# Local triviality of equivariant algebras

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This paper was motivated by the observation that in the presence of some additional structure for an algebra $A$ over a commutative ring $R$ one can prove that the reductions of $A$ modulo two maximal ideals of $R$ are isomorphic. In [5, 9.1] and [6, 11.8] this fact is explained by means of an analytic argument. Suppose that $B$ is a smooth bundle of algebras over a differentiable manifold $X$ of class $C^\infty$. Let $\varphi_t$ be the local flow of a smooth vector field $D$ on $X$. We interpret $D$ as a derivation of the ring $C^\infty(X)$ of smooth functions on $X$. Suppose that there exists a $D$-compatible derivation $\mathcal{D}$ of the algebra of global sections $\Gamma(X,B)$. Then $\mathcal{D}$ gives rise to a vector field on $B$, and the corresponding local flow $\psi_t$ induces an algebra isomorphism $B_x \cong B_{\varphi_t(x)}$ between the fibers of $B$ for all $t$ in a suitable neighbourhood of 0 in $\mathbb{R}$ where $x \in X$ is some fixed point. In the list of open problems [2, III.10.1, Q.4] Brown and Goodearl asked whether a purely algebraic approach to results of this kind can be found.

We will assume in the paper that the ring $R$ itself is an algebra over a field $k$. In general the term "algebra" will be used in the universal sense. More precisely, by an $R$-algebra we mean an $R$-module equipped with any indexed set of $R$-multilinear operations. An $R$-algebra $A$ will be called *finite* if its underlying $R$-module is finitely generated. Denote by $\operatorname{Der}_k R$ and $\operatorname{Der}_k A$ the Lie algebras of $k$-linear derivations of $R$ and $A$. Given $D \in \operatorname{Der}_k R$, we say that $\mathcal{D} \in \operatorname{Der}_k A$ is *$D$-compatible* if

$$\mathcal{D}(ca) = D(c)a + c\mathcal{D}(a) \quad \text{for all } c \in R,\ a \in A.$$

Let $L \subset \operatorname{Der}_k R$ be a Lie subalgebra. For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ denote by $\mathfrak{p}_L$ the largest $L$-stable ideal of $R$ contained in $\mathfrak{p}$. The equality $\mathfrak{p}_L = \mathfrak{q}_L$ for two primes $\mathfrak{p}, \mathfrak{q}$ defines an equivalence relation on $\operatorname{Spec} R$, called the *$L$-stratification.*

**Theorem.** *Suppose that the field $k$ is algebraically closed, the $R$-algebra $A$ is finite, and for each $D \in L$ there exists a $D$-compatible derivation in $\operatorname{Der}_k A$. Then there is a noncanonical isomorphism of $k$-algebras $A/\mathfrak{m}A \cong A/\mathfrak{n}A$ for any pair of maximal ideals $\mathfrak{m}, \mathfrak{n}$ of $R$ with residue field $k$ lying in the same $L$-stratum.*

Actually a stronger conclusion will be proved. If $\mathfrak{m}_L = 0$, then for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p}_L = 0$ there exists a ring homomorphism $R \to R'$ with respect to which $R'$ is a finitely generated $R$-algebra, the ring $R'_\mathfrak{p} = R' \otimes_R R_\mathfrak{p}$ is faithfully flat over the local ring $R_\mathfrak{p}$, and the $R'$-algebra $A \otimes_R R'$ is isomorphic to $A/\mathfrak{m}A \otimes_k R'$. This property may be viewed as the local triviality of $A$ in a neighbourhood of $\mathfrak{p}$ with respect to the quasicompact faithfully flat Grothendieck topology.

The technique used in this paper is Hopf algebraic, and the main results are established when an arbitrary cocommutative Hopf algebra $H$ acts on $R$ and $A$. The theorem stated in the introduction is a special case of Theorem 2.3 in which $H$ is taken to be the universal enveloping algebra of the Lie algebra

$$\widetilde{L} = \{(D, \mathcal{D}) \in L \times \operatorname{Der}_k A \mid \mathcal{D} \text{ is } D\text{-compatible}\}.$$

Since the projection $\widetilde{L} \to L$ is surjective, the $L$-stable ideals of $R$ are precisely the $H$-stable ideals, so that the $L$-stratification coincides with the $H$-stratification.

In a special case the previous theorem applies to the Poisson orders introduced by Brown and Gordon [3]. Suppose that $R$ is equipped with a $k$-bilinear Poisson bracket $\{-,-\}$. For each $a \in R$ the formula $D_a(z) = \{a, z\}$, $z \in R$, defines a derivation $D_a \in \operatorname{Der}_k R$. A Poisson $R$-order is an associative unital finite $R$-algebra $A$ such that $R$ is identified with a central subring of $A$ and each $D_a,\ a \in R$, extends to a $k$-linear derivation of $A$ (the latter is then $D_a$-compatible). The assumption that $A$ is finitely generated as a $k$-algebra is also included in [3], but we do not need it. Take $L = \{D_a \mid a \in R\}$. Then $\mathfrak{p}_L$ is the largest Poisson ideal of $R$ contained in a prime ideal $\mathfrak{p}$. The $L$-strata in this case were called the symplectic cores of the Poisson algebra $R$ in [3]. Thus $A/\mathfrak{m}A \cong A/\mathfrak{n}A$ whenever $k$ is algebraically closed and $\mathfrak{m}, \mathfrak{n}$ are two maximal ideals of $R$ with residue field $k$ lying in the same symplectic core. In the case $k = \mathbb{C}$ this result was obtained in [3, Th. 4.2]. Besides the quantized function algebras at a root of 1 considered by De Concini, Lyubashenko, Procesi other interesting examples of Poisson orders are found among the symplectic reflection algebras of Etingof and Ginzburg [8]. I thank A. Premet for mentioning the results of [3] to me.

When $\operatorname{char} k > 0$, all $L$-strata in $\operatorname{Spec} R$ reduce to single points. This cannot be said about the $H$-strata, for example, when $H$ is a group algebra or when $H$ contains an infinite sequence of divided powers for some primitive element. So the conclusion of Theorem 2.3 is nontrivial in any characteristic.

## 1. Prime lying-over in equivariant extensions

Throughout the whole paper we assume that $H$ is a cocommutative Hopf algebra over the ground field $k$ and $R$ is a commutative left $H$-module algebra. Thus $R$ is a commutative ring extension of $k$ which also has a left $H$-module structure such that the multiplication map $R \otimes_k R \to R$ is $H$-linear and the unity $1 \in R$ is $H$-invariant. We assume that $H$ acts in the tensor products via the comultiplication $H \to H \otimes_k H$ written symbolically as $h \mapsto \sum h_{(1)} \otimes h_{(2)}$. An $R$-module $M$ is *$H$-equivariant* if $M$ is equipped with a left $H$-module structure such that the action of $R$ is given by an $H$-linear map $R \otimes_k M \to M$. The compatibility of the two module structures mean precisely that $M$ may be regarded as a left module over the smash product $R\#H$, and we denote by ${}_{R\#H}\mathcal{M}$ the category of $H$-equivariant $R$-modules. We say that $M \in {}_{R\#H}\mathcal{M}$ is *$R$-finite* if $M$ is finitely generated as an $R$-module.

Given two objects $M, N \in {}_{R\#H}\mathcal{M}$, the $R$-modules $M \otimes_R N$ and $\operatorname{Hom}_R(M, N)$ will be regarded as objects of ${}_{R\#H}\mathcal{M}$ too. In the first case the action of $H$ is obtained by observing that the kernel of the canonical surjection $M \otimes_k N \to M \otimes_R N$ is $H$-stable. In the second case the action is defined by the formula

$$(hq)(v) = \sum h_{(1)} q\big(S(h_{(2)})v\big)$$

where $h \in H$, $q \in \mathrm{Hom}_R(M,N)$, $v \in M$ and $S : H \to H$ is the antipode. In particular, the $R$-dual $M_R^* = \mathrm{Hom}_R(M,R)$ is an $H$-equivariant $R$-module.

Let $\Omega$ be a fixed set written as a disjoint union of subsets $\Omega_n,\ n = 0,1,2,\ldots$. An *$R$-algebra with the operator domain* $\Omega$ is an $R$-module $A$ together with a collection of maps

$$\Omega_n \to \mathrm{Hom}_R(A_R^{\otimes n},\, A), \qquad n = 0,1,2,\ldots,$$

where $A_R^{\otimes n}$ stands for the $n$th tensor power of the $R$-module $A$. In other words, to each element of $\Omega_n$ there corresponds an $R$-multilinear operation $A^n \to A$ (cf. [4]). Further on the set $\Omega$ will not be mentioned explicitly, and we will speak simply about $R$-algebras. An *$H$-equivariant $R$-algebra* is an $R$-algebra $A$ equipped with a left $H$-module structure with respect to which $A$ is an $H$-equivariant $R$-module and each structure map $A_R^{\otimes n} \to A$ is $H$-linear.

By a *ring extension* we mean any ring homomorphism $\varphi : R \to R'$ where the ring $R'$ is assumed to be commutative, and we also call $R'$ an extension of $R$. Furthermore, $\varphi$ is an *$H$-equivariant extension* if $\varphi$ is a homomorphism of commutative $H$-module algebras. For a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$ denote

$$\mathfrak{p}_H = \{a \in R \mid Ha \subset \mathfrak{p}\}.$$

So $\mathfrak{p}_H$ is the largest $H$-stable ideal of $R$ contained in $\mathfrak{p}$. We say that two primes $\mathfrak{p}, \mathfrak{q}$ lie in the same *$H$-stratum* if $\mathfrak{p}_H = \mathfrak{q}_H$. Denote by $R_\mathfrak{p}$ the local ring of $\mathfrak{p}$ and by $M_\mathfrak{p}$ the corresponding localization $R_\mathfrak{p} \otimes_R M$ of an $R$-module $M$.

**Proposition 1.1.** *If $M \in {}_{R\#H}\mathcal{M}$ is $R$-finite, then $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for each $\mathfrak{p} \in \mathrm{Spec}\, R$ such that $\mathfrak{p}_H = 0$.*

A stronger form of this result is given in [10, Cor. 1.5]. When $H = U(L)$ is the universal enveloping algebra of a Lie algebra $L$, the proof can be obtained especially easily. In this case the actions of $L$ on $R$ and $M$ extend to actions on $R_\mathfrak{p}$ and $M_\mathfrak{p}$. So one may assume $R$ to be local and $\mathfrak{p}$ maximal. Let $v_1,\ldots,v_n \in M$ be elements whose cosets give a basis for $M/\mathfrak{p}M$ over the residue field $R/\mathfrak{p}$. By Nakayama's Lemma $M = \sum Rv_i$. Denote by $I$ the ideal of $R$ generated by all elements which occur as a coefficient in some relation $a_1v_1+\ldots+a_nv_n = 0$. If $a_1,\ldots,a_n \in R$ satisfy this relation and $D \in L$, then

$$\sum (Da_i)v_i = -\sum a_i(Dv_i) \in IM = \sum Iv_i.$$

Therefore for each $i$ there exists $b_i \in I$ such that $Da_i - b_i \in I$, but then $Da_i \in I$. This shows that $I$ is $L$-stable, hence $H$-stable. However, $I \subset \mathfrak{p}$ since $v_1,\ldots,v_n$ are linearly independent modulo $\mathfrak{p}$. The assumption $\mathfrak{p}_H = 0$ now forces $I = 0$. In other words, $v_1,\ldots,v_n$ are a basis for $M$ over $R$.

**Corollary 1.2.** *Suppose that $M \in {}_{R\#H}\mathcal{M}$ has a chain $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ of $R$-finite ${}_{R\#H}\mathcal{M}$-subobjects such that $M = \bigcup M_i$. Then $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module for each $\mathfrak{p} \in \mathrm{Spec}\, R$ with $\mathfrak{p}_H = 0$. If $N \subset M$ is an arbitrary ${}_{R\#H}\mathcal{M}$-subobject, then $N_\mathfrak{p}$ is a direct summand of $M_\mathfrak{p}$ for any such $\mathfrak{p}$.*

*Proof.* Each quotient $M_i/M_{i-1},\ i > 0$, is an $R$-finite object of ${}_{R\#H}\mathcal{M}$. Therefore $(M_i)_\mathfrak{p}/(M_{i-1})_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module, and $(M_{i-1})_\mathfrak{p}$ is a direct summand of $(M_i)_\mathfrak{p}$. It follows that $M_\mathfrak{p} \cong \bigoplus (M_i)_\mathfrak{p}/(M_{i-1})_\mathfrak{p}$ is free. Since $M/N$ satisfies the same assumptions as $M$, the $R_\mathfrak{p}$-module $M_\mathfrak{p}/N_\mathfrak{p}$ is also free, whence the final conclusion. □

**Proposition 1.3.** *Assume $M \in {}_{R\#H}\mathcal{M}$ to be $R$-finite. If $N$ is any ${}_{R\#H}\mathcal{M}$-subobject of $M$, then for each $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p}_H = 0$ the $R_\mathfrak{p}$-module $N_\mathfrak{p}$ is finitely generated free and its rank $\operatorname{rk} N_\mathfrak{p}$ does not depend on $\mathfrak{p}$.*

*Proof.* It has been verified in [10, Prop. 1.1] that all Fitting invariants $\operatorname{Fitt}_i M$ of the $R$-module $M$ are $H$-stable ideals of $R$ (when $H = U(L)$ this can be done straightforwardly). Hence $\operatorname{Fitt}_i M = 0$ whenever $\operatorname{Fitt}_i M \subset \mathfrak{p}$ for at least one $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p}_H = 0$. There is an integer $r \geq 0$ such that $\operatorname{Fitt}_{r-1} M = 0$, but $\operatorname{Fitt}_r M \not\subset \mathfrak{p}$ for each $\mathfrak{p}$ with $\mathfrak{p}_H = 0$. By [7, Cor. 20.5] the Fitting invariants of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ are extensions of the ideals $\operatorname{Fitt}_i M$. Thus $\operatorname{Fitt}_i M_\mathfrak{p}$ is zero for $i = r-1$ and equals $R_\mathfrak{p}$ for $i = r$. By [7, Prop. 20.8] $M_\mathfrak{p}$ is projective, hence free, of rank $r$. This shows that the ranks of $M_\mathfrak{p}$'s have the same value for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p}_H = 0$. Replacing $M$ with $M/N$, we deduce the same conclusion for the ranks of $M_\mathfrak{p}/N_\mathfrak{p}$'s. Now $N_\mathfrak{p}$ is a direct summand of $M_\mathfrak{p}$. Hence $N_\mathfrak{p}$ is free with $\operatorname{rk} N_\mathfrak{p} = \operatorname{rk} M_\mathfrak{p} - \operatorname{rk} M_\mathfrak{p}/N_\mathfrak{p}$. □

In the next result we may regard $R'$ as an $H$-equivariant $R$-module, and it is already clear from Corollary 1.2 that $R'_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. If $\varphi$ is injective then $R'_\mathfrak{p}$ is faithfully flat over $R_\mathfrak{p}$, which implies the existence of a prime $\mathfrak{p}' \in \operatorname{Spec} R'$ lying over $\mathfrak{p}$. It will later be essential to find $\mathfrak{p}'$ with additional properties.

**Proposition 1.4.** *Let $\varphi : R \to R'$ be an $H$-equivariant extension of commutative rings such that $R' = \varphi(R)[V]$ (i.e. $R'$ is generated by $\varphi(R) \cup V$) where $V \subset R'$ is an $H$-stable finitely generated $R$-submodule, and let $J$ be an $H$-stable ideal of $R'$. Put $X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p}_H = 0\}$ and*

$$X(J) = \{\mathfrak{p} \in X \mid \textit{there exists } \mathfrak{p}' \in \operatorname{Spec} R' \textit{ such that } J \not\subset \mathfrak{p}' \textit{ and } \varphi^{-1}(\mathfrak{p}') = \mathfrak{p}\}.$$

*If $X(J) \neq \varnothing$ then $X(J) = X$.*

*Proof.* Setting $V_0 = \varphi(R)$ and inductively $V_i = V_{i-1} + V_{i-1}V$ for $i > 0$, we get a chain of $R$-finite ${}_{R\#H}\mathcal{M}$-subobjects whose union equals $R'$. Suppose that $\mathfrak{p} \in X$. By Corollary 1.2 $J_\mathfrak{p}$ is an $R_\mathfrak{p}$-module direct summand of $R'_\mathfrak{p}$. Hence

$$\mathfrak{p}R'_\mathfrak{p} \cap J_\mathfrak{p} = \mathfrak{p}J_\mathfrak{p}.$$

The inclusion $J_\mathfrak{p} \subset \mathfrak{p}R'_\mathfrak{p}$ holds if and only if $J_\mathfrak{p} = \mathfrak{p}J_\mathfrak{p}$. Since the $R_\mathfrak{p}$-module $J_\mathfrak{p}$ is projective, and even free by Kaplansky's Theorem, the last equality is equivalent to $J_\mathfrak{p} = 0$. Since $J = \bigcup (J \cap V_i)$, the equality $J_\mathfrak{p} = 0$ amounts to the condition that $(J \cap V_i)_\mathfrak{p} = 0$ for all $i > 0$. Applying Proposition 1.3 to the subobjects $J \cap V_i$ of the $R$-finite ${}_{R\#H}\mathcal{M}$-objects $V_i$, we conclude that the inclusion $J_\mathfrak{p} \subset \mathfrak{p}R'_\mathfrak{p}$ holds for some prime in $X$ if and only if it holds for all primes in $X$.

Suppose that $\mathfrak{q} \in X(J)$, and let $\mathfrak{q}' \in \operatorname{Spec} R'$ be such that $J \not\subset \mathfrak{q}'$ and $\varphi^{-1}(\mathfrak{q}') = \mathfrak{q}$. Then $\mathfrak{q}'$ coincides with the preimage of $\mathfrak{q}'R'_\mathfrak{q}$ in $R'$, which forces $J_\mathfrak{q} \not\subset \mathfrak{q}'R'_\mathfrak{q}$. Since $\mathfrak{q}R'_\mathfrak{q} \subset \mathfrak{q}'R'_\mathfrak{q}$, we get $J_\mathfrak{q} \not\subset \mathfrak{q}R'_\mathfrak{q}$. As we have seen, this implies that $J_\mathfrak{p} \not\subset \mathfrak{p}R'_\mathfrak{p}$ for all $\mathfrak{p} \in X$. Moreover, $J^n$ is an $H$-stable ideal of $R'$ and $J^n \not\subset \mathfrak{q}'$ for any $n > 0$ since $\mathfrak{q}'$ is prime. Replacing $J$ with $J^n$ in the preceding arguments, we deduce that $J^n_\mathfrak{p} \not\subset \mathfrak{p}R'_\mathfrak{p}$ for all $n > 0$ and all $\mathfrak{p} \in X$.

Now fix some $\mathfrak{p} \in X$ and denote by $I$ the extension of $J$ in the ring $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$. Then $I^n$ is the extension of $J^n$. So $J^n_\mathfrak{p} \not\subset \mathfrak{p}R'_\mathfrak{p}$ yields $I^n \neq 0$. In other words, the ideal $I$ is not nilpotent. The ring $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ is noetherian since it is a finitely generated algebra over the residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. It follows that $I$ is finitely generated, and therefore $I$ is not contained in the nil radical of $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$. Hence $R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ has a prime ideal which does not contain $I$. Take the preimage $\mathfrak{p}'$ in $R'$ of such an ideal. Then $\mathfrak{p}' \in \operatorname{Spec} R'$ and $J \not\subset \mathfrak{p}'$. Since the composite $R \to R' \to R'_\mathfrak{p}/\mathfrak{p}R'_\mathfrak{p}$ factors through $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, we also have $\varphi^{-1}(\mathfrak{p}') = \mathfrak{p}$. □

## 2. Construction of trivializing extensions

Let $\mathfrak{m}$ be a maximal ideal of $R$. For each $R$-module $M$ put

$$\mathcal{J}_\mathfrak{m}(M) = \operatorname{Hom}_k(H, M/\mathfrak{m}M).$$

In particular, $\mathcal{J}_\mathfrak{m}(R) = \operatorname{Hom}_k(H, R/\mathfrak{m})$ is a commutative ring, extension of $k$, with respect to the convolution multiplication (see [9] or [11]). The unity 1 of $\mathcal{J}_\mathfrak{m}(R)$ is the counit $\varepsilon : H \to k$. We will view $\mathcal{J}_\mathfrak{m}(M)$ as a module over $\mathcal{J}_\mathfrak{m}(R)$ and a module over $H$ with respect to the actions

$$(\xi\eta)(h) = \sum \xi(h_{(1)})\eta(h_{(2)}), \qquad (g \rightharpoonup \eta)(h) = \eta(hg)$$

where $\xi \in \mathcal{J}_\mathfrak{m}(R)$, $\eta \in \mathcal{J}_\mathfrak{m}(M)$ and $g, h \in H$. It is checked straightforwardly that $\mathcal{J}_\mathfrak{m}(R)$ is an $H$-module algebra and $\mathcal{J}_\mathfrak{m}(M)$ is an $H$-equivariant $\mathcal{J}_\mathfrak{m}(R)$-module. Each $H$-invariant element in $\mathcal{J}_\mathfrak{m}(M)$ is of the form $h \mapsto \varepsilon(h)u$ for some $u \in M/\mathfrak{m}M$. Thus $M/\mathfrak{m}M$ may be identified with the $k$-subspace of $H$-invariant elements in $\mathcal{J}_\mathfrak{m}(M)$. This gives rise to a homomorphism of $H$-equivariant $\mathcal{J}_\mathfrak{m}(R)$-modules

$$M/\mathfrak{m}M \otimes_k \mathcal{J}_\mathfrak{m}(R) \to \mathcal{J}_\mathfrak{m}(M)$$

It is an isomorphism whenever $R/\mathfrak{m} = k$ and $\dim_k M/\mathfrak{m}M < \infty$. Assuming further that $M \in {}_{R\#H}\mathcal{M}$, define

$$\theta^M_\mathfrak{m} : M \to \mathcal{J}_\mathfrak{m}(M)$$

by the rule

$$\theta^M_\mathfrak{m}(v)(h) = \pi^M_\mathfrak{m}(hv), \quad v \in M,\ h \in H,$$

where $\pi^M_\mathfrak{m} : M \to M/\mathfrak{m}M$ is the canonical map. Then $\theta^M_\mathfrak{m}$ is $H$-linear and

$$\theta^M_\mathfrak{m}(av) = \theta^R_\mathfrak{m}(a)\,\theta^M_\mathfrak{m}(v) \quad \text{for all } a \in R,\ v \in M.$$

This means that $\theta^R_\mathfrak{m} : R \to \mathcal{J}_\mathfrak{m}(R)$ is a homomorphism of $H$-module algebras, while $\theta^M_\mathfrak{m}$ is a morphism in ${}_{R\#H}\mathcal{M}$ when $\mathcal{J}_\mathfrak{m}(M)$ is viewed as an $R$-module via $\theta^R_\mathfrak{m}$.

Each $R/\mathfrak{m}$-linear map $f : M/\mathfrak{m}M \to R/\mathfrak{m}$ induces a homomorphism of $H$-equivariant $\mathcal{J}_\mathfrak{m}(R)$-modules $\mathcal{J}_\mathfrak{m}(f) : \mathcal{J}_\mathfrak{m}(M) \to \mathcal{J}_\mathfrak{m}(R)$. Then the map

$$c_f : M \to \mathcal{J}_\mathfrak{m}(R), \qquad c_f = \mathcal{J}_\mathfrak{m}(f) \circ \theta^M_\mathfrak{m},$$

is $H$-linear and

$$c_f(av) = \theta^R_\mathfrak{m}(a)\,c_f(v) \quad \text{for all } a \in R,\ v \in M.$$

We call $c_f$ the *coefficient function* on $M$ associated with $f$. Denote by $\mathcal{E}_\mathfrak{m}(M)$ the subring of $\mathcal{J}_\mathfrak{m}(R)$ generated by the image of $\theta^R_\mathfrak{m}$ and the images of all $c_f$'s, when $f$ runs over the dual of the $R/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Clearly $\mathcal{E}_\mathfrak{m}(M)$ is stable under the action of $H$. We will always view $\mathcal{E}_\mathfrak{m}(M)$ as an $H$-equivariant extension of $R$ via $\theta^R_\mathfrak{m} : R \to \mathcal{E}_\mathfrak{m}(M)$.

**Lemma 2.1.** *Suppose that $M \in {}_{R\#H}\mathcal{M}$ is $R$-finite and $R/\mathfrak{m} = k$. Then $\mathcal{E}_\mathfrak{m}(M)$ is generated as an $R$-algebra by an $H$-stable finite $R$-submodule. In particular, $\mathcal{E}_\mathfrak{m}(M)$ is the union of a chain of $H$-stable finite $R$-submodules. Also,*

$$\operatorname{Im}\theta_\mathfrak{m}^M \subset M/\mathfrak{m}M \otimes_k \mathcal{E}_\mathfrak{m}(M).$$

*Proof.* Let $e_1, \ldots, e_r$ be any $k$-basis for $M/\mathfrak{m}M$ and $f_1, \ldots, f_r$ the dual basis for the dual vector space $(M/\mathfrak{m}M)^*$. Each $f \in (M/\mathfrak{m}M)^*$ is a $k$-linear combination of $f_1, \ldots, f_r$, whence $c_f$ is a $k$-linear combination of $c_{f_1}, \ldots, c_{f_r}$. Now it follows that $\mathcal{E}_\mathfrak{m}(M) = \theta_\mathfrak{m}^R(R)[V]$ where $V = \sum c_{f_i}(M)$ is an $H$-stable finite $\theta_\mathfrak{m}^R(R)$-submodule in $\mathcal{J}_\mathfrak{m}(R)$. By definitions

$$\theta_\mathfrak{m}^M(v)(h) = \sum c_{f_i}(v)(h)\, e_i \quad \text{for all } v \in M,\ h \in H,$$

i.e. $\theta_\mathfrak{m}^M(v) = \sum e_i \otimes c_{f_i}(v) \in M/\mathfrak{m}M \otimes_k \mathcal{E}_\mathfrak{m}(M)$ under the identification of $\mathcal{J}_\mathfrak{m}(M)$ with $M/\mathfrak{m}M \otimes_k \mathcal{J}_\mathfrak{m}(R)$. □

If $A$ is an $H$-equivariant $R$-algebra, then $\mathcal{J}_\mathfrak{m}(A)$ will be viewed as an $H$-equivariant $\mathcal{J}_\mathfrak{m}(R)$-algebra (and as an $R$-algebra via $\theta_\mathfrak{m}^R$) with the same operator domain. The structure map $\hat{\mu}$ for $\mathcal{J}_\mathfrak{m}(A)$ corresponding to a structure map $\mu : A_R^{\otimes n} \to A$ is defined as

$$\hat{\mu}(\eta_1 \otimes \cdots \otimes \eta_n)(h) = \sum \mu_\mathfrak{m}\big(\eta_1(h_{(1)}) \otimes \cdots \otimes \eta_n(h_{(n)})\big).$$

where $\eta_1, \ldots, \eta_n \in \mathcal{J}_\mathfrak{m}(A)$, $h \in H$ and $\mu_\mathfrak{m}$ is the structure map of the $R/\mathfrak{m}$-algebra $A/\mathfrak{m}A$ induced by $\mu$. It is checked straightforwardly that $\theta_\mathfrak{m}^A : A \to \mathcal{J}_\mathfrak{m}(A)$ is a homomorphism of $H$-equivariant $R$-algebras.

In the next theorem $R'_t$ stands for the ring of fractions of $R'$ with respect to the multiplicatively closed set of powers of $t$ and the functor $? \otimes_{R'} R'_t$ on the category of $R'$-modules is denoted by the subscript $t$.

**Theorem 2.2.** *Let $A$ be an $H$-equivariant finite $R$-algebra and $\mathfrak{m}$ a maximal ideal of $R$ such that $R/\mathfrak{m} = k$ and $\mathfrak{m}_H = 0$. Put $R' = \mathcal{E}_\mathfrak{m}(A)$. For each $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p}_H = 0$ there exists $t \in R'$ such that $(R'_t)_\mathfrak{p}$ is faithfully flat over $R_\mathfrak{p}$ and there is an isomorphism of $R'_t$-algebras*

$$A \otimes_R R'_t \cong A/\mathfrak{m}A \otimes_k R'_t.$$

*Proof.* The map $R' \to R/\mathfrak{m}$, $\xi \mapsto \xi(1)$, is a ring homomorphism whose composite with $\theta_\mathfrak{m}^R$ coincides with the canonical map $R \to R/\mathfrak{m}$. Therefore the kernel

$$\mathfrak{m}' = \{\xi \in R' \mid \xi(1) = 0\}$$

is a maximal ideal of $R'$ lying over $\mathfrak{m}$. Since $\theta_\mathfrak{m}^A : A \to \mathcal{J}_\mathfrak{m}(A) \cong A/\mathfrak{m}A \otimes_k \mathcal{J}_\mathfrak{m}(R)$ has image in $A/\mathfrak{m}A \otimes_k R'$, it extends to a homomorphism of $H$-equivariant $R'$-algebras

$$\psi : A \otimes_R R' \to A/\mathfrak{m}A \otimes_k R'.$$

Put $K = \operatorname{Ker}\psi$, $C = \operatorname{Coker}\psi$, and denote by $J$ the annihilator of $C$ in $R'$. Since $C$ is an $H$-equivariant $R'$-module, $J$ is an $H$-stable ideal of $R'$. Note that $C$ is

$R'$-finite because $\dim_k A/\mathfrak{m}A < \infty$. Since $R'/\mathfrak{m}' \cong R/\mathfrak{m}$, the map $\psi \otimes_{R'} R'/\mathfrak{m}'$ is identified with the canonical isomorphism $A \otimes_R R/\mathfrak{m} \to A/\mathfrak{m}A$. Hence $C = \mathfrak{m}'C$, and therefore $J \not\subset \mathfrak{m}'$ by Nakayama's Lemma. This entails $\mathfrak{m} \in X(J)$ in the notation of Proposition 1.4. But then $\mathfrak{p} \in X(J)$ as well. Let $\mathfrak{p}' \in \operatorname{Spec} R'$ be any prime ideal lying over $\mathfrak{p}$ such that $J \not\subset \mathfrak{p}'$.

Suppose that $t \in J$, but $t \notin \mathfrak{p}'$. From the exact sequence of $R'_t$-modules

$$0 \longrightarrow K_t \longrightarrow A \otimes_R R'_t \xrightarrow{\psi_t} A/\mathfrak{m}A \otimes_k R'_t \longrightarrow C_t = 0$$

we deduce that $\psi_t$ (i.e. $\psi \otimes_{R'} R'_t$) is surjective. Since $A/\mathfrak{m}A \otimes_k R'_t$ is a free $R'_t$-module, $K_t$ is a direct summand of $A \otimes_R R'_t$. In particular, $K_t$ is $R'_t$-finite. By Proposition 1.3 the $R_\mathfrak{p}$-module $A_\mathfrak{p}$ and the $R_\mathfrak{m}$-module $A_\mathfrak{m}$ are both free of equal rank $r$. Clearly $r = \dim_k A/\mathfrak{m}A$. Then the $R'_{\mathfrak{p}'}$-module $A \otimes_R R'_{\mathfrak{p}'} \cong A_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}$ is also free of rank $r$. Thus

$$\psi \otimes_{R'} R'_{\mathfrak{p}'} : A \otimes_R R'_{\mathfrak{p}'} \to A/\mathfrak{m}A \otimes_k R'_{\mathfrak{p}'}$$

is a homomorphism between two free modules of equal rank. Since $R' \to R'_{\mathfrak{p}'}$ factors through $R'_t$, this homomorphism is surjective. But then $\psi \otimes_{R'} R'_{\mathfrak{p}'}$ has to be bijective, and it follows that $K_t \otimes_{R'_t} R'_{\mathfrak{p}'} \cong K_{\mathfrak{p}'} = 0$. By Nakayama's Lemma $K_t$ is annihilated by some element $s \in R' \smallsetminus \mathfrak{p}'$. Replacing $t$ with $st$, we obtain an element $t \in J \smallsetminus \mathfrak{p}'$ for which $K_t = 0$, so that $\psi_t$ is an isomorphism.

By Corollary 1.2 $R'_\mathfrak{p}$ is flat over $R_\mathfrak{p}$. Hence $(R'_t)_\mathfrak{p} \cong (R'_\mathfrak{p})_t$ is also flat over $R_\mathfrak{p}$. Since $\mathfrak{p}'R'_t$ is a prime ideal of $R'_t$ lying over $\mathfrak{p}$, we have faithful flatness. □

**Theorem 2.3.** *Assume the field $k$ to be algebraically closed. If $A$ is an $H$-equivariant finite $R$-algebra, then $A/\mathfrak{m}A \cong A/\mathfrak{n}A$ for any pair of maximal ideals $\mathfrak{m}, \mathfrak{n}$ of $R$ with residue field $k$ lying in the same $H$-stratum.*

*Proof.* By the hypothesis $\mathfrak{m}_H = \mathfrak{n}_H$. Passing to the $H$-equivariant $R/\mathfrak{m}_H$-algebra $A/\mathfrak{m}_H A$, we may assume that $\mathfrak{m}_H = 0$ and apply Theorem 2.2. There exists $t \in R'$ such that $(R'_t)_\mathfrak{n}$ is faithfully flat over $R_\mathfrak{n}$ and

$$A \otimes_R R'_t \cong A/\mathfrak{m}A \otimes_k R'_t.$$

Faithful flatness ensures that $\mathfrak{n}R'_t \neq R'_t$. Since $R'_t$ is a finitely generated $R$-algebra, $R'_t/\mathfrak{n}R'_t$ is a finitely generated algebra over $R/\mathfrak{n} = k$. As the latter is nonzero, it has at least one maximal ideal. The residue field of such an ideal coincides with $k$ by Hilbert's Nullstellensatz. Taking preimage, we get a maximal ideal $\mathfrak{n}'$ of $R'_t$ lying over $\mathfrak{n}$ such that $R'_t/\mathfrak{n}' = k$. Finally,

$$A/\mathfrak{n}A \cong (A \otimes_R R'_t) \otimes_{R'_t} R'_t/\mathfrak{n}' \cong (A/\mathfrak{m}A \otimes_k R'_t) \otimes_{R'_t} R'_t/\mathfrak{n}' \cong A/\mathfrak{m}A. \qquad \square$$

## 3. Removing the localization

We want to obtain a stronger conclusion compared with that in Theorem 2.2. Let $\mathfrak{m}$ be a maximal ideal of $R$ such that $R/\mathfrak{m} = k$. Suppose that $M \in {}_{R\#H}\mathcal{M}$ is $R$-finite and $R$-projective. Then $M/\mathfrak{m}M$ and $M^*_R/\mathfrak{m}M^*_R$ are mutually dual vector spaces. As explained in section 2, each $p \in M^*_R/\mathfrak{m}M^*_R$ and each $u \in M/\mathfrak{m}M$ determine coefficient functions

$$c_p : M \to \mathcal{J}_{\mathfrak{m}}(R), \qquad c_u : M_R^* \to \mathcal{J}_{\mathfrak{m}}(R).$$

**Lemma 3.1.** *Let* $x_1, \dots, x_n \in M$ *and* $y_1, \dots, y_n \in M_R^*$ *satisfy* $\sum_{j=1}^n \langle y_j, v\rangle x_j = v$ *for all* $v \in M$*. Then*

$$\sum_{j=1}^n c_p(x_j)c_u(y_j) = \langle p, u\rangle 1 \quad \text{for all } p \in M_R^*/\mathfrak{m}M_R^*,\ u \in M/\mathfrak{m}M.$$

*Proof.* Consider $T = M \otimes_R M_R^* \in {}_{R\#H}\mathcal{M}$. Identifying $p \otimes u \in M_R^*/\mathfrak{m}M_R^* \otimes_k M/\mathfrak{m}M$ with an element of $(T/\mathfrak{m}T)^*$, we get the coefficient function $c_{p\otimes u} : T \to \mathcal{J}_{\mathfrak{m}}(R)$ such that

$$c_{p\otimes u}(v \otimes q) = c_p(v)c_u(q) \quad \text{for } v \in M,\ q \in M_R^*.$$

Now

$$\sum_{j=1}^n c_p(x_j)c_u(y_j) = c_{p\otimes u}(z) \quad \text{where } z = \sum_{j=1}^n x_j \otimes y_j \in T.$$

Note that $z$ corresponds to the identity transformation $\mathrm{Id}_M$ under the canonical isomorphism of $H$-equivariant $R$-modules $T \cong \mathrm{End}_R(M, M)$. Since $\mathrm{Id}_M$ is an $H$-invariant element, so too is $z$. Hence $\theta_{\mathfrak{m}}^T(z)(h) = \varepsilon(h)\pi_{\mathfrak{m}}^T(z)$ for $h \in H$. Identifying $\mathcal{J}_{\mathfrak{m}}(T)$ with $T/\mathfrak{m}T \otimes_k \mathcal{J}_{\mathfrak{m}}(R)$, we get

$$\theta_{\mathfrak{m}}^T(z) = \pi_{\mathfrak{m}}^T(z) \otimes 1 = \sum_{j=1}^n \left(\pi_{\mathfrak{m}}^M(x_j) \otimes \pi_{\mathfrak{m}}^{M_R^*}(y_j)\right) \otimes 1,$$

and so

$$c_{p\otimes u}(z) = \langle p \otimes u,\, \pi_{\mathfrak{m}}^T(z)\rangle 1 = \sum_{j=1}^n \langle p,\, \pi_{\mathfrak{m}}^M(x_j)\rangle\langle u, \pi_{\mathfrak{m}}^{M_R^*}(y_j)\rangle 1.$$

Since the pairing $M_R^*/\mathfrak{m}M_R^* \times M/\mathfrak{m}M \to k$ coincides with the modulo $\mathfrak{m}$ reduction of the pairing $M_R^* \times M \to R$ we have

$$\sum_{j=1}^n \langle u, \pi_{\mathfrak{m}}^{M_R^*}(y_j)\rangle \pi_{\mathfrak{m}}^M(x_j) = u,$$

and the previous formula yields $c_{p\otimes u}(z) = \langle p, u\rangle 1$. □

**Theorem 3.2.** *Let* $A$ *and* $\mathfrak{m}$ *be as in Theorem 2.2. In addition assume that the underlying* $R$*-module of* $A$ *is projective of constant rank. With* $R'' = \mathcal{E}_{\mathfrak{m}}(A \oplus A_R^*)$ *there is an isomorphism of* $H$*-equivariant* $R''$*-algebras* $A \otimes_R R'' \cong A/\mathfrak{m}A \otimes_k R''$*.*

*Proof.* We have a homomorphism of $H$-equivariant $R''$-algebras

$$\psi : A \otimes_R R'' \to A/\mathfrak{m}A \otimes_k R'', \qquad a \otimes b \mapsto \theta_{\mathfrak{m}}^A(a)b.$$

Both algebras are projective $R''$-modules of constant rank $r = \dim_k A/\mathfrak{m}A$. Next we apply Lemma 3.1 to $M = A$. The required elements in $A$ and $A_R^*$ exist by the dual basis lemma. Let $e_1, \dots, e_r$ be any $k$-basis for $A/\mathfrak{m}A$ and $f_1, \dots, f_r$ the dual basis for $(M/\mathfrak{m}M)^*$. Recall that $\theta_{\mathfrak{m}}^A(x_j) = \sum_{l=1}^r e_l \otimes c_{f_l}(x_j)$. Lemma 3.1 yields

$$e_i \otimes 1 = \sum_{l=1}^{r} \langle f_l, e_i \rangle e_l \otimes 1 = \sum_{l=1}^{r} \sum_{j=1}^{n} e_l \otimes c_{f_l}(x_j) c_{e_i}(y_j) = \sum_{j=1}^{n} \psi\big(x_j \otimes c_{e_i}(y_j)\big)$$

for each $i$. This shows that $\psi$ is surjective. But then $\psi$ splits as an epimorphism of $R''$-modules. Hence $\operatorname{Ker}\psi$ is a projective $R''$-module of rank 0, i.e. $\operatorname{Ker}\psi = 0$. In other words, $\psi$ is bijective. □

Any $H$-equivariant $R$-module $M$ is an $R$-algebra with the empty operator domain. When $M$ is $R$-finite and $R$-projective of constant rank, Theorem 3.2 says just that there is an isomorphism of $H$-equivariant $R''$-modules $M \otimes_R R'' \cong M/\mathfrak{m}M \otimes_k R''$. The question about splittings of equivariant modules arises naturally in the Picard-Vessiot theory. A Hopf algebraic approach to this theory was developed by Takeuchi [12]. Originally it dealt with a field acted upon by a cocommutative Hopf algebra or, more generally, just a coalgebra, and later it was extended to $H$-simple artinian module algebras [1]. There the construction of a splitting extension for a given equivariant module followed a different path.

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