.

Let S be the inverse function of φ continued as unity for $\varphi > 1$. Then, the substitution of the inverse relationship (4) into (3) reduces problem (1), (2) to solution of the ordinary differential equation for $y(t,z)$

$$\frac{\partial y}{\partial z} = \int_0^\infty S\left(\frac{t-y}{a^2}\right) f(a)da. \tag{5}$$

The boundary condition

$$y(t,0) = 0 \tag{6}$$

follows from (2). Here the time t is a parameter.

2. Problem solution. Let us introduce the function k

$$k(\tau) = \int_0^\infty S\left(\frac{\tau}{a^2}\right) f(a)da,$$

which monotonically increases from zero at $\tau = 0$ as

$$k(\tau) \sim \sqrt{\tau}, \quad \tau \to 0$$

and tends to unity at $\tau \to \infty$. Accordingly, Eq. (5) takes the form of the ordinary differential equation with separable variables

$$\frac{\partial y}{\partial z} = k(t-y). \tag{7}$$

The solution to this equation with boundary condition (6) consists of two branches: $y = t$ when $z > z_-(t)$, and $y < t$ when $z < z_-(t)$. Integration of (7) in the latter case with account of (6) gives

$$z < z_-(t) : z = \int_{t-y}^{t} \frac{d\tau}{k(\tau)}.$$

The boundary $z = z_-(t)$ which separates two regimes is defined by the solution continuity

$$z_-(t) = \int_0^t \frac{d\tau}{k(\tau)}.$$

For the bounded particle size ($a \le a_{\max}$) there exists another boundary $z = z_+(t)$ in the parametric (t, z)-plane which outlines the inlet part of the reactor with fully extracted particles

$$z < z_+(t) : y = z,$$

and

$$z_+ = \int_{t-z_+}^{t} \frac{d\tau}{k(\tau)}.$$

Due to properties of k, this equality is equivalent to $k(t - z_+) = 1$. And from the definition of k it follows that the minimum value of $t - z_+$ is $a_{\max}^2$, i.e.,

$$z_+ = t - a_{\max}^2.$$

Along with z_-, z_+ let us also introduce the duration t_- of the initial (linear) extraction stage and the time t_+ of the complete extraction determined as

$$z_-(t_-) = 1,\ z_+(t_+) = 1 \ \Leftrightarrow \ \int_0^{t_-} \frac{d\tau}{k(\tau)} = 1, \quad t_+ = 1 + a_{\max}^2.$$

In certain particular cases the above solution derived in quadratures can be rewritten in an explicit analytical form. As an example, let us consider a problem of oil extraction from a monodisperse packed bed of flat particles of half-thickness a_0 when $\varphi(s) = s^2$, $k(\tau) = \min\left(1,\ \tau^{1/2} a_0^{-1}\right)$, and distribution density is expressed by the Dirac delta-function $f(a) = \delta(a - a_0)$. Invariance of problem (5), (6) with respect to z-coordinate scaling allows in this case to express $y(t, z, a_0^2)$ via $Y(t; a_0^2)$ as $y(t, z, a_0^2) = zY(tz^{-1}, a_0^2 z^{-1})$. Hence, the complete solution is fully represented by the OEC $Y(t, a_0^2)$.

The duration of the initial (linear) extraction stage is

$$t_- = \begin{cases} 1 - a_0^2, & a_0^2 < 0.5; \\ 0.25 a_0^{-2}, & a_0^2 > 0.5. \end{cases}$$

Boundary curves $t = t_-(a_0)$, $t = t_+(a_0) = 1 + a_0^2$ along with the curve $t = t_0(a_0) = a_0^2$ divide the plane (t, a_0) (solid lines on Fig. 1) into four parts A, B, C, and D. $Y = t$ in the linear-extraction zone A; the packed bed is depleted, $Y = 1$, in domain B; the respective non-linear part of OEC is determined in zones C and D by

$$(t, a_0) \in C : Y(t) = \frac{\sqrt{t}}{a_0} - \frac{1}{4a_0^2}; \quad (t, a_0) \in D : Y(t) = t - \left(\frac{t - 1 + a_0^2}{2a_0}\right)^2.$$

It is easy to verify that $Y(t)$ and its first derivative are continuous across the domain boundaries for all $t \ge 0$.

Typical curves $y(t)$ at different z (numbers above the lines) are presented on the Fig. 2. The upper line ($z = 1$) is the OEC. All curves correspond to $a_0 = 1$. Markers on the dashed line $y = t$ indicate the end of initial stage for the packed bed of height z, and markers on the dashed line $y = t - a_0^2$ do the time of full extraction for the fixed cross-section z of the packed bed.

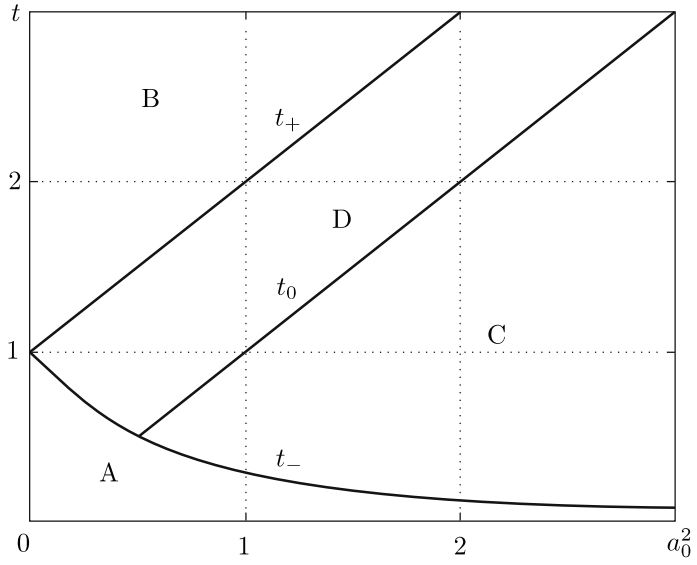

Fig. 1. A, B, C and D parts of the $(t; a_0)$-plane for monodisperse pack of flat particles.

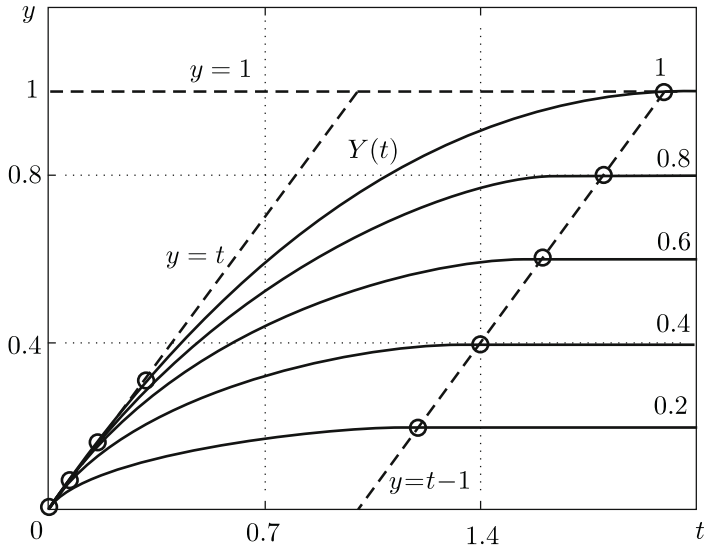

Fig. 2. Dependence of zonal oil fraction y of t at different z for the monodisperse packed bed of flat particles of size $a_0 = 1$.

3. Optimal particles packing along the vessel. Above results have been derived for the typical in practice uniform distribution of particles along the vessel, when ground material is packed uniformly, without being sorted in advance. The particle size distribution density $f(a)$ is same for every cross-section z. However, preliminary sorting, e.g., by sieving or using cyclones allows to obtain variable particle-size distribution densities along the extraction vessel, versus spatial coordinate z. Hereinafter this generalized density is designated as $\chi(z, a)$, and its variability is restricted by the mass conservation of particles of a fixed size a

$$f(a) = \int_0^1 \chi(z, a)dz. \tag{8}$$

Obviously, for the uniform particle distribution $\chi(z,a) = f(a)$, whereas to describe the general case, it is only necessary to replace $f(a)$ by $\chi(z,a)$ in the introduced SC model, in Eq. (5)

$$\frac{\partial y}{\partial z} = \int_0^\infty S\left(\frac{t-y}{a^2}\right)\chi(z,a)da, \tag{9}$$

leaving the same boundary condition (6) for y.

Among other properties of S-function, it is assumed that the related function

$$\phi(x) = \frac{x}{S(x)}\frac{dS(x)}{dx}$$

does not increase in the interval $0 < x < 1$. For spherical and cylindrical particles $\phi(x)$ monotonically decreases, while for flat particles it remains constant in the interval.

Now we can consider a problem of OEC maximization on the set of packing functions under constraint (8).

With this in mind, let us introduce a special locally monodisperse packing of the ground material when each cross-section z of the packed bed contains particles of only one size decreasing with z from the vessel inlet to its outlet. We call it locally monodisperse stratified (LMS) pack. The corresponding particle-volume distribution density along the vessel is defined by the Dirac delta-function

$$\chi(z,a) = \delta\left(a - a_s(z)\right),$$

where $a_s(z)$ is determined implicitly by the following equation:

$$z = 1 - F(a_s), \quad 0 < F(a_s) < 1. \tag{10}$$

Theorem 1. *For every t LMS pack maximizes extraction yield $Y(t)$.*

Proof. First, let us assume the function $\phi(x)$ to be monotonically decreasing, and introduce a discrete pack of particle sets with different size $a_1 < a_2 < \cdots < a_n$ and respective particle-volume distribution densities $p_1(z), p_2(z), \ldots, p_n(z)$ along the vessel. Every nonincreasing function can be considered as a limit of decreasing functions, and every continuous particle distribution can be closely approximated by a sequence of discrete packs. Consequently, in the general case of nonincreasing function $\phi(x)$ and continuous distribution $\chi(z,a)$, the proof of the theorem can be directly deduced from the described particular case.

For a discrete approximation of the distribution density, Eq. (8) takes the form of a differential equation (DE)

$$\frac{\partial y}{\partial z} = \sum_{k=1}^{n} S\left(\frac{t-y}{a_k^2}\right)p_k(z) \tag{11}$$

and the proof of the theorem is based on the fact that any local rearrangement of particle distribution by shifting a bigger-size fraction upflow, towards the vessel inlet, would lead to an increase of Y for every moment t. Corresponding statements are formulated as two following lemmas. □

Lemma 1 (local stratification)**.** *For any (defined above) basic discrete packing, let us introduce a new locally stratified packing, which satisfies condition* (8)*, coincides with the basic one outside a small interval $(z_0, z_0 + h)$ where $0 \le z_0 < 1$, and is given as*

$$\begin{gathered}\chi(z,a) = \delta(a - a_{n-k+1}), \quad z_{k-1} < z < z_k, \quad k = 1,2,\ldots,n,\\ z_k = z_{k-1} + h\overline{p}_{n-k+1}, \quad \overline{p}_k = h^{-1}\int_{z_0}^{z_0+h} p_k(z)dz,\end{gathered} \tag{12}$$

inside this interval. Then the new packing provides for not smaller values of $Y(t)$ for every moment of time.

Lemma 2 (swapping layers)**.** *Let the basic packing be defined in the h-interval at z_0 coordinate as*

$$\chi(z,a)=\begin{cases}\delta(a-a_1), & z_0<z<z_0+\xi h;\\ \delta(a-a_2), & z_0+\xi h<z<z_0+h,\end{cases}$$

where $0<\xi<1$, $a_1<a_2$.

Then for the new locally swapped packing characterized by

$$\chi(z,a)=\begin{cases}\delta(a-a_2), & z_0<z<z_0+(1-\xi)\,h;\\ \delta(a-a_1), & z_0+(1-\xi)\,h<z<z_0+h,\end{cases}$$

the amount of extracted oil at any moment of time is greater or equal to the one than that for the basic one.

According to Lemma 1 it is possible to switch from any discrete basic packing to the locally monodisperse pack and then, following Lemma 2, rearrange it to the LMS pack. Both steps do not decrease OEC for every moment t.

Since proofs of both lemmas are practically identical, only the proof of the first one is given below.

Proof of Lemma 1. The notation y is kept for the sought-for function related to the basic packing, and notation $\widetilde{y}$ is used for the modified packing (12). Subscripts "in" and "out" mean the values of any function at the inlet ($z=z_0$) and outlet ($z=z_0+h$) cross-sections of the considered interval $z_0<z<z_0+h$. Since both packs are identical at $z<z_0$, it follows $y_{\text{in}}=\widetilde{y}_{\text{in}}$. And if we show that $y_{\text{out}}\le\widetilde{y}_{\text{out}}$, then the lemma will be proved, because for both packings DEs (11) are also identical at $z>z_0+h$, and according to the comparison theorem [9] for DEs $y\le\widetilde{y}$ for $z\ge z_0+h$ (and for $z=1$) when $y_{\text{out}}\le\widetilde{y}_{\text{out}}$.

To deduce the power series expansion for y_{out} to the order of $O\left(h^3\right)$, Eq. (11) is presented in the discrete form

$$y_{\text{out}}-y_{\text{in}}=h\sum_{k=1}^{n}S\left(\frac{t-\overline{y}}{a_k^2}\right)\overline{p}_k+O\left(h^3\right),\quad \overline{y}=\frac{y_{\text{out}}+y_{\text{in}}}{2}. \tag{13}$$

Taking into account that

$$S\left(\frac{t-\overline{y}}{a_k^2}\right)=S_k-S_k'\frac{y_{\text{out}}-y_{\text{in}}}{2a_k^2}+O\left(h^2\right),\ \ S_k=S\left(\frac{t-y_{\text{in}}}{a_k^2}\right),\ \ S_k'=S'\left(\frac{t-y_{\text{in}}}{a_k^2}\right),$$

instead of Eq. (13), we obtain

$$y_{\text{out}}-y_{\text{in}}=h\sum_{k=1}^{n}S_k\overline{p}_k-\frac{h^2}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{S_iS_k'}{a_k^2}\overline{p}_i\overline{p}_k+O\left(h^3\right). \tag{14}$$

Next, $\widetilde{y}_{\text{out}}$ is calculated with the same accuracy level, as follows.

Repeating similar calculations for $\widetilde{y}_k=\widetilde{y}(z_k)$ in each interval $(z_0,z_1),(z_1,z_2),\dots$ we sequentially obtain

$$\widetilde{y}_1-y_{\text{in}}=hS_n\overline{p}_n-\frac{h^2}{2}\frac{S_nS_n'}{a_n^2}\overline{p}_n\overline{p}_n+O\left(h^3\right),$$

$$\widetilde{y}_2-y_{\text{in}}=h\sum_{k=n-1}^{n}S_k\overline{p}_k\left(1-\frac{h}{2}\frac{S_k'}{a_k^2}\overline{p}_k\right)-h^2\frac{S_nS_{n-1}'}{a_{n-1}^2}\overline{p}_n\overline{p}_{n-1}+O\left(h^3\right),\dots,$$

$$\widetilde{y}_{\text{out}}-y_{\text{in}}=h\sum_{k=1}^{n}S_k\overline{p}_k\left(1-\frac{h}{2}\frac{S_k'}{a_k^2}\overline{p}_k\right)-h^2\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\frac{S_iS_k'}{a_k^2}\overline{p}_k\overline{p}_i+O\left(h^3\right). \tag{15}$$

Finally, Eqs. (14) and (15) yield

$$\widetilde{y}_{\text{out}} - y_{\text{out}} = \frac{h^2}{2\,(t - y_{\text{in}})} \sum_{k=2}^{n} \sum_{i=1}^{k-1} S_i S_k \left(\phi(x_k) - \phi(x_i)\right) \overline{p}_i \overline{p}_k + O\left(h^3\right), \tag{16}$$

$$\phi(x) = \frac{xS'(x)}{S(x)}, \quad x_k = \frac{t - y_{\text{in}}}{a_k^2}.$$

Generally, when at least one of x_k is in the interval $(0, 1)$, the positiveness of $\widetilde{y}_{\text{out}} - y_{\text{out}}$ follows from the monotonic decrease of $\phi(x)$. Two limiting cases have to be considered separately. The first one is $y_{\text{in}} = t$, and original equations show that $\widetilde{y}(z) = y(z) = t$ for $z \geq z_0$.

The second case is $x_1 > x_2 > \cdots > x_n \geq 1$. Hence, functions $y(z)$ and $\widetilde{y}(z)$ differ from y_{in} by values of the $O\,(h)$-order in the interval $z_0 < z < z_0 + h$, and for sufficiently small h

$$S\left(\frac{t - y}{a_k^2}\right) = S\left(\frac{t - \widetilde{y}}{a_k^2}\right) = 1, \quad k < n. \tag{17}$$

Let us assume for simplicity $a_n = 1$, $z_0 = 0$. The function $w = y - z$ is introduced further, and a new spatial coordinate $\overline{z} = \overline{p}_n z$ is defined for the basic packing. Omitting the bar over $\overline{z}$ and taking into account (17) the following DE is derived for w:

$$0 < z < h_1 : \frac{\partial w}{\partial z} = g(w, z) = S\left(t - w - \overline{p_n}^{-1} z\right) - 1. \tag{18}$$

The function $\widetilde{w} = \widetilde{y} - z$ satisfies the following equation

$$0 < z < h_1 : \frac{\partial \widetilde{w}}{\partial z} = \widetilde{g}(\widetilde{w}, z) = S\left(t - \widetilde{w} - z\right) - 1 \tag{19}$$

in the interval $(0, h_1)$, and according to (17) is constant in the interval (h_1, h). Functions w and $\widetilde{w}$ determined by Eqs. (18) and (19) have the same initial values y_{in} at $z = 0$, and $\widetilde{g}(w, z) \geq g(w, z)$. According to the comparison theorem [9] for ordinary DE $\widetilde{w}(h_1) \geq w(h_1)$. Consequently, $y_{\text{out}} = w(h_1) + h$ is not greater than $\widetilde{y}_{\text{out}} = \widetilde{w}(h_1) + h$. □

Following similar considerations as in the case of Theorem 1, it shows that the LMS pack of particles is the optimal one among other possible distributions. Repeating the proof of the Theorem 1 one can show that the inverse LMS (when particle size monotonically increases with z) is the worst one.

4. Minimization of the full extraction time. Full extraction time t_+ is one of the most important characteristics of the SFE efficiency. The smaller it is, the more efficient is the extraction process. Obviously, for any overall particle distribution $F(a)$, the LMS packing minimizes t_+, and only such packed beds are studied further.

The full extraction time can be defined in two different ways. On one hand, it is the minimum time when function $Y(t) = y(t, 1)$ becomes a unity. On the other hand, t_+ is the maximum of the function $\tau(z)$ (the moment of full extraction of all particles at the cross-section z). Equation (4) is valid for every pack, and can be rewritten as

$$\tau(z) = y(\tau, z) + a_M^2(z),$$

where $a_M(z)$ is the maximum particle size at the cross-section z. Consequently,

$$t_+ = \max_{0 \leq z \leq 1} \left(y(\tau, z) + a_M^2(z)\right). \tag{20}$$

For $0 \leq y \leq 1$, Eq. (20) results in $a_{\max}^2 \leq t_+ \leq 1 + a_{\max}^2$, and, finally, gives

$$\max\left\{1; a_{\max}^2\right\} \leq t_+ \leq 1 + a_{\max}^2. \tag{21}$$

Universal inequalities (21) constrain the full extraction time for any overall distribution function $F(a)$ and different ways of packing. As it follows from (21) and Item 3, the uniform pack, $\chi(z, a) = f(a)$, for every $f(a)$ is the worst one with respect to the full extraction time. The same is true for the inverse LMS pack. The following theorem, which estimates the full extraction time for the LMS pack, shows that t_+ can take any value from the interval (21).

Theorem 2. *The full extraction time t_+ for the LMS pack is*

$$t_+ = \max_{a \le a_{\max}} \left(1 - F(a) + a^2\right).$$

Proof. Taking into account that $y(t,z) \le z$ and that for LMS pack $a_M(z) = a_s(z)$, and replacing z in (20) by (10), we have

$$t_+ \le \max_z \left(z + a_s^2(z)\right) = \max_a \left(1 - F(a) + a^2\right). \tag{22}$$

On the other hand, $y(t_+, z) \equiv z$, and Eq. (9) at $t = t_+$ gives

$$1 = S\left(\frac{t_+ - z}{a_s^2(z)}\right).$$

This means that the argument of S is greater or equal to unity for every z, and

$$t_+ \ge \max_z \left(z + a_s^2(z)\right) = \max_a \left(1 - F(a) + a^2\right). \tag{23}$$

Simultaneous inequalities (22), (23) prove Theorem 2. □

Theorem 2 shows that the only distribution which delivers maximum value (21) to t_+ is the monodisperse distribution $f(a) = \delta(a - a_{\max})$ with particle size $a = a_{\max}$. But the lower boundary (which is of interest) is attained on a wide set of distributions defined by the inequality $F(a; a_{\max}) \ge F_*(a, a_{\max})$. For $a_{\max} > 1$ the limiting distribution function is determined by the formula

$$a_{\max} > 1 : F_*(a; a_{\max}) = \max(0, 1 - a_{\max}^2 + a^2).$$

However, for $a_{\max} \le 1$ the limiting distribution function does not depend on $a_{\max}$ and is given as

$$a_{\max} \le 1 : F_*(a) = \min\left(1,\ a^2\right).$$

As mentioned above, particle distribution function depends on milling time and conditions. By definition, grinding 1 is more coarse (finer) than grinding 2 if the respective distribution functions are such that $F_1(a) \le F_2(a)$ ($F_1(a) \ge F_2(a)$), and the inequality becomes strict for at least one particle size a. In terms of the above definitions, Theorem 2 can be reformulated as two following statements.

Corollary 1. Minimum value of the full extraction time for LMS pack is $t_+ = a_{\max}^2$ at $a_{\max} > 1$. It is attained if the grinding is not more coarse than that for $F_*(a; a_{\max})$.

Corollary 2. Global minimum of the full extraction time for the LMS pack is $t_+ = 1$. It is attained if the grinding is not more coarse than that for $F_*(a)$.

It is of interest to note that for $F(a) = F_*(a; a_{\max})$ all particles in the vessel with LMS pack become fully extracted simultaneously at $t = t_+$.

Denoting the ratio of full extraction times for uniform and LMS packs as η, this value has been estimated to evaluate the superiority of the LMS pack in comparison with the uniform packing.

Corollary 3. The advantage η is always lesser or equal to two. The upper limit is attained if and only if $a_{\max} = 1$ and the corresponding grinding is not more coarse than $F_*(a)$.

It should be emphasized that the minimum extraction time $t_+ = 1$ and the maximum advantage $\eta = 2$ which results from the substitution of the LMS pack for the uniform pack is attained when the concentration of oil at the outlet cross-section of the vessel equals the equilibrium (set as unity here) value throughout the whole extraction process

5. Conclusions. The principal results can be directly used for improving SFE technology. They show that the normalized maximum particle radius $a_{\max}$ is the principal parameter which determines the full extraction time t_+. For high values of $a_{\max}$ this time only slightly depends on density $f(a)$ and type of particles packing, and it is on the order of $a_{\max}^2$.

The essential (by two times) decrease in t_+ can be attained only if the grinding is fine enough and $a_{\max} \le 1$. In order to minimize full extraction time in this case, the LMS pack must be used and the grinding should be not coarser than $F_*(a)$. In the case of $a_{\max}$ being less than unity further milling and additional reduction of $a_{\max}$ do not lead to decrease of the full extraction time t_+.

The practical realization of the LMS pack assumes ground particles fractionation before the extraction, i.e., by sieving or using cyclones. Obviously, it requires additional inputs, and the question of their payback should be studied in advance.

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