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Affiliated subspaces and infiniteness of von Neumann algebras

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We show that the structural properties of von Neumann algebras are connected with the metric and order theoretic properties of various classes of affiliated subspaces. Among others we show that properly infinite von Neumann algebras always admit an affiliated subspace for which (1) closed and orthogonally closed affiliated subspaces are different; (2) splitting and quasi-splitting affiliated subspaces do not coincide. We provide an involved construction showing that concepts of splitting and quasi-splitting subspaces are non-equivalent in any GNS representation space arising from a faithful normal state on a Type I factor. We are putting together the theory of quasi-splitting subspaces developed for inner product spaces on one side and the modular theory of von Neumann algebras on the other side.

1 Introduction

Various classes of subspaces in inner product spaces have been intensely studied since interesting results of Piron [9] and Amemiya nad Araki [1]. These authors discovered a connection between order theoretic properties of subspace classes in a pre Hilbert space and metric completeness (see [5], [6]). Recently, a few new directions have appeared in this research line. New type of a subspace, called quasi-splitting subspace, was introduced and studied by Buhagiar, Chetcuti, Weber and others [3], [4]. On the other hand, the investigation of subspace classes invariant with respect to a given von Neumann algebra has brought a new look at the link between the properties of subspace structures and structure theory of von Neumann algebras. In particular, based on initial analysis in [10], [14], [15], the authors have shown in [8], that von Neumann algebra is properly infinite if, and only if, closed affiliated subspaces and operator closed affiliated subspaces coincide in the GNS representation space generated by a faithful normal state. In this note we would like to continue this analysis in the following new directions. Firstly, we consider general spaces affiliated with von Neumann algebras rather than spaces resulting from GNS construction only. Secondly, for the first time we study quasi-splitting affiliated subspaces.

The paper is organised as follows. In the second section we study orthogonally closed affiliated subspaces. Among others we show that for any properly infinite von Neumann algebra there is an affiliated subspace for which closed and orthogonally closed subspaces differ. This is in contrast with abelian von Neumann algebras for which the above mentioned classes always coincide. In the third part we characterize quasi-splitting affiliated subspaces. Attention is also paid to affiliated quasi-splitting subspaces generated by normal faithful semifinite weights on Type I factors which have bounded Radon-Nikodym derivatives with respect to the canonical trace. By an involved construction we show that for infinite dimensional Type I factor the quasi-splitting and slitting

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