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Bifurcations and New Uniqueness Criteria for Critical Points of Hyperbolic Derivatives

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Abstract—Types of bifurcations of zeros for the gradient of a hyperbolic derivative of a holomorphic function on the unit disk are determined, provided that the derivative is embedded in the family of level lines of the given function. The character of the dependence of the motion of zeros on the curvature of the hyperbolic derivative is described, which makes it possible to extend the Poincaré—Hopf theorem so as to obtain a class of new uniqueness conditions for zeros in the form of nonnegativity conditions on curvature-type functionals. This class contains a one-parameter series of Epstein inequalities obtained from the Behnke—Peschl linear convexity condition for Hartogs domains of special form. A specific rigidity effect arises; namely, the inequalities mentioned above are meaningful only on a finite interval of parameters.

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INTRODUCTION

According to Riemann's theorem, a normalized conformal mapping F from a hyperbolic domain $D \subset \hat{\mathbb{C}}$ onto a disk E_R generates a surface (in $\mathbb{R}^3$) over D [1, p. 32]. This surface, $R = R_D(z)$, is characterized by that each of its level lines is the radius R of the target E_R, whose center is the F-image of the current point on the line. The value $R_D(z)$ is called the (inner) *conformal radius* of the domain D at the point z [2, p. 26].

By means of a biholomorphism $f : \mathbb{D} \to D$, this situation can be transferred to the space over the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, so that the conformal radius at the point $f(\omega)$ turns out to be equal to the value of the function

$$h_f(\zeta) = (1 - |\zeta|^2)\big|f'(\zeta)\big| \tag{1}$$

at the point $\zeta = \omega \in \mathbb{D}$ [2, p. 28], [3]). For a given domain D, the choice of such f is unique up to an automorphism of $\mathbb{D}$; thus, (1) give a complete information about the conformal radius of the domain D.

The study of quantity (1) with an *arbitrary* holomorphic function f on $\mathbb{D}$ puts correspondences $f \mapsto h_f$, which substitute correspondences $D \mapsto R_D$ for various classes of domains in the initial D-setting, in the forefront. In this f-approach, quantity (1) is called the *hyperbolic derivative*, or the *Bloch derivative*, of the function f (see, e.g., [4, 5]). As is known, the extrema of (1) "formalize obstructions" in studying the well-posedness of a number of problems in mathematical physics and function theory (see [6] and the bibliography in [7]). The difference between the representations $R = R_D(z)$ and $h = h_f(\zeta)$ (which are usually identified by means of $z = f(\zeta)$) plays an essential role in expressing their Gaussian curvatures in terms of $\zeta \in \mathbb{D}$; the nonnegativity of these curvatures leads to the conditions [8]

$$\big|\{f,\zeta\}\big| \le \big|-2/(1-|\zeta|^2)^2 + (1/2)|(f''/f')(\zeta) - 2\overline{\zeta}/(1-|\zeta|^2)|^2\big|, \quad \zeta \in \mathbb{D}, \tag{2}$$

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where $\{f,\zeta\} = (f''/f')'(\zeta) - (f''/f')^2(\zeta)/2$ is the Schwarz derivative of the function f at the point ζ, and $|\{f,\zeta\} + 2(\ln h_f(\zeta))_\zeta (f''/f')(\zeta)| \le |-2/(1-|\zeta|^2)^2 + 2|(\ln h_f(\zeta))_\zeta|^2|$ for $\zeta \in \mathbb{D}$. A similar "recalculation" for the logarithms $\ln R$ and $\ln h$ gives the inequalities

$$\left|\{f,\zeta\} - (1/2)((f''/f')(\zeta) - 2\overline{\zeta}/(1-|\zeta|^2))^2\right| \le 2/(1-|\zeta|^2)^2, \quad \zeta \in \mathbb{D}, \tag{3}$$

and $|(f''/f')'(\zeta) - 2\overline{\zeta}^2/(1-|\zeta|^2)^2| \le 2/(1-|\zeta|^2)^2$, $\zeta \in \mathbb{D}$.

A general approach to constructing similar conditions was outlined by Kinder in [9] in relation to the uniqueness problem for critical points of function (1) (see Section 1); in [8], case (2) was completely studied. In this paper, in the framework of this problem, we study conditions of the form

$$J(f,\zeta) \ge 0, \quad \zeta \in \mathbb{D}, \tag{4}$$

for holomorphic functions f on $\mathbb{D}$ and $J = G_\alpha, I_\beta$ and K_γ, where

$$\begin{aligned} G_\alpha(f,\zeta) + \left|\{f,\zeta\} + (3/2-\alpha)[(f''/f')^2(\zeta) - 4\overline{\zeta}^2/(1-|\zeta|^2)^2]\right| \\ = I_\beta(f,\zeta) + \left|\{f,\zeta\} + 2(2\beta-1)(\ln h_f(\zeta))_\zeta^2\right| \\ = K_\gamma(f,\zeta) + \left|(f''/f')'(\zeta) - \gamma\overline{\zeta}^2/(1-|\zeta|^2)^2\right| = 2/(1-|\zeta|^2)^2. \end{aligned}$$

One of the reasons why we take the class H of holomorphic functions on $\mathbb{D}$ for the domain of the functionals J is that condition (4) (unlike, e.g., (2)) is surely violated for meromorphic functions on $\mathbb{D}$ when J coincides with one of the functionals G_α (with $\alpha \ne 3/2$), I_β (with $\beta \ne 1/2$), and K_γ. In the case $J = G_{3/2} = I_{1/2}$, we obtain Nehari's well-known inequality

$$\left|\{f,\zeta\}\right| \le 2/(1-|\zeta|^2)^2, \quad \zeta \in \mathbb{D}, \tag{5}$$

which ensures the uniqueness of the critical point of (1) for meromorphic f (this point coincides with the pole of f). For establishing uniqueness in the case of holomorphic f in (5), a number of methods were suggested in [5, 10–13]. We mention two of them, namely, the method of radial superposition used in [10] and developed in [11] and the method of bifurcations of parametric families [13, 14], which is related to a version of the Poincaré–Hopf theorem constructed in [15, 16] for the vector field ∇h_f. In this paper, both methods are compared as applied to the proof of the (at most) uniqueness of the critical point of (1) for functions $f \in H$ satisfying the inequality $I_\beta(f,\zeta) \ge 0, \zeta \in \mathbb{D}$, in which the weak linear convexity condition on special Hartogs domains in $\mathbb{C}^2$ being a version of Epstein's condition [17] arises (see Section 3). The application of the bifurcation method is simplified at the expense of the a generalization of the above-mentioned version of the Poincaré–Hopf theorem (Theorem 1).

An accompanying question is whether condition (4) is well defined for $f \in H$. Let $H_0 = \{f \in H : f'(\zeta) \ne 0, \zeta \in \mathbb{D}\}$ be the class of holomorphic locally univalent functions on $\mathbb{D}$. It is easy to verify that, for $J = G_\alpha$, I_β, or K_γ, the fulfillment of condition (4) for a holomorphic function f on $\mathbb{D}$ not being identically constant implies $f \in H_0$. Thus, unless otherwise specified, we assume that all functions under consideration are locally univalent in $\mathbb{D}$. An important aspect of conditions of the form (4) being well defined is the question of whether such conditions are meaningful. Answering this question turns out to require the application (in the spirit of [18]) of Plesner's classical theorem; nevertheless, it is convenient to use the following definition.

A functional $J : H_0 \times \mathbb{D} \times \mathbb{R} \to \mathbb{R} : (f,\zeta,\omega) \mapsto J_\omega(f,\zeta)$ generating the family of classes $\mathcal{J}_\omega = \{f \in H_0 : J_\omega(f,\zeta) \ge 0, \zeta \in \mathbb{D}\}$ is said to be *rigid in the parameter* ω, or simply *rigid*, if the set $\Omega = \Omega(J) = \{\omega \in \mathbb{R} : \mathcal{J}_\omega \ne \emptyset\}$, which is called the *support* of the functional J, is an interval of $\mathbb{R}$.

The crucial role played by the Nehari functional

$$G_{3/2}(f,\zeta) = I_{1/2}(f,\zeta) = 2/(1-|\zeta|^2)^2 - \left|\{f,\zeta\}\right|$$

and the related inequality (5) in setting the problem of constructing functionals J with the property that $J \ge 0$ implies the uniqueness of the extremum of (1) can be clarified by, e.g., the following observation. At the elements of

$$M_f = \{a \in \mathbb{D} : (\partial h_f/\partial\zeta)(a) = 0\},$$

that is, the sets of critical points of (1) for $f \in H_0$, the Nehari functional coincides both with the functionals G_α, I_β, and K_γ (no matter what the parameter values are) and with the functionals generating inequality (2) and (3) and their analogues. The effectiveness of the setting mentioned above is related to the extension of the set of situations in which the following "metatheorem" is valid; already during the work on papers [11] and [9], this assertion proved a guiding conjecture [19] rather than an ordinary assumption.

M.I. Kinder's conjecture. *Let* $J : H_0 \times \mathbb{D} \to \mathbb{R}$ *be a functional with the property*

$$\mathrm{sign} J(f, a) = \mathrm{sign} I_{1/2}(f, a), a \in M_f, \quad f \in H_0, \tag{6}$$

and let the class $\mathcal{J} = \{f \in H_0 : J(f, \zeta) \geq 0, \zeta \in \mathbb{D}\}$ *be nonempty. Then* $f \in \mathcal{J}$ *and* $k_f < \infty$ *imply* $k_f \leq 1$.

Here, $k_f = \# M_f$ is the number of elements in M_f. The case $k_f = \infty$ can occur in only two situations, when M_f contains analytic arcs (with endpoints in $\partial\mathbb{D}$ for $f \in H_0$) [20] and when M_f is discrete and has a limit point in $\partial\mathbb{D}$ [21].

1. VERIFICATION OF THE CONJECTURE

The special role played by the functional $I_{1/2}$ in the setting under consideration is based on that this functional, as well as K_2, is a sign-defining factor in the expression for the curvature $K_f(\zeta)$ of the function $\ln h_f(\zeta)$ at the elements of M_f:

$$K_f(a) = \left[2/(1 - |a|^2)^2 + \left|\{f, a\}\right|\right] I_{1/2}(f, a), \quad a \in M_f. \tag{7}$$

We also mention the relation $K_f(\zeta) = [1 + |F|^2]^{-2} J_f(\zeta)$ for $\zeta \in \mathbb{D}$, where $J_f(\zeta) = (|F_\zeta| + |F_{\overline{\zeta}}|) K_2(f, \zeta)$ is the Jacobian of the vector field $\nabla \ln h_f(\zeta) \simeq \overline{F} = 2(\ln h_f)_{\overline{\zeta}}$ and $F = F(\zeta, \overline{\zeta}) = f''(\zeta)/f'(\zeta) - 2\overline{\zeta}/(1 - \zeta\overline{\zeta})$ is the *Gakhov mapping* with zero set M_f [6].

A more delicate characteristic of the surface $h = h_f(\zeta)$ in a neighborhood of the isolated elements of M_f is the index

$$\gamma_f(a) = -(2\pi i)^{-1} \int\limits_{|\zeta - a| = \rho} d\ln(h_f)_{\overline{\zeta}}$$

of a point $a \in M_f$ singular for the vector field $\nabla h_f(\zeta)$; $M_f \bigcap \{|\zeta - a| \leq \rho\} = \{a\}$. As is known [15, 16], $\gamma_f(M_f) \subset \{\pm 1, 0\}$. Setting $m_f^\varepsilon = \#\{a \in M_f : \gamma_f(a) = \varepsilon 1\}$, we obtain $k_f = m_f^+ + m_f^0 + m_f^-$. A united classification of the isolated elements M_f ($f \in H_0$) is as follows (see, e.g., [6]).

Proposition 1. *On the discrete part of* M_f, *the relations* $\gamma_f(\mathrm{sgn} K_f = \pm 1) = \mathrm{sgn} K_f$ *and* $\gamma_f(\mathrm{sgn} K_f = 0) \subseteq \{-1, 0 + 1\}$ *hold; moreover,* $\mathrm{sgn} K_{f\circ\phi}(a) = \mathrm{sgn} K_f(\phi(a))$ *and* $\gamma_{f\circ\phi}(a) = \gamma_f(\phi(a))$, *where* $a \in M_f$ *and* ϕ *is an automorphism of* $\mathbb{D}$. *The surface* $h = h_f(\zeta)$ *over the elements* $M_f \ni a \simeq (\mathrm{sgn} K_f(a), \gamma_f(a))$ *admits the following structure:* $(+1, +1)$ *is an elliptic maximum (a umbilic if* $\{f, 0\} = 0$); $(0, +1)$ *is a parabolic maximum;* $(0, 0)$ *is a parabolic half-saddle;* $(0, -1)$ *is a parabolic saddle; and* $(-1, -1)$ *is a hyperbolic saddle. All of these cases are realizable.*

As the point of departure in studying the conjecture stated above the following version [15, 16] of the classical Poincaré–Hopf theorem ([22, p. 223]) can be considered, in which $\mathcal{B}_0 = \{f \in H : \lim_{\zeta \to \partial\mathbb{D}} h_f(\zeta) = 0\}$ is the small Bloch class.

Proposition 2. *If* $f \in \mathcal{B}_0 \bigcap H_0$ *and* $k_f < \infty$, *then* $\sum_{a \in M_f} \gamma_f(a) = m_f^+ - m_f^- = 1$.

Remark 1. The example of a function $f \in \mathcal{B}_0 \bigcap H_0$ with countable M_f constructed in [21] shows that the second condition $k_f < \infty$ in this proposition cannot be removed at the expense of the first condition.

To prove the conjecture in a special case, in [9], the class $\mathcal{F}$ of functions $f \in H$, reconstructible from the representations $\ln f'(\zeta) = (1/2\pi) \int_0^{2\pi} p(\theta)(e^{i\theta} + \zeta)/(e^{i\theta} - \zeta) d\theta$ with $p \in C[0, 2\pi]$ was introduced. The following assertion is valid.

Proposition 3. *Let* $J : H_0 \times \mathbb{D} \to \mathbb{R}$ *be a functional with property (6), and let* $f \in \mathcal{F}$ *be a function satisfying the strict inequality (4), that is, such that*

$$J(f, \zeta) > 0, \quad \zeta \in \mathbb{D}. \tag{8}$$

Then $k_f = 1$.

The relation $f \in \mathcal{F}$ ensures the simultaneous fulfillment of both conditions in Proposition 2. The application of this proposition with taking into account (6) (as well as (7) and Proposition 1) proves Proposition 3: $m_f^- = m_f^0 = 0$ and $k_f = m_f^+ = 1$.

As an illustration of this assertion, in [9], strict versions of the following inequalities with $\zeta \in \mathbb{D}$ were suggested:

(A) $G_0(f,\zeta) \geq 0$;

(B) $L_{3/2}(f,\zeta) \geq 0$, where $L_\delta(f,\zeta) = 1/(1-|\zeta|^2) - |\zeta|\big|(f'/f'')(\zeta)\{f,\zeta\} + 2\delta(\ln h_f(\zeta))_\zeta\big|$;

(C) $|\zeta|\big|(f'''/f'')(\zeta) - 3/2(f''/f')(\zeta)\big| \leq 1/(1-|\zeta|^2)$;

(D) $|\zeta|^2\big|2(f'/f'')'(\zeta)+1\big| \leq 1$.

The following proposition is valid.

Proposition 4. *The functionals* $G : (f,\zeta,\alpha) \mapsto G_\alpha(f,\zeta)$ *and* $L : (f,\zeta,\delta) \mapsto L_\delta(f,\zeta)$ *are rigid with respect to theirs parameters with supports* $|2\alpha - 3| \leq 1$ *and* $|\delta| \leq 1/2$*, respectively. In particular, inequalities* (A) *and* (B) *are not meaningful (in the class* H*).*

Proof. According to Plesner's classical theorem [23], there exists a point on $\partial\mathbb{D}$ and a sequence of elements of $\mathbb{D}$ converging to this point such that the corresponding sequence of values of the holomorphic function $(f''/f')' + (1-\alpha)(f''/f')^2$ has a *finite* limit. Performing the corresponding passage to the limit in the inequality $(1-|\zeta|^2)^2 G_\alpha(f,\zeta) \geq 0$ ($\zeta \in \mathbb{D}$), we obtain $|2\alpha - 3| \leq 1$. The rigidity of the functional L is proved in a similar way. □

The functionals corresponding to (C) and (D) are defined on $H \setminus \{a\zeta + b : a, b \in \mathbb{C}\}$; in case (C), the condition $f \in H_0$ holds automatically, and in case (G), it is imposed additionally; then, in both cases, $\zeta = 0$ cannot be a umbilic. The fulfillment of conditions (C) and (D) is obvious for, e.g., fractional-linear f (both inequalities are strict) and for functions of the form $f(\zeta) = a + bf_s(\varepsilon\zeta)$, where $f_s(\zeta) = (1/2)\ln((1+\zeta)/(1-\zeta))$ ($f(\mathbb{D})$ is a strip), $a, b \in \mathbb{C}$, and $|\varepsilon| = 1$; in (C), the equality is attained on a diameter of $\mathbb{D}$, and in (D), everywhere in $\mathbb{D}$. If $0 \in M_f$, then the strict inequality (C) can hold only if $0 < |\zeta| < 1$: at the point $\zeta = 0$, the left-hand side of (C) equals $\big|\{f,\zeta\}\big|/\big|(f''(\zeta)/f'(\zeta))/\zeta\big|_{\zeta=0} = 1$.

If $f''(0) \neq 0$, then case (D) can be simplified by applying Schwarz' lemma to the inequality $|\zeta^2 u'(\zeta)/u^2(\zeta)| \leq 1$, where $\zeta \in \mathbb{D}$, which is equivalent to (D), because

$$f''(\zeta)/f'(\zeta) = 2u(\zeta)/(1-\zeta u(\zeta)). \tag{9}$$

Estimate (D) turns out to be strict; the special cases $u'/u^2 \equiv -1$, $u'/u^2 \equiv 1$, and $u'/u^2 \equiv 0$ are exemplified by, respectively, $f(\zeta) = e^{\tau\zeta} \in \mathcal{F}$ ($\tau \in \mathbb{R} \setminus \{0\}$) with $k_f = 1$,

$$f(\zeta) = \ln(1/(1-\zeta)) \in \mathcal{B} \setminus \mathcal{B}_0, \quad \text{and} \quad f(\zeta) = 1/(1-\zeta) \notin \mathcal{B}, \tag{10}$$

where $\mathcal{B} = \{f \in H : \sup_{\zeta \in \mathbb{D}} h_f(\zeta) < \infty\}$ is the Bloch class. In the two last cases, we have $k_f = 0$. Thus, the strictness of (D) alone does not ensure the presence of critical points for the function (1) in $\mathbb{D}$.

When $f''(0) = 0$, a new effect arises; namely, the point $0 \in M_f$ is essentially nonelliptic: the condition $|\{f,0\}| \geq 2$ coincides with (D) for $\zeta = 0$. The parabolic case $|\{f,0\}| = 2$ is exhausted by a family outside $\mathcal{F}$ containing f_s, which can be written explicitly. An example of a function $f \in \mathcal{F}$ with $0 \in M_f$ and strict estimate (D) is obtained from (9) for at $1/u(\zeta) = 1/(\alpha\zeta) + \phi(\zeta)$, where $\alpha > 2$, $\phi'(\zeta) = (1-1/\alpha^2)(1-\zeta^2/\alpha)^{-1}$, and $\phi(0) = 0$. Here,

$$f''(\zeta)/f'(\zeta) = 2\alpha\zeta/(\varphi(\zeta) + \psi(\zeta))$$

with

$$\varphi(\zeta) = 1 - \zeta^2/\alpha, \quad \psi(\zeta) = (1-1/\alpha^2)\zeta\int_0^\zeta t^2(1-t^2/\alpha)^{-1}dt.$$

It is easy to show that $\big|\varphi\big|_{\partial\mathbb{D}} \geq 1 - 1/\alpha > (1+1/\alpha)/3 \geq \big|\psi\big|_{\partial\mathbb{D}}$ at $\alpha > 2$; the absence of poles for f''/f' in $\overline{\mathbb{D}}$ is now established by using Rouché's theorem.

As a result, we obtain the following sharpening of the corollary to Theorem 5 from [9].

Proposition 5. *For $f \in \mathcal{F}$, the strict estimate (C) gives $0 \notin M_f$ and $k_f = 1$, and the strict estimate (D) gives $k_f = 3$ if $0 \in M_f$ and $k_f = 1$ if $0 \notin M_f$.*

Proof. In both cases, if $0 \notin M_f$, then we can apply at once Proposition 3. Suppose that, for $f \in \mathcal{F}$, the strict estimate (D) holds and $0 \in M_f$. As mentioned above, in this case, we have $|\{f, 0\}| > 2$, that is, $K_f(0) < 0$. By virtue of Proposition 1, $\gamma_f(0) = -1$. We have $m_f^- = 1$ and $m_f^0 = 0$, because for $a \in M_f \setminus \{0\}$, the strict inequality (D) is precisely the strict inequality (5), that is, $K_f(a) > 0$, and hence, $\gamma_f(a) = +1$. Proposition 2 implies $m_f^+ = m_f^- + 1 = 2$ and $k_f = m_f^+ + m_f^- = 3$. □

Remark 2. Change (9) allowed S.R. Nasyrov to simplify the proof of (5) in the class S^0 of normalized convex functions on $\mathbb{D}$ (see [11, 24]) and F.G. Avkhadiev, who revealed this property of S^0 in [25], to completely describe the class of functions satisfying condition (2) [8].

2. GENERALIZATION OF THE POINCARÉ–HOPF THEOREM

In [26, p. 117], the following version of the Poincaré–Hopf theorem for the unit disk is given.

Lemma 1. *Suppose that a vector field continuous on $\overline{\mathbb{D}}$ and continuously differentiable on $\mathbb{D}$ is directed outside $\mathbb{D}$ at all points of $\partial\mathbb{D}$ and vanishes on a set $M \subset \mathbb{D}$. If the Jacobian of the field is positive on M, then M is a singleton.*

For the vector field $\nabla \ln h_f$ (considered in this paper) with $f \in H_0$, Lemma 1 can be strengthened by removing the boundary condition. Namely, the following generalization of Proposition 3 is valid.

Theorem 1. *If $f \in H_0$, M_f is nonempty, and $\gamma_f(M_f) = +1$, then $k_f = 1$.*

We need the following result of [13] about bifurcations of the elements of the sets $M_r := M_{f_r}$ for the family $f_r(\zeta) = f(r\zeta)$ of "level lines" of a function $f \in H_0$, in which every f_r is defined for $|\zeta| < 1/r$, where $r \in (0, +\infty)$. We set $\mathfrak{R} = \mathfrak{R}_f = \bigcup_{r \in (0,+\infty)} M_r \times \{r\}$, $K_r = K_{f_r}$, $\gamma_r = \gamma_{f_r}$, $g(\zeta) = \zeta f''(\zeta)/f'(\zeta)$, and $g_r(\zeta) = g(r\zeta)$.

Lemma 2. *Suppose that $f \in H_0$ and α is an isolated element M_ρ with $0 < \rho \le 1$ being a zero of multiplicity k for the function $g_\rho(\zeta) - g_\rho(\alpha)$.*

(1) If $\alpha \ne 0$ or $\alpha = 0$ and $K_\rho(\alpha) = 0$, then the foliation $\mathfrak{R}$ near the point (α, ρ) consists of k ($k = 2$ for $\alpha = 0$) analytic curves intersecting in this point. Moreover, the index $\gamma : (a, r) \mapsto \gamma_r(a)$ does not vanish on $\mathfrak{R} \setminus \{(\alpha, \rho)\}$ near (α, ρ), and the number $k_r = \#\{M_r \bigcap$ (sufficiently small neighborhood of α)$\}$ equals k for all $r \ne \rho$ close to ρ or has a jump of 2 at ρ.

(2) The relation $K_\rho(\alpha) \ne 0$ is stable with respect to the "perturbations" f_r of the function f_ρ at r close to ρ: $K_r(a_r) \ne 0$, where a_r is the (unique) element of M_r such that $a_\rho = \alpha$ ($a_r = 0$ for $\alpha = 0$). If $\alpha \ne 0$, then, as r increases near ρ, the absolute value $|a_r|$ increases (if $K_\rho(a_\rho) > 0$) or decreases (if $K_\rho(a_\rho) < 0$).

(3) Let $K_\rho(\alpha) = 0$. For $k = 1$ (except in the "stable" situation where $k_r = 1$ for r close to ρ, in which $|\gamma_\rho(\alpha)| = 1$), the birth or annihilation of one maximum and one saddle at (α, ρ) may occur provided that $\gamma_\rho(\alpha) = 0$. If $\alpha = 0$, then $0 \in M_f$ for all $r \in (0, +\infty)$ with $\gamma_r(0) = \operatorname{sgn}(\rho - r)$, $r \ne \rho$, and $k_r = 2 + \gamma_\rho(\alpha)\operatorname{sgn}(r - \rho)$ for $r \ne \rho$ near ρ.

In what follows, we denote birth by the symbol $\cup$; the symbol Ψ is used when a maximum (for $r \le \rho$) decomposes into two maxima and a saddle (for $r > \rho$). We set $\mathfrak{R}(I) = \bigcup_{r \in I} M_r \times \{r\}$, where $I \subseteq (0, 1]$ and consider the functional $\overline{r} = \overline{r}_f = \sup\{\xi \in (0, 1] : r \in (0, \xi] \Longrightarrow k_{f_r} = 1\}$ of the first exit from the set $\mathcal{H} = \{h \in H_0 : k_h = 1\}$ along the level lines of the function f. The quantity $\overline{r}$ is bounded away from zero by the convexity radius of the function f; therefore, $\overline{r} > 0$.

Let $R = \{r \in (0, 1) : 0 \in K_r(M_r)\}$. It is easy to show by using Lemma 2 that the set R is at most countable and can have at most one limit point ($r = 1$). Next, $\mathfrak{R}(0, \overline{r})$ is a simple C^ω-curve admitting the parameterization $(a(r), r)$, where $r \in (0, \overline{r})$, in which $\zeta = a(r)$ is a continuous function such that $\lim_{r \to 0+} a(r) = 0$, $\gamma_r(a(r)) = +1$, and its absolute value and argument are (piecewise if $(0, \overline{r}) \bigcap R \ne \emptyset$) real analytic. Moreover, either $a(r) \equiv 0$ or $|a(r)|$ increases with respect to the parameter r.

If $\overline{r} < 1$, then $M_{\overline{r}}$ consists of the point $a(\overline{r}) = \lim_{r \to \overline{r}-} a(r)$ with $\gamma_{\overline{r}}(a(\overline{r})) = +1$ and at most finitely many points with index zero. The latter points, if exist, give bifurcations of type $\cup$. The point $a(\overline{r})$ can generate a bifurcation only of type Ψ (this always occurs if $M_{\overline{r}} = \{a(\overline{r})\}$). In the case $f''(0) = 0$, for $\rho \geq \overline{r}$, we define the additional set $\mathfrak{R}'_\rho = \mathfrak{R}[\overline{r}, \rho] \setminus (\{0\} \times [\overline{r}, \rho])$ and the quantity $\mu_\rho = \inf_{(a,r)\in\mathfrak{R}'_\rho} |a|$.

Lemma 3. If $f \in H_0$, $0 \in M_f$, $\gamma_f(0) = +1$, and $\overline{r} < 1$, then the function $\mu : \rho \mapsto \mu_\rho$ is right continuous and decreases on $[\overline{r}, 1]$. If a_ρ is an element of M_ρ with $|a_\rho| = \mu_\rho$ and $\overline{r} \leq \rho \leq 1$, then $\gamma_\rho(a_\rho) = -1$ for $\rho \notin R$ and $\gamma_\rho(a_\rho) \neq +1$ for $\rho \in R$.

Proof. It is sufficient to verify the nonemptiness of $\mathfrak{R}'_\rho$ for $\rho \in [\overline{r}, 1]$ for $\rho = \overline{r}$. According to Lemma 2 ($\alpha = 0$), the conditions $\gamma_f(0) = +1$ and $\overline{r} < 1$ imply that $M_{\overline{r}} \times \{\overline{r}\}$ has points of type $\cup$, that it, $M_{\overline{r}} \setminus \{0\} \neq \emptyset$. Therefore, μ is well defined and $0 \leq \mu_\rho < 1$ for $\rho \in [\overline{r}, 1]$. The relation $\mu_\rho = 0$ would mean that $\mathfrak{R}'_\rho$ has a limit point in $\{0\} \times [\overline{r}, 1]$. By Lemma 2, this point must be $(0, 1)$, which is impossible, because $\gamma_f(0) = +1$. Next, take $(a_n, r_n) \in \mathfrak{R}'_\rho$ such that $|a_n| \to \mu_\rho$. The convergence of the subsequences $a_{n'} \to a_\rho$ and $r_{n'} \to r_\rho (\in [\overline{r}, \rho])$ implies $|a_\rho| = \mu_\rho \in (0, 1)$, whence $a_\rho \in \mathbb{D} \setminus \{0\}$. Finally, the relation $(a_\rho, r_\rho) \in \mathfrak{R}[\overline{r}, \rho]$ follows from the continuity of the Gakhov mapping, and $a_\rho \neq 0$ implies $(a_\rho, r_\rho) \in \mathfrak{R}'_\rho$, that is, $a_\rho \in M_{r_\rho} \setminus \{0\}$ with $r_\rho \leq \rho$.

Let us show that a_ρ is an element of $M_\rho \setminus \{0\}$ with $\rho \in [\overline{r}, 1]$. Suppose that $r_\rho < \rho$. Then, by Lemma 2, there are two possibilities for some neighborhood $U \times V \subset \mathbb{D} \times [\overline{r}, \rho)$ of the point (a_ρ, r_ρ): $1 \in \gamma_r(M_r \bigcap U)$ for all $r < r_\rho$ from V or $-1 \in \gamma_r(M_r \bigcap U)$ for all $r > r_\rho$ from V. In each of these cases, there exists a branch of a C^ω-curve from $\mathfrak{R}$ of the form $(a(r), r)$ with $a(r_\rho) = a_\rho$, where r ranges over the corresponding half-neighborhood $V \bigcap \{r \gtrless r_\rho\}$. For $r \neq r_\rho$, the derivative $d|a(r)|/dr$ has sign $\operatorname{sgn} K_r(a(r)) = \gamma_r(a(r)) = \operatorname{sgn}(r_\rho - r)$ (see Lemma 2 in Section 2 and Proposition 1); therefore, in the half-neighborhood mentioned above, the inequality $|a(r)| < \mu_\rho$ must hold, which contradicts the definition of μ_ρ. Thus, $r_\rho = \rho$ and $a_\rho \in M_\rho \setminus \{0\}$. Moreover, $|a_\rho| = \min\{|a| : a \in M_\rho \setminus \{0\}\}, \rho \in [\overline{r}, 1]$ by the definition of the quantity μ_ρ.

It follows from these considerations that $\gamma_\rho(a_\rho) \neq +1$ and the function μ is monotone. Indeed, by virtue of Lemma 2, if $\gamma_\rho(a_\rho) = +1$, then one of the possibilities specified above ($1 \in \gamma_r(M_r \bigcap U)$ for all $r < r_\rho = \rho$ from V) is realized. For $\rho \notin R$, the obtained inequality is sharpened as $\gamma_\rho(a_\rho) = -1$, because in this case, $0 \notin \gamma_\rho(M_\rho)$. Now, let us prove the implication $r < \rho \Rightarrow \mu_r > \mu_\rho$. It was established above that if $a \in M_r \setminus \{0\}$ for $r \leq \rho$ and $|a| = \mu_\rho$, then $r = \rho$. This means that $a \in M_r \setminus \{0\}$ with $r < \rho$ implies $|a| > \mu_\rho$, whence $\mu_r = |a_r| > \mu_\rho$. Since the function μ decreases, it is right continuous at the points of $[\overline{r}, 1) \bigcap R$. We use Lemma 2 and the continuity of μ outside R as the lower envelope of a finite family of C^ω-functions on each of the intervals forming $[\overline{r}, 1) \setminus R$. □

Proof of Theorem 1. By virtue of Proposition 1, we can assume that $0 \in M_f$. Suppose that $k_f > 1$. If $\overline{r} = 1$, then the elements of the set $M_f \setminus \{0\}$ generate bifurcations of type $\cup$, that is, $\gamma_f(M_f \setminus \{0\}) = 0$. If $\overline{r} < 1$, then, according to Lemma 3, the index of the element of $M_f \setminus \{0\}$ nearest to zero is different from $+1$. In both cases, we arrive at a contradiction, which proves the theorem.

Lemma 2 makes it also possible to prove the star-shapedness of the class $\mathcal{H} \bigcap \{f \in H_0 : f''(0) = 0\}$ along the level lines. Namely, is the following corollary is valid.

Corollary 1. *If* $f \in H_0$, $M_f = \{0\}$, *and* $\gamma_f(0) = +1$, *then* $k_{f_r} = 1$ *for* $r \in (0, 1)$.

Proof. Suppose that there exists an $\widetilde{r} \in (0, 1)$ such that $k_{f_{\widetilde{r}}} > 1$. Then $\overline{r} \leq \widetilde{r} < 1$; therefore, by Lemma 3, for any $\rho \in [\overline{r}, 1]$, the set $M_\rho \setminus \{0\}$ is nonempty, which cannot be for $\rho = 1$. □

The following example demonstrates that the condition $f''(0) = 0$ in Corollary 1 is essential.

Example 1. Consider the level lines of a function $f (\in H_0)$ with $f''(\zeta)/f'(\zeta) = (1/2)/(1 - \zeta)^2$. A simple routine analysis shows that the foliation $\mathfrak{R}_f$ over $(0, 1]$ consists of only one C^ω-curve $(\rho, r(\rho))$, where $\rho \in (0, 1]$. In addition to the end extrema 0 and 1, the function $r = r(\rho)$ has two interior extrema, namely, the maximum $r_m = \sqrt{3}/2$ at the point $\rho = 1/\sqrt{3}$ and the minimum $\overline{r} = (2/3)\sqrt{5/3}$ at $\rho = \sqrt{3/5}$. Take any $r_0 \in (r_m, 1)$. We have $k_{f_{r_0}} = 1$, but $k_{f_{tr_0}} \geq 2$ for $\overline{r}/r_0 \leq t \leq r_m/r_0$.

Theorem 1 fully confirms the strict version of Kinder's conjecture. Namely, the following assertion is valid.

Corollary 2. *Let $J : H_0 \times \mathbb{D} \to \mathbb{R}$ be a functional with property (6), and let $f \in H_0$ be a function satisfying inequality (8). Then $k_f \le 1$.*

Proof. If $M_f \ne \emptyset$, then inequality (8) and property (6), together with (7) and Proposition 1, ensure the discreteness of M_f [27, p. 209] and the fulfillment of the condition $\gamma_f(M_f) = +1$. By Theorem 1, this implies $k_f = 1$. □

Corollary 3. *Let $J : H_0 \times \mathbb{D} \to \mathbb{R}$ be a functional with the following properties:*

(1) $f \in \mathcal{J} \Rightarrow I_{1/2}(f_r, a) > 0$, where $a \in M_{f_r}$ and $r \in (r_0, 1)$, for some $r_0 = r_0(f) \in [0, 1)$;

(2) $f \in \mathcal{J} \Rightarrow f \circ \phi \in \mathcal{J}$ for each automorphism ϕ of the disk $\mathbb{D}$.

Then, for any function $f \in \mathcal{J}$, either $k_f \le 1$ or M_f contains a continuum.

Proof. Suppose that M_f is nonempty ($k_f \ge 1$) and discrete (otherwise, according to [20], M_f contains a continuum). Let us show that $k_f = 1$.

Note at once that, by virtue of Proposition 1 and Theorem 1, property (1) can be extended as $f \in \mathcal{J} \Rightarrow k_{f_r} = 1$ for $r \in (r_0(f), 1)$.

Take any $a \in M_f$. If $K_f(a) > 0$, then $\gamma_f(a) = +1$. Suppose that $K_f(a) = 0$. Consider the function $\widetilde{f} = f \circ \phi$ ($\in \mathcal{J}$ by virtue of condition (2) in this corollary), where ϕ is an automorphism of $\mathbb{D}$ such that $\phi(0) = a$. We have $0 \in M_{\widetilde{f}}$ and $K_{\widetilde{f}}(0) = 0$ by proposition 1; moreover, it follows from the above considerations that $k_{\widetilde{f}_r} = 1$ for $r \in (r_0(\widetilde{f}), 1)$. According to Lemma 2, this implies that $\zeta = 0$ is a bifurcation point of type Ψ (for $r = 1$) in the foliation $\mathfrak{R}$, and hence (by Proposition 1), $\gamma_f(a) = \gamma_{\widetilde{f}}(0) = +1$.

Thus, $\gamma_f(M_f) = +1$, which implies $k_f = 1$ by Theorem 1. □

Remark 3. The result obtained above remains valid when implications (1) and (2) are replaced by the conditions (1′) $J(f, a) \ge 0\,(> 0) \Rightarrow I_{1/2}(f, a) \ge 0\,(> 0)$ for $a \in M_f$ and $f \in H_0$; (1″) $J(f_r, M_{f_r}) > 0$ for $r \in (r_0, 1)$ and $f \in \mathcal{J}$; and (2′) $J(f, a) = 0 \Rightarrow g'(a) = 0$ for $a \in M_f$ and $f \in \mathcal{J}$ ($g = \zeta f''/f'$). It can be shown that the functionals G_α and K_γ satisfy (1″) with $r_0 = 0$ (provided that the parameters values are in the supports) and even the more restrictive condition $J(f_r, \zeta) > r^2 J(f, r\zeta)$ for $r \in (0, 1)$ and $\zeta \in \mathbb{D}$ (which is equivalent to Ahlfors' inequality [17] in the case $J = G_1 = K_2$). Corollary 3 and its analogue thus obtained generalize the constructions of uniqueness conditions of the form (4) given in [13] and [28] for the functionals $J = I_{1/2}$ and $J = 2/(1 - |\zeta|^2) - |(f''/f')'(\zeta)|$, respectively, and distinguish the situation in which Kinder's conjecture is confirmed under modified condition (6). Obviously, in any case, the validity of this conjecture is related to the elimination of elements $a \in M_f$ of index zero.

3. LINEAR CONVEXITY OF HARTOGS DOMAINS AND THE EPSTEIN INEQUALITY

Consider the class $\mathcal{N}(\beta)$ ($\beta \in \mathbb{R}$) of normalized holomorphic functions f satisfying the condition $I_\beta(f, \zeta) \ge 0$ for $\zeta \in \mathbb{D}$, or, in more detail,

$$\left|\{f, \zeta\} + (\beta - 1/2)((f''/f')(\zeta) - 2\overline{\zeta}/(1 - |\zeta|^2))^2\right| \le 2/(1 - |\zeta|^2)^2, \quad \zeta \in \mathbb{D}. \tag{11}$$

Suppose that D is a hyperbolic Riemann surface and $f : \mathbb{D} \to D$ is its holomorphic parameterization by the unit disk. The Hartogs domain over D is defined as $H = \{(z, w) \in D \times \mathbb{C} : |w| < \Omega(z)\}$, where the function $\Omega \in C^2(D)$ is positive and satisfies the inequality $(\ln\Omega)_{z\overline{z}} < 0$ in D (that is, H is strictly pseudoconvex). For the defining function for H we take $r(z, w) = \ln|w| - \ln\Omega(z)$. One of the versions of the definition of linear convexity, which goes back to the classical work [29], means as applied to H that the real Hessian $\mathrm{Hess}r(z, w)(\lambda, \mu)$ of the function r is nonnegative at any point $(z, w) \in r^{-1}(0) \bigcap (D \times \mathbb{C})$ and any vector (λ, μ) from the complex tangent plane $T^{\mathbb{C}}_{(z,w)}(\partial H)$.

We have $(1/2)\mathrm{Hess}r(z, w)(\lambda, \mu) = -(\ln\Omega)_{z\overline{z}}|\lambda|^2 - \mathrm{Re}\{\mu^2/2w^2 + (\ln\Omega)_{zz}\lambda^2\}$ and $T^{\mathbb{C}}_{(z,w)}(\partial H) = \{(\lambda, \mu) \in \mathbb{C}^2 : \mu/w = 2(\ln\Omega)_z\lambda\}$ (see, e.g., [30]). Thus, according to the version mentioned above, we have $\mathrm{Re}\{[(\ln\Omega)_{zz} + 2(\ln\Omega)_z^2]\lambda^2\} \le -(\ln\Omega)_{z\overline{z}}|\lambda|^2$, $\zeta \in \mathbb{D}$, $\lambda \in \mathbb{C}$, or, equivalently,

$$\left|(\ln\Omega)_{zz} + 2(\ln\Omega)_z^2\right| \le -(\ln\Omega)_{z\overline{z}}, \quad \zeta \in \mathbb{D}. \tag{12}$$

The change $\Omega = \sqrt{R/e^s}$ (where $R = R(z)$ denotes conformal radius) followed by the passage to the unit disk $z = f(\zeta)$, $\sigma(\zeta) := s(f(\zeta))$ transforms estimate (12) into the Epstein inequality [17]

$$\left|\sigma_{\zeta\zeta} - \sigma_\zeta^2 - \{f,\zeta\}/2 - (2\overline{\zeta}/(1-|\zeta|^2))\sigma_\zeta\right| \le \sigma_{\zeta\overline{\zeta}} + 1/(1-|\zeta|^2)^2, \quad \zeta \in \mathbb{D}. \tag{13}$$

As is known [17, 31], if $|\sigma_\zeta| \le |\zeta|/(1-|\zeta|^2)$, where $r_0 \le |\zeta| < 1$ for some $r_0 \in (0,1)$, then, under condition (13), the function f is univalent in $\mathbb{D}$. This suggests that if the Hartogs domain H is linearly convex over D (in the sense of a suitable version of the definition), then any holomorphic covering of the Riemann surface (the Riemann domain over $\mathbb{C}^n$ in the case $n \ge 2$) D is simple.

We return to condition (12) in the special case $\Omega = R^\beta$; passing to $\mathbb{D}$, we obtain precisely inequality (11).

Theorem 2. *If $\beta \in [0,1]$, then $\mathcal{N}(\beta)$ is a linearly invariant family of order ord$\mathcal{N}(\beta) \le (1-\beta)^{-1/2}$ containing the class S^0 of convex functions. The classes $\mathcal{N}(\beta)$ are empty for $\beta \notin [0,1]$.*

Proof. The linear invariance of the classes $\mathcal{N}(\beta)$ is verified directly. Using the actions

$$\Lambda_{\phi_\zeta} f(z) = (f(\phi_\zeta(z)) - f(\phi_\zeta(0)))/(\phi_\zeta'(0)f'(\phi_\zeta(0))) = z + \sum_{n=2}^{\infty} A_n(f,\zeta)z^n$$

on the functions $f \in \mathcal{N}(\beta)$ by the Möbius automorphisms $\phi_\zeta(z) = (z+\zeta)/(1+\overline{\zeta}z)$ (see [32]), we can rewrite (11) in terms of the coefficients $A_2(f,\zeta)$ and $A_3(f,\zeta)$ with taking into account the relations $A_3(f,\zeta) - A_2^2(f,\zeta) = (1/6)(1-|\zeta|^2)^2\{f,\zeta\}$ and $A_2(f,\zeta) = -\overline{\zeta} + ((1-|\zeta|^2)/2)(f''/f')(\zeta)$ as follows:

$$\left|3A_3(f,\zeta) + 2(\beta-2)A_2^2(f,\zeta)\right| \le 1, \quad \zeta \in \mathbb{D}. \tag{14}$$

Under the passage from $f \in \mathcal{N}(\beta)$ to the function $f_r^\varepsilon(\zeta) = \overline{\varepsilon}f(\varepsilon r\zeta)/r$, where $r \in (0,1)$ and $|\varepsilon| = 1$, this inequality transforms into the more complicated inequality

$$\left|3A_3(f_r^\varepsilon,\zeta) - 3A_2^2(f_r^\varepsilon,\zeta) + (2\beta-1)[A_2(f_r^\varepsilon,\zeta) + \overline{\zeta}\gamma_r(\zeta)]^2\right| \le r^2(1-|\zeta|^2\gamma_r(\zeta))^2, \tag{15}$$

where $\gamma_r(\zeta) = (1-r^2)/(1-r^2|\zeta|^2)$.

Let us show that the order ord$f = \sup_{\zeta \in \mathbb{D}}|A_2(f,\zeta)|$ of any function $f \in \mathcal{N}(\beta)$ is finite for any $\beta \ne 1$ with nonempty $\mathcal{N}(\beta)$. Take any $\beta \in \mathbb{R}$ and suppose that there exists an $f \in \mathcal{N}(\beta)$ such that $\alpha = $ ord$f = +\infty$. In this case, as well as for finite α [33], for the dilations $f_r(\zeta) = f(r\zeta)/r$, the passage to the limit $\alpha_r :=$ ord$f_r \to \alpha$ as $r \to 1-$ can be performed. It is easy to show that $A_2(f^\varepsilon, \overline{\varepsilon}\zeta) = \varepsilon A_2(f,\zeta)$ for $\zeta \in \mathbb{D}$ and $|\varepsilon| = 1$, where $f^\varepsilon(\zeta) = \overline{\varepsilon}f(\varepsilon\zeta)$. This allows us to realify the second coefficient, and the condition $\alpha_r > 1$, which holds for $r < 1$ close to 1, ensures the existence of points $\zeta_r \in \mathbb{D}$ and $\varepsilon_r \in \partial\mathbb{D}$ such that $A_2(f_r^{\varepsilon_r}, \zeta_r) = \alpha_r (< +\infty)$. By virtue of Theorem 2.3 a from [32], we have $A_3(f_r^{\varepsilon_r}, \zeta_r) = (2\alpha_r^2+1)/3$. Substituting the obtained expressions for the coefficients in (15) with $\varepsilon = \varepsilon_r$ and $\zeta = \zeta_r$, we see that $|1 - \alpha_r^2 + (2\beta-1)[\alpha_r + \overline{\zeta}_r\gamma_r(\zeta_r)]^2| \le r^2(1-|\zeta_r|^2\gamma_r(\zeta_r))^2$ for all $r < 1$ close to 1. Dividing both sides by α_r^2, passing to the limit as $r \to 1-$, and taking into account the boundedness of $\gamma_r(\zeta)$ in $\mathbb{D}$ and the convergence $\alpha_r \to +\infty$, we obtain $\beta = 1$. Thus, for any $\beta \ne 1$, $f \in \mathcal{N}(\beta)$ implies ord$f < +\infty$.

Now, let us determine β for which the classes $\mathcal{N}(\beta)$ are nonempty. Suppose that a class $\mathcal{N}(\beta)$ with $\beta \ne 1$ is nonempty. Let us show that, in this case, β belongs to the interval $[0,1)$. Note that the nonemptiness of $\mathcal{N}(\beta)$ for $\beta \in [0,1]$ follows from the inclusion $S^0 \subset \mathcal{N}(\beta)$ for $\beta \in [0,1]$, which was proved on the basis of the well-known estimate $|A_3 - A_2| \le (1-|A_2|^2)/3$ for the coefficients $A_2 = A_2(f,\zeta)$ and $A_3 = A_3(f,\zeta)$ with $f \in S^0$ in [25].

Thus, suppose that $\mathcal{N}(\beta)$ with $\beta \ne 1$ contains a function f with ord$f = \alpha < +\infty$. The compactness principle for the sequence $\{f_n = \Lambda_{\phi_{\zeta_n}} f : n \in \mathbb{N}\}$, for which $|A_2(f,\zeta_n)| \to \alpha$ ($\zeta_n \in \mathbb{D}$) as $n \to \infty$ and $|A_2(f_n,\zeta)| \le \alpha$ for $\zeta \in \mathbb{D}$, and the linear invariance of the class $\mathcal{N}(\beta)$ ensure the existence of a function $g(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots \in \mathcal{N}(\beta) \bigcap \mathfrak{A}_\alpha$ such that $a_2 = \alpha$, where $\mathfrak{A}_\alpha$ is the universal linearly invariant family of order α [32]. Applying the same Theorem 2.3 a from [32] to $a_2 = \alpha = A_2(g,0)$ and $a_3 = A_3(g,0)$ and substituting the resulting relation $a_3 = (2\alpha^2+1)/3$ into (14) with g instead of f and with $\zeta = 0$, we obtain the inequality $|2(\beta-1)\alpha^2 + 1| \le 1$, that is, $0 \le (1-\beta)\alpha^2 \le 1$, which immediately implies the estimate $\beta < 1$ (recall that $\beta \neq 1$); taking into account the inequality $\alpha \geq 1$ (see [32]), we obtain the estimate $\beta \geq 0$.

Moreover, if $f \in \mathcal{N}(\beta) \bigcap \mathfrak{S}_\alpha$ for $\beta \in [0,1)$, where $\mathfrak{S}_\alpha = \{h \in \mathfrak{A}_\alpha : \text{ord} h = \alpha\}$, then $\alpha = \text{ord} f \leq (1-\beta)^{-1/2}$, that is, the intersection $\mathcal{N}(\beta) \bigcap \mathfrak{S}_\alpha$ is empty for $\beta \in [0,1)$ and $\alpha > (1-\beta)^{-1/2}$. As shown above, $\mathcal{N}(\beta)$ with $\beta \in [0,1)$ contains no functions of infinite order; it follows that $\text{ord}\mathcal{N}(\beta) \leq (1-\beta)^{-1/2}$, $\beta \in [0,1]$. This completes the proof of Theorem 2. □

Remark 4. The relation $\mathcal{N}(0) = S^0$, which was obtained as a byproduct in the proof given above, was proved by a different method in [34].

Corollary 4. The functional $I : (f, \zeta, \beta) \mapsto I_\beta(f, \zeta)$ is rigid with respect to the parameter β with support $[0,1]$.

The following example shows that the equality in the estimate $\text{ord}\mathcal{N}(\beta) \leq (1-\beta)^{-1/2}$ is partially attained.

Example 2. The function $f_q(\zeta) = [((1+\zeta)/(1-\zeta))^q - 1]/(2q)$, where $q \geq 1$, with $\text{ord} f_q = q$ belongs to the class $\mathcal{N}(\beta)$ with $q \leq (1-\beta)^{-1/2}$ if $\beta \in [0, 2/3]$ and with $q \leq \sqrt{H(\beta)}$ if $\beta \in [2/3, 1]$, where $H(\beta) = \beta/(8\beta^2 - 11\beta + 4)$. These bounds cannot be unimproved.

Theorem 3. *If $\beta \in [0,1]$ and $f \in \mathcal{N}(\beta)$, then either $k_f \leq 1$ or $f(\mathbb{D})$ is a strip.*

Proof. The case $k_f = 0$ is meaningful: functions (10) belong to $\mathcal{N}(\beta)$ for any $\beta \in [0,1]$. Suppose that $k_f \geq 1$ and consider the following cases (cf. [13]).

1. *M_f is discrete in $\mathbb{D}$.*

Substituting $\zeta = ra$ with $a \in M_{f_r}$ into condition (11), we obtain $\big|\{f_r, a\} + 2(2\beta-1)\gamma_r(a)^2\overline{a}^2/(1-|a|^2)^2\big| \leq 2r^2/(1-r^2|a|^2)^2$, whence $I_{1/2}(f_r, a) > 0$ for $r \in (0,1)$ and $\beta \in [0,1]$; it remains to apply Corollary 3.

2. *M_f contains its limit points.*

As is known [20], any limit point M_f in $\mathbb{D}$ is contained in an analytic arc $\{\zeta = \zeta(t), t \in T \subset \mathbb{R}\} \subset M_f$. Thus, $(\ln h_f)_\zeta\big|_{\zeta=\zeta(t)} \equiv 0$ for $t \in T$, or $(f''/f')(\zeta(t)) \equiv 2\overline{\zeta}(t)/(1-|\zeta(t)|^2)$ for $t \in T$, whence

$$\{f, \zeta(t)\}\zeta'(t) \equiv 2\overline{\zeta}'(t)/(1-|\zeta(t)|^2)^2, \quad t \in T. \tag{16}$$

Substituting the two last identities into (11), we see that the function $\varkappa_f(\zeta) = (1-|\zeta|^2)^2\big|\{f,\zeta\} + 2(2\beta-1)(\ln h_f)^2_\zeta\big|$ attains its maximum, which equals 2, at the points $\zeta = \zeta(t)$, $t \in T$, which satisfy the equation $(\ln \varkappa_f)_\zeta = 0$:

$$\{f,\zeta\}'/\{f,\zeta\}\big|_{\zeta=\zeta(t)} \equiv 4\overline{\zeta}(t)/(1-|\zeta(t)|^2), \quad t \in T. \tag{17}$$

Without loss of generality, we assume that $\zeta(t_0) = \overline{\zeta}(t_0) = 0$ for some $t_0 \in T$; since the arc is analytic, we have $\zeta'(t_0) \neq 0$.

To integrate system (16), (17) with given initial data, we pass to complexifications of the real analytic functions $\zeta(t)$ and $\overline{\zeta}(t)$ on T, that is, to holomorphic functions $u(\tau)$ and $u(\tau)$ on the strip $T \subset \mathfrak{T} \subset \mathbb{C}$ such that $u_{|T} = \zeta$ and $v_{|T} = \overline{\zeta}$. The conditions $u(t_0) = v(t_0) = 0$ and $u'(t_0) \neq 0$ allow us to pass to the superposition $w(u) := v(\tau(u))$, which is holomorphic in some neighborhood of the point $u = 0$ and satisfies the condition $w(0) = 0$. The complexification of identities (16) and (17) in terms of $w = w(u)$ yields, respectively,

$$\{f,u\} = 2w'(u)/(1-uw(u))^2 \quad \text{and} \quad \{f,u\}'/\{f,u\} = 4w(u)/(1-uw(u)). \tag{18}$$

The former relation gives $|w'(0)| = (1/2)\big|(1-uw(u))^2\{f,u\}\big|_{|u=0} = 1$ (because $\varkappa_f(\zeta(t)) = 2$); thus, we can assume without loss of generality that $w'(0) = 1$.

Identities (18) imply $w''(u)/w'(u) = 2(w(u) - uw'(u))/(1-uw(u))$. Therefore, $w''(0) = 0$ and $\{w,u\} = 0$, and taking into account the relations $w(0) = 0$ and $w'(0) = 1$, we obtain $w(u) = u$, which implies that $f(\mathbb{D})$ is a strip. This completes the proof of Theorem 3. □

An easy modification of the method used in the first proof of the implication (5) $\Rightarrow k_f \le 1$ or $f(\mathbb{D})$ is a strip given in [10, 11] proves the following theorem.

Theorem 4. *Suppose that $a > 0$ and $f \in H_0$ satisfies the conditions $f''(0) = 0$ and*

$$\mathrm{Re}\big(e^{i2\theta}\{f,\zeta\}\big) \le 2/(1-r^2)^2 + 4(a-1)\big[\mathrm{Re}e^{i\theta}(\ln h_f(\zeta))_\zeta\big]^2 + 2\big|(\ln h_f(\zeta))_\zeta\big|^2 \tag{19}$$

for $\zeta = re^{i\theta} \in \mathbb{D}$. If $f(\mathbb{D})$ is not a strip, then

$$\mathrm{Re}e^{i\theta}(f''/f')(\zeta) < 2r/(1-r^2), \quad \zeta \in \mathbb{D}. \tag{20}$$

Proof. Consider the functions $g(t,\theta) = f(r(t)e^{i\theta})$ and $u(t,\theta) = |g'_t(t,\theta)|^{-a}$, where $r = r(t)$ is the inverse to $t = f_s(r)$ (in [10], the dependence on θ was not used). For $a > 0$, condition (19) is equivalent to the inequality

$$a^{-1}u_{tt}/u \equiv -\mathrm{Re}\{g,t\} + (a-1/2)\big(\mathrm{Re}(g_{tt}/g_t)\big)^2 + (1/2)\big(\mathrm{Im}(g_{tt}/g_t)\big)^2 \ge 0, \tag{21}$$

where $(t,\theta) \in [0,+\infty) \times \mathbb{R}$. Together with $f''(0) = 0$, this inequality implies that the function u does not decrease in t for any fixed θ. In terms of g, this means that

$$\mathrm{Re}g_{tt}/g_t \le 0, \quad (t,\theta) \in [0,+\infty) \times \mathbb{R}, \tag{22}$$

which is the nonstrict inequality (20). The equality in (22) at some point (t_0,θ_0) with $t_0 \ne 0$ extends over the interval $[0,t_0] \times \{\theta_0\}$ (thanks to (21)), in which we have $u \equiv \equiv c(= |f'(0)|^{-a})$. By virtue of the relation $u_\theta/u = (ar/(1-r^2))\mathrm{Im}g_{tt}/g_t$, the assumption of the existence of $\overline{t} \in (0,t_0]$ for which $\mathrm{Im}(g_{tt}/g_t)(\overline{t},\theta_0) \ne 0$ implies the inequality $u(\overline{t},\theta) < c(= u(0,\theta))$ for those θ which are “adjacent” to θ_0 from one side. Obviously, this contradicts the function u being nondecreasing (in t), which was established above. Thus, the equality in (22) for $(t,\theta) = (t_0,\theta_0)$ implies the identity $g_{tt}/g_t \equiv 0$ on $[0,t_0] \times \{\theta_0\}$, and, by the uniqueness theorem, $f(\mathbb{D})$ is a strip. □

Theorem 3 can be proved by using the following assertion.

Corollary 5. Suppose that $\beta \in (-\infty,1]$, $f \in H_0$, $f''(0) = 0$, and

$$\mathrm{Re}\big\{e^{i2\theta}\{f,\zeta\} + (\beta-1/2)(e^{i\theta}(f''/f')(\zeta) - 2r/(1-r^2))^2\big\} \le 2/(1-r^2)^2 \tag{23}$$

for $\zeta = re^{i\theta} \in \mathbb{D}$. Then the assertion of Theorem 4 holds.

Proof. For $\beta \le 1$, estimate (23) implies (19) with $a \ge 1-\beta$. □

The proof of Theorem 3 is simple: the condition $f \in \mathcal{N}(\beta)$, where $\beta \in [0,1]$, ensures the fulfillment of inequality (23), estimate (20) implies the equality $k_f = 1$, and the relation $f''(0) = 0$ follows from the linear invariance of the class $\mathcal{N}(\beta)$.

Note that this approach does not work for the condition $G_\alpha(f,\zeta) \ge 0$, where $\zeta \in \mathbb{D}$ ($|2\alpha - 3| \le 1$), except in the case $G_{3/2} = I_{1/2}$. At present, we can only assert that the above inequality implies $k_{f_r} = 1$ for $r \in (0,1)$ and that Corollary 2 holds for $J = G_\alpha$.

4. THE INEQUALITY $K_\gamma(f,\zeta) \ge 0$ FOR $\zeta \in \mathbb{D}$

Lemma 4. Suppose that a real-valued function $\Omega \in C^2(\mathbb{D})$ satisfies the condition

$$\big|\Omega_{\zeta\zeta}(\zeta)\big| \le -\Omega_{\zeta\overline{\zeta}}(\zeta), \quad \zeta \in \mathbb{D}. \tag{24}$$

If Ω_ζ vanishes at two different points $\zeta_0, \zeta_1 \in \mathbb{D}$, then $\Omega_\zeta = 0$ on the rectilinear segment joining ζ_0 and ζ_1.

Proof. Setting $\Phi(\rho,\theta) = \Omega(\zeta_0 + \rho e^{i\theta})$, we obtain $\Phi_\rho - i\Phi_\theta/\rho = 2e^{i\theta}\Omega_\zeta$ and $\Phi_{\rho\rho} - i(\Phi_\theta/\rho)_\rho = 2[e^{i2\theta}\Omega_{\zeta\zeta} + \Omega_{\zeta\overline{\zeta}}]$. Let $\zeta_1 = \zeta_0 + \rho_1 e^{i\theta_1}$. Then the equality of the values Ω_ζ at the points ζ_0 and ζ_1 implies $\Phi_\rho(\rho_1,\theta_1) = \Phi_\rho(0,\theta_1)$, which implies the vanishing of the integral over the interval $T = \{\zeta_0 + \tau e^{i\theta_1} : \tau \in [0,\rho_1]\}$ of the nonpositive function $\mathrm{Re}e^{i2\theta_1}\Omega_{\zeta\zeta} + \Omega_{\zeta\overline{\zeta}}$; thus, this function vanishes on this interval. Therefore, by virtue of (24), we have $\mathrm{Re}e^{i2\theta_1}\Omega_{\zeta\zeta} = \big|e^{i2\theta_1}\Omega_{\zeta\zeta}\big| = -\Omega_{\zeta\overline{\zeta}}$; therefore, $\mathrm{Im}e^{i2\theta_1}\Omega_{\zeta\zeta} = 0$ on T.

Integrating the resulting identity $e^{i2\theta_1}\Omega_{\zeta\zeta}+\Omega_{\zeta\overline{\zeta}}=0$ (with $\zeta\in T$) with respect to τ from 0 to $\rho\in[0,\rho_1]$, we arrive at the required conclusion $\Omega_\zeta=0$ for $\zeta\in T$. □

The following theorem is valid.

Theorem 5. *If a function $f\in H$ satisfies the condition*

$$\left|(1-|\zeta|^2)^2(f''/f')'(\zeta)-\gamma\overline{\zeta}^2\right|\le 2,\quad \zeta\in\mathbb{D}, \tag{25}$$

where $-2\le\gamma\le 2$, then $k_f\le 1$. For $|\gamma|>2$, conditions (25) and $f\in H$ are inconsistent.

Proof. The rigidity of the functional K_γ with respect to the parameter γ with support $[-2,2]$ is proved in the same way as in Proposition 4.

Let $\Omega=\ln h_f$. Then condition (25) with $\zeta=\rho e^{i\theta}(\in\mathbb{D})$ acquires the form $\left|2e^{i2\theta}\Omega_{\zeta\zeta}+(2-\gamma)\rho^2/(1-\rho^2)^2\right|\le -2\Omega_{\zeta\overline{\zeta}}$, whence $\mathrm{Re}\, e^{i2\theta}\Omega_{\zeta\zeta}+\Omega_{\zeta\overline{\zeta}}\le 0$ on $\mathbb{D}$, provided that $\gamma\le 2$.

Suppose that $\Omega_\zeta=0$ at points $\zeta_0,\zeta_1\in\mathbb{D}$ ($\zeta_0\ne\zeta_1$). As in Lemma 4, for $\zeta\in T$, the latter inequality turns out to be an identity; substituting it into the former, we obtain the estimate $\left|-2\Omega_{\zeta\overline{\zeta}}+2i\mathrm{Im}\, e^{i2\theta_1}\Omega_{\zeta\zeta}+(2-\gamma)\rho^2/(1-\rho^2)^2\right|\le\le -2\Omega_{\zeta\overline{\zeta}}$ for $\zeta\in T$, which, surely, does not hold for $\gamma<2$. Thus, $k_f\le 1$ if $\gamma\in[-2,2)$.

Suppose that $\gamma=2$. Then (25) coincides with (24), and, according to Lemma 4, we have $\Omega_\zeta=0$ on T. This means that $(f''/f')(\zeta(\tau))\equiv 2\overline{\zeta}(\tau)/(1-|\zeta(\tau)|^2)$, where $\zeta(\tau)=\zeta_0+\tau e^{i\theta_1}$, $\tau\in[0,\rho_1]$, is a parametric representation of the interval T.

The analytic continuation of the last identity with respect to τ into the disk $\zeta^{-1}(\mathbb{D})$, which extends the interval T of the critical points of the function h_f to the chord $S=\zeta(\zeta^{-1}(\mathbb{D})\bigcap\mathbb{R})$, yields an explicit form of the pre-Schwarzian f''/f'. We can be assume that, up to a rotation in the ζ plane, $\zeta_0=ih$ and $S=\{ih+\tau:\tau\in(-\sqrt{1-h^2},\sqrt{1-h^2})\}$ ($h\in(-1,1)$); thus, $(f''/f')(\zeta)=2(\zeta-ih)/(1-\zeta(\zeta-ih))$. A cumbersome analysis shows that, for any function $f(\zeta)$ with pre-Schwarzian of such a form (and, therefore, for all rotations $\varepsilon^{-1}f(\varepsilon\zeta)$, where $|\varepsilon|=1$, of this function), inequality (25) is violated at points of $\mathbb{D}$ close to $(\sqrt{1-h^2}+ih)(\overline{\varepsilon})\in\partial\mathbb{D}$. Thus, functions f for which $k_f>1$ do not belong to the class determined by condition (25), as required. □

Part of the results of this paper were announced in [35, 36].

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