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Solution of the Stability Problem for a Thin Shell under Impulsive Loading

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Abstract—The stability problem for a thin shell under an axial impulsive load is considered. A new approach to building a mathematical model is presented, which is based on the Ostrogradskii–Hamilton principle of stationary action. It is shown that the problem reduces to a system of nonlinear differential equations that can be solved numerically and by using an approximate calculation algorithm developed by the authors. A formula determining the dependence between the load intensity and the initial conditions of the problem is derived. In the above setting, the stability problem for a circular cylindrical shell is solved. To determine the critical value of the load impulse, the Lyapunov theory of dynamic stability is used.

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INTRODUCTION

This paper continues the series of papers [1, 3] devoted to the setting of stability problems for thin shells. The study of special features of the behavior of shell constructions (such as aircrafts, ships, motor vehicles, etc.) under the action of bouncing loads is interesting both theoretically and practically. Constructing a mathematical model for such problems involves certain difficulties. The authors of theoretical studies [4] known to us interpret loading as “imparting an initial velocity” to the end walls of the shell and study the wave processes thus arising. In this paper, we propose a new mathematical model of the problem, which allows us to determine the critical value of the impulsive load and the parameters of wave formation.

1. CONSTRUCTION OF A MATHEMATICAL MODEL

Suppose that a shell is subject to a longitudinal compressive load q, which we specify in the form

$$q(t) = I\Delta(t),$$

where I is the impulse intensity and $\Delta(t)$ is the Dirac delta function.

The Ostrogradskii–Hamilton variational principle gives

$$\delta \int_0^t L dt = 0, \tag{1}$$

where t is time, $L = K - P + A$ is the Lagrangian function, and P is the potential energy of deformation determined by

$$P = \frac{h}{2E} \iint \left[\left(\nabla^2 F\right)^2 - 2(1+\nu) \left(\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 \right) \right] dxdy$$

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$$+\frac{D}{2}\iint\left[\left(\nabla^2 w\right)^2-2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right)\right]dxdy; \tag{2}$$

here and in what follows, double integration is over the axial section S of the shell; $D=\dfrac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity of the shell, h is its thickness, E is the elasticity modulus, and ν is the Poisson coefficient; w and F denote the unknown deflection and stress functions to be determined, which are related by the deformation continuity equation

$$\frac{1}{E}\nabla^4 F+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+k_x\frac{\partial^2 w}{\partial y^2}+k_y\frac{\partial^2 w}{\partial x^2}=0, \tag{3}$$

where k_x and k_y are the curvatures of the coordinate lines of the shell.

The kinetic energy K and the work A of external forces satisfy the relations

$$K=\frac{\rho h}{2}\iint(\dot{w})^2\,dxdy,\quad \dot{w}=\frac{\partial w}{\partial t}, \tag{4}$$

$$A=-\frac{h}{E}\iint\left(\frac{\partial^2 F}{\partial y^2}\right)\Bigg|_{x=0,L}\left[\frac{\partial^2 F}{\partial y^2}-\nu\frac{\partial^2 F}{\partial x^2}-\frac{E}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dxdy, \tag{5}$$

where $\rho=E/V^2$ is the density and V is the speed of sound in the shell material.

Theorem 1. *Suppose that the sought deflection function can be represented in the form*

$$w(x,y,t)=\sum_{i=1}^{m}f_i(t)\varphi_i(x,y),\quad m\geq 2, \tag{6}$$

where the $\varphi_i(x,y)$ are orthogonal basis functions on S and the $f_i(t)$ are twice differentiable functions on $(0,t)$. Then the functions $f_i(t)$ are determined by the system of differential equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial\dot{f}_i}\right)+\frac{\partial P}{\partial f_i}=0,\quad i=1,2,\ldots,m, \tag{7}$$

with initial values $f_i(0)$ and $\dot{f}_i(0)$ related to the load impulse intensity by

$$\dot{f}_i(0)=\frac{If_i(0)\displaystyle\iint\left(\frac{\partial\varphi_i}{\partial x}\right)^2dxdy}{2\rho\displaystyle\iint\varphi_i^2dxdy},\quad i=1,2,\ldots,m. \tag{8}$$

Proof. Let $L'=K-P$. It follows from (1) that

$$\delta\int_0^t L'dt+\delta\int_0^t Adt=0. \tag{9}$$

Taking into account (2) and (4)–(6), we obtain

$$\delta L'=\sum_{i=1}^{m}\frac{\partial L'}{\partial\dot{f}_i}\delta\dot{f}_i+\sum_{i=1}^{m}\frac{\partial L'}{\partial f_i}\delta f_i.$$

The integration by parts of the first term in (9) yields

$$\delta\int_0^t L'dt=\sum_{i=1}^{m}\left[\frac{\partial L'}{\partial\dot{f}_i}\delta f_i\Bigg|_0^t+\int_0^t\left(\frac{\partial L'}{\partial f_i}-\frac{d}{dt}\left(\frac{\partial L'}{\partial\dot{f}_i}\right)\right)\delta f_i dt\right].$$

Using (5) and (6), we find the expression

$$A = q(t)\sum_{i=1}^{m} c_i f_i^2, \quad c_i = \frac{h}{2}\iint \left(\frac{\partial \varphi_i}{\partial x}\right)^2 dxdy, \quad i = 1, 2, \ldots, m,$$

for the external work, which, together with the filtering property of the Dirac delta function, implies

$$\delta \int_0^t A dt = I \sum_{i=1}^{m} c_i f_i(0)\delta f_i(0).$$

Equation (9) and the relations

$$\frac{\partial L'}{\partial \dot{f}_i} = \frac{\partial K}{\partial \dot{f}_i}, \quad \frac{\partial L'}{\partial f_i} = -\frac{\partial P}{\partial f_i}$$

found above lead to system (7) on the interval $(0, t_k)$ (t_k is the moment of stability loss) and the additional conditions

$$\left.\frac{\partial K}{\partial \dot{f}_i}\delta f_i\right|_0^{t_k} + I c_i f_i(0)\delta f_i(0) = 0, \quad i = 1, 2, \ldots, m. \tag{10}$$

Using (4) and (6), we obtain

$$\frac{\partial K}{\partial \dot{f}_i} = \rho h \dot{f}_i(t) \iint \varphi_i^2(x, y) dxdy.$$

In accordance with Sachenkov's criterion for dynamical stability [5], we set

$$\dot{f}_i(t_k) = 0, \quad i = 1, 2, \ldots, m,$$

and derive the following relations from conditions (10):

$$-\rho h \dot{f}_i(0) \iint \varphi_i^2(x, y) dxdy + I c_i f_i(0) = 0, \quad i = 1, 2, \ldots, m.$$

These relations imply (8), which completes the proof of the theorem. □

Equations (7) with initial conditions (8) make it possible to determine the critical value of the impulsive load and the parameters of wave formation. Note that it is desirable to select the functions $\varphi_i(x, y)$ in (6) so as to satisfy the boundary conditions of the problem.

Consider a particular example.

2. SOLUTION OF THE STABILITY PROBLEM FOR A CYLINDRICAL SHELL

For a hingedly supported cylindrical shell ($k_x = 0, k_y = 1/R$), we seek the approximating deflection function (6) in the form

$$w(x, y, t) = f_1(t) \sin \alpha x \sin \beta y + f_2(t) \sin^2 \alpha x,$$

where $\alpha = m\pi/L$, $\beta = n/R$, R is the shell radius, L is the shell length, and m and n are wave numbers to be determined. Note that this function satisfies the boundary conditions of the problem and fairly adequately describes the diamond-like shape of the experimentally observed [4] dents appearing after the loss of stability.

We proceed to construct the system of equations (7). Substituting the function w into the deformation continuity equation (3) and integrating the result, we obtain the following expression for the stress function:

$$F = C_1 \cos 2\alpha x + C_2 \cos 2\beta y + C_3 \sin \alpha x \sin \beta y + C_4 \sin 3\alpha x \sin \beta y - qy^2/2;$$

the coefficients in this expression are calculated by

$$C_1 = \frac{\beta^2 f_1^2}{32\alpha^2} - \frac{f_2}{8R\alpha^2}, \quad C_2 = \frac{\alpha^2 f_1^2}{32\beta^2},$$

$$C_3 = \frac{\alpha^2 f_1}{R(\alpha^2+\beta^2)^2} - \frac{\alpha^2\beta^2 f_1 f_2}{(\alpha^2+\beta^2)^2}, \quad C_4 = \frac{\alpha^2\beta^2 f_1 f_2}{(9\alpha^2+\beta^2)^2}.$$

Using the above expression, we determine the potential energy of deformation by (2) as

$$\widehat{P} = A_1\xi_1^4 + A_2\xi_1^2 + A_3\xi_2^2 + A_4\xi_1^2\xi_2^2 + A_5\xi_1^2\xi_2 + \widehat{q}^2.$$

Here we use the dimensionless quantities

$$\widehat{P} = \frac{PR}{\pi E h^3 L}, \quad \xi_i = \frac{f_i}{h}, \quad A_1 = \frac{\eta^2(1+\theta^4)}{128}, \quad A_2 = \frac{1}{4}\left(\frac{\theta^4}{s_1^2} + \frac{\eta^2 s_1^2}{12(1-\nu^2)}\right),$$

$$A_3 = \frac{1}{8} + \frac{\eta^2\theta^4}{6(1-\nu^2)}, \quad A_4 = \frac{\eta^2\theta^4}{4}\left(\frac{1}{s_1^2}+\frac{1}{s_2^2}\right), \quad A_5 = -\frac{\eta}{16}\left(1+\frac{8\theta^4}{s_1^2}\right),$$

$$\widehat{q} = \frac{qR}{Eh}, \quad \eta = \frac{n^2 h}{R}, \quad \theta = \frac{m\pi R}{nL}, \quad s_1 = 1+\theta^2, \quad s_2 = 1+9\theta^2.$$

Let us calculate the kinetic energy by (4):

$$\widehat{K} = \frac{KR}{\pi E h^3 L} = \left(\frac{R}{V}\right)^2 \left(\frac{\dot{\xi_1}^2}{4} + \frac{3\dot{\xi_2}^2}{8}\right).$$

System (7) takes the form

$$\frac{d^2\xi_1}{d\tau^2} + 4(2A_1\xi_1^3 + A_2\xi_1 + A_4\xi_1\xi_2^2 + A_5\xi_1\xi_2) = 0, \tag{11}$$

$$\frac{d^2\xi_2}{d\tau^2} + \frac{4}{3}(2A_3\xi_2 + 2A_4\xi_1^2\xi_2 + A_5\xi_1^2) = 0, \quad \tau = tV/R.$$

Conditions (8) give the initial values

$$\dot{\xi_1}(0) = 0, \quad \dot{\xi_2}(0) = \frac{2(m\pi)^2\xi_2(0)}{3}\frac{IVR}{EL^2}. \tag{12}$$

Problem (11), (12) can be solved numerically. Calculations show that, at small initial velocity $\dot{\xi_2}(0)$ (and, therefore, at small intensity of the load impulse; see (8)), the shell oscillates with an amplitude of order $\xi_2(0)$ (see Fig. 1).

As $\dot{\xi_2}(0)$ approaches the critical value, a sharp increase in the deflection amplitude is observed (see Fig. 2), that is, the loss of Lyapunov stability of motion occurs [6].

Calculating the critical impulse intensity for various m and n and minimizing the load, we find the parameters of wave formation. From the graph in Fig. 2 we determine the critical moment of time corresponding to the first maximum of the deflection amplitude (according to Sachenkov's criterion [5]) and the critical value of deflection. Thus, knowing the solution of problem (11), (12), we can find all characteristics corresponding to the loss of stability of the shell.

We propose the following algorithm for approximate calculations. From the static analogue of the first equation in system (11), we find

$$\xi_1^2 = -\frac{1}{2A_1}\big(A_2 + A_5\xi_2 + A_4\xi_2^2\big).$$

Expressing the right-hand side of the second equation in (11) in terms of ξ_2 by using the expression found above, multiplying both sides of the resulting equality by $\dot{\xi_2}$, and integrating from 0 to t_k, we obtain

$$\dot{\xi_2}^2(0) = B_0 + B_1\xi_2(t_k) + B_2\xi_2^2(t_k) + B_3\xi_2^3(t_k) + B_4\xi_2^4(t_k),$$

where the coefficients $B_0, \ldots, B_4$ are determined from $\xi_2(0)$ and $A_1, \ldots, A_5$.

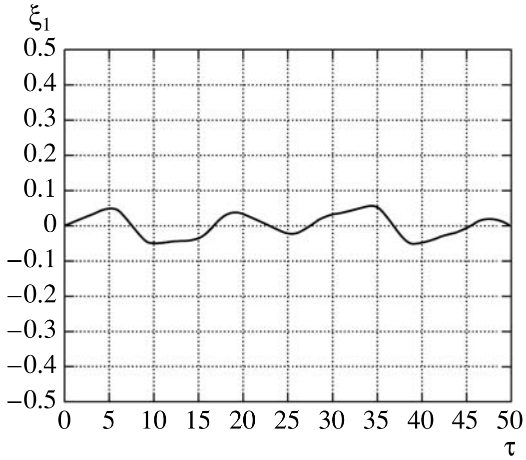

Fig. 1. The time dependence of the deflection amplitude under small values of load impulse.

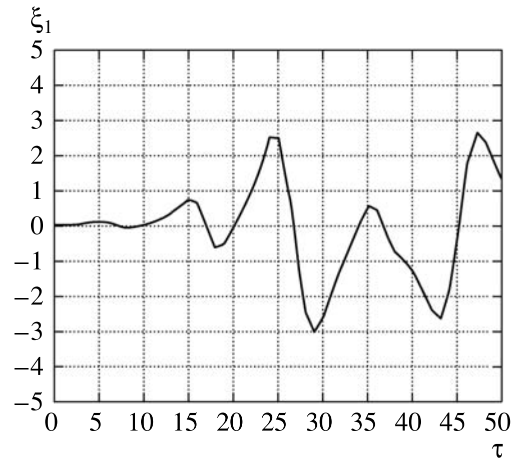

Fig. 2. The time dependence of the deflection amplitude at the critical value of the load impulse.

Minimizing $\dot{\xi}_2(0)$ with respect to the amplitude $\xi_2(t_k)$ and the wave numbers m and n, we determine the critical value $v_{kr} = \dot{\xi}_2(0)$ of the initial velocity and the critical impulse value by

$$\widehat{I}_{kr} = I\frac{2\pi^2 V\xi_2(0)}{3ER} = \frac{v_{kr}L^2}{m^2R^2}.$$

Below, we give some results of calculations:

$$R/h = 100, \quad L/R = 1: \quad v_{kr} = 1.61, \quad m = 2, \quad n = 8, \quad \widehat{I}_{kr} = 0.40,$$
$$R/h = 100, \quad L/R = 2: \quad v_{kr} = 1.60, \quad m = 4, \quad n = 8, \quad \widehat{I}_{kr} = 0.40,$$
$$R/h = 200, \quad L/R = 1: \quad v_{kr} = 1.59, \quad m = 3, \quad n = 12, \quad \widehat{I}_{kr} = 0.18.$$

An analysis of the obtained data shows that increasing the length of the shell does not affect the critical impulse value and the wave number n but results in an increase in the number of dents along the axis of the shell. Decreasing the thickness of the shell results, as expected, in a decrease in the critical impulse value and an increase in the wave numbers m and n.

Thus, the proposed approach to solving stability problems for shells under impulsive loading proves very effective and makes it possible to obtain results which agree well with experimental data [4].

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