MAPPING CONNECTING CRITIAL VALUES OF RATIONAL FUNCTIONS TO THEIR CRITICAL POINTS AND POLES

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Every rational function R in the complex plane $\mathbb{C}$ can be represented in the form

$$R(z) = C_0 \int_{z_0}^{z} \frac{\prod_{k=1}^{M}(\zeta - a_k)^{m_k - 1}\, d\zeta}{\prod_{j=1}^{N}(\zeta - b_j)^{n_j+1}} + C_1. \tag{1}$$

Here a_k be its critical points of order m_k and b_j be poles of order n_j. Denote by $A_k = R(a_k)$ critical values of R. It is well known that R defines a Riemann surface S, i.e., branched covering of the Riemann sphere non-ramified over $\mathbb{C} \setminus \{A_1, \dots, A_M\}$. Given S, an important problem is to determine a rational function R uniformizing S. In essence, it is equivalent to the problem of finding parameters a_k and b_j in (1). Without loss of generality we can assume that $C_0 = 1$, $z_0 = a_1$, $C_1 = A_1$, and

$$\sum_{k=1}^{M}(m_k - 1)a_k - \sum_{j=1}^{N}(n_j + 1)b_j = 0. \tag{2}$$

Consider the set $\mathfrak{P}$ in $\mathbb{C}^{M+N}$ consisting of all $(a_1, \dots, a_M, b_1, \dots, b_N)$ satisfying (2) and the conditions

$$\operatorname{res}_{z=b_l} \frac{\prod_{k=1}^{M}(\zeta - a_k)^{m_k-1}}{\prod_{j=1}^{N}(\zeta - b_j)^{n_j+1}} = 0, \quad 1 \le l \le N,$$

and the mapping $f : \mathfrak{P} \to \mathbb{C}^M$ defined by

$$(a_1, \dots, a_M, b_1, \dots, b_N) \mapsto (A_1, \dots, A_M).$$

Theorem. *The set $\mathfrak{P}$ is an M-dimensional submanifold in $\mathbb{C}^{M+N}$. The mapping f is non-degenerate at every point of $\mathfrak{P}$ and the differential of the locally inverse mapping is given by the formulas:*

$$da_l = \frac{H_l^{(m_l-1)}(a_l)}{(m_l-1)!}\, dA_l + \sum_{k=1, k\neq l}^{M} \frac{G_{kl}^{(m_k-2)}(a_k)}{(m_k-2)!}\, dA_k, \; db_j = \sum_{k=1}^{M} \frac{I_{kj}^{(m_k-2)}(a_k)}{(m_k-2)!}\, dA_k,$$

where

$$H_l(x) = \frac{\prod_{j=1}^{N}(x - b_j)^{n_j+1}}{\prod_{k=1, k\neq l}^{M}(x - a_k)^{m_k-1}}, \quad G_{kl}(x) = \frac{H_k(x)}{x - a_l}, \quad I_{kj}(x) = \frac{H_k(x)}{x - b_j}.$$

The theorem allows us to find approximately rational functions uniformizing given Riemann surface over the Riemann sphere (see [1] for the case of simple critical points).

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References

[1] Nasyrov S.R. Uniformization of simply connected compact Riemann surfaces by rational functions // Proc. of VIII Petrozavodsk Int. Conf. 'Complex Analysis and Appls'. Petrozavodsk, Petr. Univ., 2016. P. 54–56.