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Doubly Periodic Riemann Boundary Value Problem for Non-Rectifiable Curves

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Abstract—The known results on the doubly periodic Riemann boundary value problem concern the case of piecewise-smooth contours. In the present paper we study it for non-rectifiable curves in terms of so called Marcinkiewicz exponents.

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INTRODUCTION

Let τ_1, τ_2 be non-zero complex values such that $\operatorname{Im}\frac{\tau_1}{\tau_2} \neq 0$. In what follows $B_{m,n}$ stands for $(m\tau_1 + n\tau_2)$−transfer of set $B \subset \mathbb{C}$, i.e., $B_{m,n} := \{z \in \mathbb{C} : z - m\tau_1 - n\tau_2 \in B\}$, $m, n \in \mathbb{Z}$. We consider open parallelogram P with vertices at the points $(\pm\tau_1 \pm \tau_2)/2$, and domain D with Jordan boundary Γ such that $0 \in \overline{D} \subset P$. We put

$$D^+ := \bigcup_{m,n=-\infty}^{m,n=+\infty} D_{m,n}, \quad D^- := \overline{\mathbb{C}} \setminus D^+, \quad \mathbf{\Gamma} := \bigcup_{m,n=-\infty}^{+\infty} \Gamma_{m,n}.$$

The doubly periodic Riemann boundary value problem is stated as follows. Given Hölder continuous functions $G(t) \neq 0$ and $g(t)$ on Γ. To find a function $\Phi(z)$ analytic in D^+ and in D^-, satisfying the periodicity conditions

$$\Phi(z + \tau_j) = \Phi(z), \quad j = 1, 2, \tag{1}$$

and the conjugation condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \mathbf{\Gamma}. \tag{2}$$

Here $G(t)$ and $g(t)$ are extended onto $\mathbf{\Gamma}$ by periodicity, and $\Phi^+(t)$ and $\Phi^-(t)$ are the limit values of $\Phi(z)$ for z tending to $t \in \mathbf{\Gamma}$ from D^+ and D^- correspondingly.

This problem is one of versions of the Riemann boundary value problem (see [1, 2]). Its detailed solution for piecewise smooth curves Γ was obtained by Chibrikova [3, 4] and later by Lu Jianke [5, 6]. The periodic Riemann problem admits various applications in elasticity theory (see [7]). Let us note also that related with doubly periodic Riemann problem ideas are helpful in theory of elliptic functions (see,

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for instance, [8]). Various modifications and generalizations of periodic and doubly periodic Riemann boundary value problems for piecewise smooth curves keep interest until our days; see, for instance, [9]. In all studies of this class of boundary value problems assumption of piecewise smoothness of the curves is essential, because their solutions base on certain properties of curvilinear integrals over Γ. That integrals are defined in customary sense for rectifiable Γ only, and properties of corresponding integral operators are connected with smoothness of contours of integration.

The Riemann boundary value problem (for non-periodic case) on non-rectifiable curves was solved in earlier 1980th (see, for instance, the pioneer work [10] and recent survey [11]). Recently these results were improved by means of new metric characteristics of non-rectifiable curves, so called Marcinkiewicz exponents (see [12, 13, 14]). In the present paper we apply these characteristics for solution of the problem (2). In the sections 1 and 2 we describe properties of the Marcinkiewicz exponents and solve the jump problem, i.e., the problem (2) with $G(t) \equiv 1$. The last section contains solution of the Riemann problem.

1. THE MARCINKIEWICZ EXPONENTS

Here we introduce the Marcinkiewicz exponents and study their properties.

Let a closed Jordan curve Γ be boundary of finite domain D. We fix a finite measurable domain Ω such that $D \subset \Omega$, and denote $D^* = \Omega \setminus D$. We assume that Γ has null square, i.e. the domains D and D^* are measurable, and put

$$I_p(D) = \iint\limits_D \frac{dxdy}{\operatorname{dist}^p(z,\Gamma)}, \quad z = x + iy.$$

Definition 1. *The values*

$$\mathfrak{m}^+(\Gamma) = \sup\{p : I_p(D) < \infty\}, \quad \mathfrak{m}^-(\Gamma) = \sup\{p : I_p(D^*) < \infty\}$$

are called inner and outer Marcinkiewicz exponents of the curve Γ *correspondingly. The most of them* $\mathfrak{m}(\Gamma) := \max\{\mathfrak{m}^+(\Gamma), \mathfrak{m}^-(\Gamma)\}$ *is called its Marcinkiewicz exponent.*

The term “Marcinkiewicz exponent” is explained by the fact that characterization of plane sets in terms of certain integrals over their complements was proposed first by Marcinkiewicz (see, for instance, [15]).

Clearly, the outer Marcinkiewicz exponent does not depend on domain Ω. In what follows we put $\Omega = P$.

Theorem 1. *Any curve* Γ *satisfies inequalities* $1 \geq \mathfrak{m}^+(\Gamma) \geq 2 - \operatorname{dm}\Gamma$, $1 \geq \mathfrak{m}^-(\Gamma) \geq 2 - \operatorname{dm}\Gamma$, *where* $\operatorname{dm}\Gamma$ *is upper metric dimension of* Γ (*see its definition below*). *If the curve* Γ *is rectifiable, then* $\mathfrak{m}^+(\Gamma) = \mathfrak{m}^-(\Gamma) = 1$.

Proof. The upper metric dimension of compact set $F \subset \mathbb{C}$ equals to

$$\operatorname{dm} F := \limsup_{\varepsilon \to 0} \frac{\ln N(\varepsilon, F)}{-\ln \varepsilon},$$

where $N(\varepsilon, F)$ is the least number of disks of diameter ε covering F. It is known also as box counting dimension, Kolmogorov dimension and so on (see, for instance, [18, 19, 20]). This dimension has equivalent definition. We divide the complex plane into dyadic squares with sides 2^{-n}. Let $M(F, n)$ stand for number of that squares intersecting F. Then

$$\operatorname{dm} F = \limsup_{n \to \infty} \frac{\log_2 M(F, n)}{n}.$$

We consider the Whitney decomposition of domain D^+. It consists (see [15]) of dyadic squares Q such that $\operatorname{diam} Q \leq \operatorname{dist}(Q, \Gamma) \leq C \operatorname{diam} Q$, where C is a constant. Hence, for any square Q with side 2^{-n} belonging to this decomposition we obtain

$$\iint\limits_Q \frac{dxdy}{(dist(z,\Gamma))^p} \leq C \frac{2^{-2n}}{(2^{-n})^p} \quad \text{and} \quad \iint\limits_{D^+} \frac{dxdy}{(dist(z,\Gamma))^p} \leq \sum_{n=1}^{\infty} w_n \cdot 2^{n(p-2)},$$

where w_n is number of squares with side 2^{-n} in the Whitney decomposition. By the second definition of the upper metric dimension we have $w_n \le 2^{nd}$ for any d greater than $\operatorname{dm}\Gamma$ and sufficiently large n. Consequently, the latter integral can be majorized by series $\sum\limits_{n=1}^{\infty} 2^{n(p-2+d)}$, which converges for $p < 2-d$. Thus, this integral converges for $p \ge 2-\operatorname{dm}\Gamma$, i.e., $\mathfrak{m}^+(\Gamma) \ge 2-\operatorname{dm}\Gamma$. The proof of inequality for $\mathfrak{m}^-(\Gamma)$ is analogous.

Now let us prove inequality $\mathfrak{m}^+(\Gamma) \le 1$. It suffices to show that $\iint\limits_{D^+} \frac{dxdy}{dist(z,\Gamma)} = \infty$. Clearly, we can choose on Γ two points $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ such that $x_1 < x_2$, and these points can be connected in D^+ by a curve λ with real equation $y = \psi(x)$, $\quad x_1 \le x \le x_2$. We drop a perpendicular from a point $z \in \lambda$ on the real axis, and consider its segment connecting z with a point of curve Γ inside D^+. Let $d(z)$ stand for the length of this segment, and $\varphi(x)$ for ordinate of its end point on curve Γ. Obviously, $d(z) \ge dist(z,\Gamma)$, and $\varphi(x) = \psi(x) - d(z)$ (for definiteness we assume that λ lies above Γ). Let Δ be part of domain D^+ concluded between curves λ and Γ. We have

$$\iint\limits_{D^+} \frac{dxdy}{dist(z,\Gamma)} \ge \iint\limits_{\Delta} \frac{dxdy}{d(z)} \ge \int\limits_{x_1}^{x_2} dx \int\limits_{\varphi(x)}^{\psi(x)} \frac{dy}{y-\varphi(x)}.$$

But the last integral diverges, what proves the desired inequality.

The proof of inequality $\mathfrak{m}^-(\Gamma) \le 1$ is analogous.

The equality $\mathfrak{m}^+(\Gamma) = \mathfrak{m}^-(\Gamma) = 1$ for rectifiable curves is consequence of the fact that $\operatorname{dm}\Gamma = 1$ for any rectifiable curve Γ (see [19, 20]).

Theorem is proved.

Example . *Let us fix values $\alpha_1, \alpha_2, \beta \ge 1$ and construct two families of rectangles.*

First family. We consider segments $I_n = [2^{-n}, 2^{-n+1}]$ of real axis, $n = 1, 2, 3, \ldots$. Every segment we divide into $2^{[n\beta]}$ equal parts; here $[n\beta]$ is entire part of $n\beta$. We denote the points of division of I_n by $x_{n,j}$, where j is number in decreasing order. The first family consists of rectangles $p_{n,j} = \{x, y : x_{n,j} - C_n \le x \le x_{nj}, 0 \le y \le 2^{-n}\}$, where $C_n = \frac{1}{2}a_n^{\alpha_1}$, where a_n is distance between the division points on the segment I_n, I.e., $2^{-n-[n\beta]}$. We denote the union of all these rectangles by R^+. The set R^+ belongs to the first quarter of the complex plane.

Second family. Now let $I_n = [-2^{-n+1}, -2^{-n}]$, $n = 1, 2, 3, \ldots$. As above, we divide I_n into $2^{[n\beta]}$ equal parts. We denote the points of division of I_n by $x_{n,j}$, where j is number in increasing order. The second family consists of rectangles $q_{n,j} = \{x, y : x_{n,j} + C_n \ge x \ge x_{nj}, 0 \ge y \ge -2^{-n}\}$, where $C_n = \frac{1}{2}a_n^{\alpha_2}$, where $a_n = 2^{-n-[n\beta]}$. We denote the union of all these rectangles by R^-. The set R^- belongs to the forth quarter of the complex plane.

Now we put $D(\alpha_1, \alpha_2, \beta) := \{z = x + iy : -1 \le x \le 1, -2 \le y \le 0\} \bigcup R^+ \setminus R^-$ and $\Gamma(\alpha_1, \alpha_2, \beta) := \partial D(\alpha_1, \alpha_2, \beta)$. Immediate calculation (see details in [14]) shows that $\operatorname{dm}\Gamma(\alpha_1, \alpha_2, \beta) = \dfrac{2\beta}{\beta+1}$, $\mathfrak{m}^+(\Gamma(\alpha_1, \alpha_2, \beta)) = 1 - \dfrac{\beta-1}{(\beta+1)\alpha_1}$, $\mathfrak{m}^-(\Gamma(\alpha_1, \alpha_2, \beta)) = 1 - \dfrac{\beta-1}{(\beta+1)\alpha_2}$. *Thus,* $\mathfrak{m}^+(\Gamma(1,1,\beta)) = \mathfrak{m}^-(\Gamma(1,1,\beta)) = 2 - \operatorname{dm}\Gamma(1,1,\beta)$, *but if parameters $\alpha_{1,2}$ increase from* 1 *and tends to* ∞, *then the upper metric dimension is constant, but the Marcinkiewicz exponents run from* $2 - \operatorname{dm}\Gamma$ to 1.

2. DOUBLY PERIODIC JUMP PROBLEM

The jump problem is a special case of the Riemann problem with $G(t) \equiv 1$, i.e., we seek analytic in $\mathbb{C} \setminus \mathbf{\Gamma}$, continuous in $\overline{D^+}$ and in $\overline{D^-}$ function Φ satisfying periodicity condition (1) and boundary conjugation condition

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \mathbf{\Gamma}. \tag{3}$$

As above, $g(t)$ is Hölder continuous, i.e.,

$$\sup\left\{\frac{|g(t')-g(t'')|}{|t'-t''|^\nu} : t', t'' \in \Gamma, t' \neq t''\right\} := h_\nu(g,\Gamma) < \infty$$

for certain constant exponent $\nu \in (0,1]$. Let $H_\nu(\Gamma)$ be a class of all functions satisfying this condition.

There exist important difference between jump problems in the cases of rectifiable and non-rectifiable curves Γ. If Γ is rectifiable, then by virtue of well known Painleve theorem [16] it is removable in class of continuous functions, i.e., any continuous in a domain $B \supset \Gamma$ and analytic in $B \setminus \Gamma$ function $F(z)$ is analytic in B. Consequently, a solution of the jump problem on rectifiable curve in unique up to additive constant. But E.P. Dolzhenko (see [17]) proved that non-rectifiable curve Γ is removable only in classes $H_\mu(\Gamma)$ for $\mu > \mathrm{dm_H}\,\Gamma - 1$; here $\mathrm{dm_H}$ stands for Hausdorff dimension. In this connection we have to include into formulation the following smoothness condition

$$\Phi^\pm \in H_\mu(\Gamma), \quad \mu > \mathrm{dm_H}\,\Gamma - 1. \tag{4}$$

Then solution of the doubly periodic jump problem (1), (3), (4) for non-rectifiable curve Γ is also unique up to additive constant.

Now let us study existence of the solution. We apply to function $g(t)$ (as defined on Γ) the Whitney extension operator $\mathbb{E}_0$ (see [15]) and multiply the result by infinitely smooth function $\psi(z)$ equaling 1 in D with compact support S. We assume without loss of generality that $S \subset P$. The obtained extension $\tilde{g}(z) = \psi\mathbb{E}_0 g$ is defined on the whole complex plane and coincides with $g(t)$ on Γ. If $g(t) \in H_\nu(\Gamma)$ then $\tilde{g}(z) \in H_\nu(\mathbb{C})$. Moreover, $\tilde{g}(z)$ has partial derivatives of any order in $\mathbb{C} \setminus \Gamma$ and

$$|\nabla \tilde{g}(z)| \le \frac{Ch_\nu(g,\Gamma)}{\mathrm{dist}^{1-\nu}(z,\Gamma)}. \tag{5}$$

By definition of the Marcinkievicz exponents $|\nabla\tilde{g}|^p$ is integrable in D for $p(1-\nu) < \mathfrak{m}^+(\Gamma)$. Hence, $\nabla\tilde{g}$ is integrable in D for $\nu > 1 - \mathfrak{m}^+(\Gamma)$. Analogously, $\nabla\tilde{g}$ is locally integrable in $\mathbb{C} \setminus D$ for $\nu > 1 - \mathfrak{m}^-(\Gamma)$. We identify any locally integrable in $\mathbb{C}$ function $F(z)$ with distribution $\langle F, \omega\rangle := \iint_{\mathbb{C}} F(z)\omega(z)dz \wedge d\overline{z}$. Then at least one of distributions

$$\langle \tilde{g}\,\overline{\partial}\,\chi^+, \omega\rangle := -\iint\limits_D \frac{\partial \tilde{g}\omega}{\partial \overline{z}} dz \wedge d\overline{z}, \quad \omega \in C^\infty(\mathbb{C}),$$

$$\langle \tilde{g}\,\overline{\partial}\,\chi^-, \omega\rangle := \iint\limits_{\mathbb{C}\setminus D} \frac{\partial \tilde{g}\omega}{\partial \overline{z}} dz \wedge d\overline{z}, \quad \omega \in C_0^\infty(\mathbb{C}),$$

is defined for $\nu > 1 - \mathfrak{m}(\Gamma)$. Here χ^+ is distribution corresponding to characteristic function of set D, and $\chi^- = \chi^+ - 1$. If Γ is rectifiable, then both these distributions equals to mapping $\omega \to \int_\Gamma g(t)\omega(t)dt$. Hence, they are generalizations of curvilinear integral for non-rectifiable curves (see [21] and [11]).

Note 1. *Clearly, the Whitney extension is not unique. But under our assumptions the generalized integrations $\tilde{g}\,\overline{\partial}\,\chi^+$ and $\tilde{g}\,\overline{\partial}\,\chi^-$ do not depend on choice of extension of Whitney type* (*in* [22] *this fact is derived in terms of box counting dimension; its proof in terms of the Marcinkiewicz exponents is analogous*).

Let us consider the Weierstrass ζ-function

$$\zeta(z) = \frac{1}{z} + \sum_{h\neq 0}\left(\frac{1}{z-h} + \frac{1}{h} + \frac{z}{h^2}\right),$$

where $h = m\tau_1 + n\tau_2$, $m, n \in \mathbb{Z}$, and the sum is taken for all periods h excluding $h = 0$. It is meromorphic quasi-periodic function, i.e., $\zeta(z + \tau_{1,2}) = \zeta(z) + \eta_{1,2}$ for any $z \in \mathbb{C}$, where cyclic constants $\eta_{1,2}$ are equal to $2\zeta(\tau_{1,2}/2)$, and in parallelogram P this function has single pole at the origin point with main part z^{-1}. Chibrikova [4] constructed a solution of the doubly periodic jump problem for piecewise smooth curve Γ as integral

$$\Phi(z) = \frac{1}{2\pi i}\int\limits_\Gamma g(t)\zeta(t-z)dt.$$

If Γ is not rectifiable, then this integral is undefined, and we seek solution as convolution of distributions (see [21])

$$\Phi(z) = \frac{1}{2\pi i}\tilde{g}\,\overline{\partial}\,\chi^{\pm} * \zeta = \frac{1}{2\pi i}\left\langle \tilde{g}\,\overline{\partial}\,\chi^{\pm}, \zeta(t-z)\right\rangle, \tag{6}$$

where the distribution $\tilde{g}\,\overline{\partial}\,\chi^{\pm}$ is applied in fact to product $\psi(t-z)\zeta(t-z)$ where smooth function $\psi(z)$ vanishes in small neighborhoods of poles of ζ and equals to unit in a neighborhood of $\mathbf{\Gamma}$.

Let us study first function $\Phi(z)$ in the plus case. Easy calculations show that

$$\Phi(z) = \tilde{\mathbf{g}}(z)\mathbf{c}^{+}(z) - \frac{1}{2\pi i}\iint\limits_{D} \frac{\partial \tilde{g}(\xi)}{\partial \overline{\xi}}\zeta(\xi - z)d\xi \wedge d\overline{\xi}, \tag{7}$$

where $\tilde{\mathbf{g}}$ and $\mathbf{c}^{+}$ are periodic extensions of restrictions of functions $\tilde{g}$ and χ^{+} on parallelogram P relatively. In other words, $\mathbf{c}^{+}$ is characteristic function of set D^{+}. As $S \subset P$, since the product $\tilde{\mathbf{g}}\mathbf{c}^{+}$ has jump g on $\mathbf{\Gamma}$. The integral in the last equality exists if derivative $\frac{\partial \tilde{g}(\xi)}{\partial \overline{\xi}}$ is integrable in D with power $p \geq 2$. If $p > 2$, i.e., $\nu > 1 - \mathfrak{m}^{+}(\Gamma)/2$, then it is continuous in the whole complex plane (see [23]). and the function $\Phi(z)$ satisfies the conjugation condition (3). Clearly, the function Φ inherits quasi-periodicity of ζ-function:

$$\Phi(z + \tau_{1,2}) = \Phi(z) - \eta_{1,2}\iint\limits_{D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z},$$

i.e., it is periodic if $\iint\limits_{D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z} = 0$. Let us note finally that the integral term of equality (7) satisfies the Hölder equation with any exponent lesser than $1 - 2(1-\nu)/\mathfrak{m}^{+}(\Gamma)$. This is easy consequence of well known estimates of that integrals (see, for instance, [23]) and definition of the Marcinkiewicz exponents. Hence, if $\mathrm{dm_H}\,\Gamma - 1 < 1 - 2(1-\nu)/\mathfrak{m}^{+}(\Gamma)$, then we can choose μ such that function Φ will satisfy condition (4).

The case of distribution $\tilde{g}\,\overline{\partial}\,\chi^{-}$ is analogous.

Now we can modify considerations from [4] by terms of the Dolzhenko theorem [17] and obtain

Theorem 2. *Let $g \in H_{\nu}(\Gamma)$,*

$$\nu > 1 - \mathfrak{m}(\Gamma)/2 \quad \text{and} \quad \mathrm{dm_H}\,\Gamma - 1 < 1 - 2(1-\nu)/\mathfrak{m}(\Gamma). \tag{8}$$

Then jump problem (3), (1), (4) *for certain μ has a unique up to arbitrary additive constant solution if and only if* $\iint\limits_{D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z} = 0$ for $\mathfrak{m}(\Gamma) = \mathfrak{m}^{+}(\Gamma)$ *or* $\iint\limits_{P\setminus D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z} = 0$ *for* $\mathfrak{m}(\Gamma) = \mathfrak{m}^{-}(\Gamma)$. *In the first case the solution is given by formula* $\Phi(z) = \frac{1}{2\pi i}\langle \tilde{g}\,\overline{\partial}\,\chi^{+}, \zeta(t-z)\rangle + C$, *and in the second case* $\Phi(z) = \frac{1}{2\pi i}\langle \tilde{g}\,\overline{\partial}\,\chi^{-}, \zeta(t-z)\rangle + C$. *Here C stands for arbitrary constant.*

As known, the doubly periodic jump problem on piecewise-smooth curve has single solvability condition $\int\limits_{\Gamma} g(t)dt = 0$, unlike the customary jump problem, which is solvable unconditionally. This result has topological reasons (the doubly periodic boundary value problems are equivalent to analogous problems on torus, i.e., the Riemann surface of genus one). In the present paper curve Γ is not rectifiable, but this fact does not change topological nature of the problem, and, as a result, we obtain single solvability condition

$$\iint\limits_{D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z} = 0 \quad \text{or} \quad \iint\limits_{P\setminus D} \frac{\partial \tilde{g}}{\partial \overline{z}}dz \wedge d\overline{z} = 0. \tag{9}$$

In what follows we call it cyclic condition.

3. THE RIEMANN PROBLEM

Now let us solve the Riemann boundary value problem in class of doubly periodic functions satisfying condition (4). We consider first homogeneous problem

$$\Phi^+(t) = G(t)\Phi^-(t), \quad t \in \mathbf{\Gamma}, \tag{10}$$

where $G \in H_\nu(\Gamma)$ is given function, and it does not vanish on Γ.

We use here the Weierstrass σ-function

$$\sigma(z) := z \prod_{h \neq 0} \left(1 - \frac{z}{h}\right) \exp\left(\frac{z}{h} + \frac{z^2}{2h^2}\right),$$

where $h = m\tau_1 + n\tau_2$, $m, n \in \mathbb{Z}$, and the product is taken for all periods h excluding $h = 0$. It is odd entire function, and its unique null in domain P is simple and lies at the origin. Therefore, the coefficient $G(t)$ is representable as $G(t) = \sigma^\varpi(t) \exp f(t)$, where $f \in H_\nu(\Gamma)$ and $2\pi\varpi$ is decrement of $\arg G(t)$ when traversing of Γ counterclockwise. In addition, for any z it satisfies equality $\sigma(z + \tau_j) = -\sigma(z)\exp[\eta_j(z + \tau_j^*/2)]$, where $\tau_j^* = \tau_{3-j}$, $j = 1, 2$ (see, for instance, [4]). Let $\nu > 1 - \mathfrak{m}^+(\Gamma)/2$. We put

$$X(z) := \sigma^\varpi(z) \exp F(z), \quad F(z) := \frac{1}{2\pi i}\left\langle \tilde{f}\,\overline{\partial}\chi^+, \zeta(t - z)\right\rangle.$$

Clearly, $X^+(t) = G(t)X^-(t)$ for $t \in \Gamma$, but it is doubly periodic if and only if $\varpi = 0$ and cyclic condition (9) fulfils for function f instead of g. The minus case is analogous. Thus, there is valid

Theorem 3. *Let $G \in H_\nu(\Gamma)$. Assume that it does not vanish, and exponent ν satisfies inequalities* (8). *Then homogeneous Riemann problem* (10), (1), (4) *for certain μ has non-trivial solution if and only if $\varpi = 0$ and function f satisfies cyclic condition. If these conditions are fulfilled, then general solution contains single arbitrary constant, and otherwise identical zero is unique solution of the problem.*

According [4], we consider now doubly periodic Riemann problem for functions with poles of orders $n_1, n_2, \ldots, n_m$ at m prescribed points. As a result, we obtain the following theorems.

Theorem 4. *Assume that coefficient $G(t)$ does not vanish and belongs to Hölder class $H_\nu(\Gamma)$ with exponent ν satisfying conditions* (8). *Let us fix a value μ from interval* $(\mathrm{dm_H}\,\Gamma - 1; 2(1-\nu)/\mathfrak{m}(\Gamma))$. *Then the following propositions are valid for homogeneous Riemann problem* (10), (1), (4) *in the class of functions with poles of orders lesser or equal* $n_1, n_2, \ldots, n_m$ at *m prescribed points $z_1, z_2, \ldots, z_m$.*

i. *Let $\kappa := \varpi + n_1 + n_2 + \cdots + n_m > 0$. Then the problem has κ linearly independent solutions.*
ii. *Let $\kappa = 0$. Then the problem has non-zero solution if and only if the cyclic condition fulfils, and linear space of its solutions in this case is one-dimensional.*

iii. Let $\kappa < 0$. Then the problem has zero solution only.

Theorem 5. *Let $G, g \in H_\nu(\Gamma)$. Assume that $G(t)$ does not vanish, and exponent ν satisfies inequalities* (8). *Fix a value μ from interval* $(\mathrm{dm_H}\,\Gamma - 1; 2(1-\nu)/\mathfrak{m}(\Gamma))$. *Then the following propositions are valid for Riemann problem* (10), (1), (4) *in the class of functions with poles of orders lesser or equal* $n_1, n_2, \ldots, n_m$ at *m prescribed points $z_1, z_2, \ldots, z_m$.*

i. *Let $\kappa > 0$. Then the problem is solvable, and dimension of affine space of its solutions is κ.*

ii. *Let $\kappa = 0$. Then the problem has either unique solution or one-parametric family of solutions depending on fulfilment of condition of cyclic type.*

iii. *Let $\kappa < 0$. Then the problem is solvable if and only if g satisfies $-\kappa$ solvability conditions, and under these conditions solution is unique.*

Proof of the last two theorems is analogous to considerations of the book [4], but instead of the Cauchy type integral and its analogs we apply here constructions of previous sections and Dolzhenko theorem [17].

The results of sections 1 and 2 belong to D.B. Katz. All other results are obtained by all three authors in cooperation.

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