ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2023, Vol. 44, No. 4, pp. 1256–1263. © Pleiades Publishing, Ltd., 2023.

Symmetries of the Black–Scholes–Merton Equation for European Options

L. N. Bakirova[1*], M. A. Shurygina[2], and V. V. Shurygin Jr.[3***]**

(Submitted by M. A. Malakhaltsev)

[1]ICL Group, Kazan, 420029 Russia

[2]Tupolev Kazan National Research Technical University—KAI, Kazan, 420111 Russia

[3]Lobachevskii Institute of Mathematics and Mechanics, Kazan (Volga Region) Federal University, Kazan, 420008 Russia

Received November 29, 2022; revised January 22, 2023; accepted February 7, 2023

Abstract—The aim of the present paper is the clarification of the result of A. Paliathanasis, K. Krishnakumar, K.M. Tamizhmani and P.G.L. Leach on the symmetry Lie algebra of the Black–Scholes–Merton equation for European options.

DOI: 10.1134/S1995080223040042

Keywords and phrases: *symmetries of PDEs, Black–Scholes–Merton equation.*

1. MAIN RESULT

The Black–Scholes–Merton (BSM) [1, 2, 5] model is one of the most important concepts in modern financial theory. It is used for the valuation of stock options, taking into account the impact of time and other risk factors.

The classical BSM model is described by a second order PDE

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0.$$

In recent years the number of papers is devoted to the determining the Lie algebra of symmetries of some PDEs that generalize this model. First of all we mention the papers of Gazizov and Ibragimov [4] and Sinkala, Leach, and O'Hara [7]. The paper of Bozhkov and Dimas [3] solves the problem of group classification of the generalized BSM equation

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x + f(u) = 0,$$

where $f(u)$ is an arbitrary smooth function.

Paliathanasis, Krishnakumar, Tamizhmani, and Leach [6] considered the BSM equation for European options with stochastic volatility for which the premium term depends only upon the return-to-risk ratio. It has the form

$$\frac{1}{2}f^2(y)x^2u_{xx} + \rho\beta x f(y)u_{xy} + \frac{1}{2}\beta^2 u_{yy} + rxu_x + \left(\alpha(m-y) - \beta\rho\frac{\mu - r}{f(y)}\right)u_y - ru + u_t = 0, \quad (1)$$

where t, x, y are independent variables, $u = u(t, x, y)$ is the value of the option, $f(y)$ is an arbitrary smooth function, and r, ρ, m, μ, α, β are real parameters with $|\rho| < 1$.

*E-mail: Lbakir@mail.ru
**E-mail: m-akm@mail.ru
***E-mail: vshjr@yandex.ru

The authors calculate the Lie algebra of symmetries of this equation in the case when $f(y) = \text{const}$, and claim that in the case of a non-constant function $f(y)$ this Lie algebra is the direct sum of a three-dimensional commutative Lie algebra and the infinite dimensional commutative Lie algebra which corresponds to the solutions on the equation. It should be mentioned that there are some misprints in their paper.

In the present paper we add some corrections to this result. We show that this PDE admits additional symmetries also in the case when

$$f(y) = \frac{k}{y-m}, \quad k = \text{const}.$$

For this case we denote

$$g = 2\left(\alpha + \frac{\rho\beta(\mu - r)}{k}\right).$$

The computations were performed using the `Maple` packages `DifferentialGeometry` and `JetCalculus` by I.M. Anderson.

Theorem. *For arbitrary function $f(y)$ the equation (1) admits the Lie symmetries*

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x}, \quad X_3 = u\frac{\partial}{\partial u}, \quad X_b = b(x,y,t)\frac{\partial}{\partial u},$$

where $b(x,y,t)$ is the solution of (1). Moreover, in the case when $f = f_0 = const$, it admits three additional symmetries

$$X_4 = e^{-\alpha t}\frac{\partial}{\partial y}, \quad X_5 = f_0^2\left(\rho^2 + \alpha t\right)x\frac{\partial}{\partial x} + f_0\rho\beta\frac{\partial}{\partial y} + \frac{1}{2}\alpha\left(t(f_0^2 - 2r) + 2\ln x\right)u\frac{\partial}{\partial u},$$

$$X_6 = 2\beta f_0^2\rho e^{\alpha t}x\frac{\partial}{\partial x} + \beta^2 f_0 e^{\alpha t}\frac{\partial}{\partial y} - 2e^{\alpha t}\left(\alpha f_0(m-y) + \beta\rho(r-\mu)\right)u\frac{\partial}{\partial u}.$$

In the case $f(y) = \dfrac{k}{y-m}$ and $g \neq 0$ the equation (1) takes the form

$$\frac{1}{2}f^2(y)x^2u_{xx} + \rho\beta x f(y)u_{xy} + \frac{1}{2}\beta^2u_{yy} + rxu_x + \frac{1}{2}g(m-y)u_y - ru + u_t = 0$$

and admits two additional symmetries

$$X_4 = e^{-gt}\left(rx\frac{\partial}{\partial x} - \frac{1}{2}g(y-m)\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + ur\frac{\partial}{\partial u}\right),$$

$$X_5 = e^{gt}\left(\frac{x}{\beta}\left(\rho g k + \beta r\right)\frac{\partial}{\partial x} + \frac{1}{2}g(y-m)\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + \frac{u}{2\beta^2}\left(g^2(y-m)^2 + \beta^2(2r-g)\right)\frac{\partial}{\partial u}\right).$$

In the case $f(y) = \dfrac{k}{y-m}$ and $g = 0$ the equation (1) takes the form

$$\frac{1}{2}f^2(y)x^2u_{xx} + \rho\beta x f(y)u_{xy} + \frac{1}{2}\beta^2u_{yy} + rxu_x - ru + u_t = 0$$

and admits two additional symmetries

$$X_4 = \frac{1}{2\beta}\left(\rho k + 2\beta r t\right)x\frac{\partial}{\partial x} + \frac{y-m}{2}\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + rtu\frac{\partial}{\partial u},$$

$$X_5 = \frac{xt}{\beta}\left(\rho k + \beta r t\right)\frac{\partial}{\partial x} + t\left(y-m\right)\frac{\partial}{\partial y} + t^2\frac{\partial}{\partial t} + \frac{u}{2\beta^2}\left((y-m)^2 + \beta^2(2rt^2 - t)\right)\frac{\partial}{\partial u}.$$

2. INVARIANT SOLUTIONS

In this section, we apply the Lie symmetries in order to reduce the equation (1) and to construct the invariant solutions. To do this, we need to use 2-dimensional Lie subalgebras. As it is mentioned in [6], solutions in which u does not depend upon some of the independent variables, are not interesting. Therefore, the reductions are performed with modified symmetry vectors like $X_1 + k_1 X_3$, $X_2 + k_2 X_3$ and some other.

2.1. The Case $f = f_0 = const$

In this subsection we repeat the results of A. Paliathanasis et al. with minor corrections. The Lie Brackets of the Lie algebra are given in the table:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-\alpha X_4$	$f_0^2\alpha X_2 + \alpha\left(\frac{1}{2}f_0^2 - r\right)X_3$	αX_6
X_2	0	0	0	0	αX_3	0
X_3	0	0	0	0	0	0
X_4	αX_4	0	0	0	0	$2\alpha f_0 X_3$
X_5	$-f_0^2\alpha X_2 - \alpha\left(\frac{1}{2}f_0^2 - r\right)X_3$	$-\alpha X_3$	0	0	0	0
X_6	$-\alpha X_6$	0	0	$-2\alpha f_0 X_3$	0	0

Throughout this subsection we denote $Y_1 = X_1 + k_1 X_3$, $Y_2 = X_2 + k_2 X_3$.

1) Algebra $\{Y_1, Y_2\}$. The invariant solution has the form $u(x, y, t) = e^{k_1 t} x^{k_2} w(y)$, where $w(y)$ satisfies the equation

$$\beta^2 f_0 w'' + 2(\rho\beta f_0^2 k_2 + \alpha(m - y)f_0 + \beta\rho(r - \mu))w' + ((k_2 - 1)(f_0^2 k_2 + 2r) + 2k_1)w = 0.$$

Its solution is expressed via Kummer functions M, U, so finally we get

$$u(x, y, t) = e^{k_1 t} x^{k_2}(C_1 M(\gamma, \tfrac{1}{2}, z) + C_2 U(\gamma, \tfrac{1}{2}, z)),$$

where

$$z = \frac{(k_2\rho\beta f_0^2 + \beta\rho(r - \mu) - \alpha f_0(y - m))^2}{\alpha\beta^2 f_0^2}, \quad \gamma = \frac{k_2(1 - k_2)f_0^2 + 2r(1 - k_2) - 2k_1}{4\alpha}.$$

2) Algebra $\{Y_2, X_4 + kX_3\}$. The invariant solution has the form $u(x, y, t) = \exp(ke^{\alpha t}y)x^{k_2}w(t)$, where $w(t)$ satisfies the equation

$$2f_0 w' + \left(\beta^2 k^2 f_0 e^{2\alpha t} + 2k\beta\rho\left(k_2 f_0^2 + \frac{\alpha m f_0}{\rho\beta} - (\mu - r)\right)e^{\alpha t} + (k_2 - 1)f_0(k_2 f_0^2 + 2r)\right)w = 0.$$

Its solution is

$$w(t) = C\exp\left(-\frac{\beta^2 k^2}{4\alpha}e^{2\alpha t} - \frac{k\beta\rho}{\alpha f_0}e^{\alpha t}\left(f_0^2 k_2 + \frac{\alpha f_0 m}{\beta\rho} - \mu + r\right) - \frac{1}{2}t(k_2 - 1)(f_0^2 k_2 + 2r)\right).$$

3) Algebra $\{Y_2, X_6 + kX_3\}$. The invariant solution has the form

$$u(x, y, t) = \exp\left(\frac{\alpha}{\beta^2}y^2 - \left(\frac{2\alpha f_0 m + ke^{-\alpha t}}{\beta^2 f_0} + \frac{2\rho(r - \mu + f_0^2 k_2)}{\beta f_0}\right)y\right)x^{k_2}w(t),$$

where $w(t)$ satisfies the equation

$$2\beta^2 f_0^2 w' + (f_0^2\beta^2(2\alpha + (k_2 - 1)(f_0^2 k_2 + 2r))$$

$$+ 2k(k_2\beta f_0^2\rho + \alpha f_0 m + \beta\rho(r-\mu))e^{-\alpha t} + k^2 e^{-2\alpha t})w = 0.$$

Its solution is

$$w(t) = C\exp\Big(\frac{1}{4\alpha\beta^2 f_0^2}\big(k^2 e^{-2\alpha t} + 4k(k_2\beta f_0^2\rho + \alpha f_0 m + \beta\rho(r-\mu))e^{-\alpha t} - 2t\alpha\beta^2 f_0^2(2\alpha + (k_2-1)(f_0^2 k_2 + 2r))\big)\Big).$$

4) Algebra $\{X_5, X_4 + kX_3\}$. The invariant solution has the form

$$u(x,y,t) = \exp\left(kye^{\alpha t} + \frac{\alpha\ln^2 x}{2f_0^2(\rho^2+\alpha t)}\right)x^{\psi(t)}w(t),$$

where

$$\psi(t) = -\frac{2k\beta\rho f_0 e^{\alpha t} + (2r - f_0^2)t\alpha}{2f_0^2(\rho^2+\alpha t)}.$$

The function $w(t)$ is the solution of the linear first order ODE

$$8f_0^2(\rho^2+\alpha t)^2 w' + \Big(4e^{2\alpha t}k^2 f_0^2\beta^2(\alpha^2t^2 + \rho^2 - \rho^4) + 4e^{\alpha t}\big(kf_0\alpha^2(\beta f_0^2\rho + 2f_0\alpha m - 2\rho\beta\mu)t^2 + k\alpha\rho^2 f_0(\beta f_0^2\rho + 4\alpha f_0 m + 2\beta\rho(r-2\mu))t + (2kf_0^2\alpha m\rho^4 - 2k\beta\rho^3 f_0(\rho^2(\mu - r) + r) - 1)\big) - \big(\alpha^2(2r + f_0^2)^2t^2 + 2\alpha(f_0^4\rho^2 + 2f_0^2(2r\rho^2 - \alpha) + 4r^2\rho^2)t + 4f_0^2\rho^2(2r\rho^2 - \alpha)\big)\Big)w = 0.$$

The final form of w is too cumbersome to give it here.

5) Algebra $\{X_5, X_6\}$. The invariant solution has the form

$$u(x,y,t) = x^{\psi(t)}\exp(\varphi(x,y,t))w(t),$$

where we denoted

$$\psi(t) = \frac{\big(t(r - \frac{1}{2}f_0^2)\beta - 2\rho f_0(m-y)\big)\alpha + 2\beta\rho^2(\mu - r)}{f_0^2(\rho^2 - \alpha t)\beta},$$

$$\varphi(x,y,t) = \alpha\beta^2\ln^2 x - 2yf_0\big((\rho^2+\alpha t)(2m-y)\alpha f_0 + \alpha\beta\rho(f_0^2 - 2\mu)t + 2\rho^3\beta(r-\mu)\big).$$

The function $w(t)$ has the form

$$w(t) = \frac{C\exp\left(\dfrac{\xi(t)}{\alpha\beta^2 f_0^2(\alpha t - \rho^2)}\right)}{\sqrt{\alpha t - \rho^2}},$$

where

$$\xi(t) = \frac{1}{8}\Big(\alpha\beta^2(\alpha t - \rho^2)\big((2r + f_0^2)^2 - 8f_0^2\alpha\big) + \rho^2\big(\beta\rho(f_0^2 - 4\mu + 2r) + 4\alpha f_0 m\big)^2\Big).$$

2.2. The Case $f = \dfrac{k}{y-m}$, $g \neq 0$

For convenience we will use $g = 2\left(\alpha + \dfrac{\rho\beta(\mu - r)}{k}\right)$ instead of α. The Lie Brackets of the Lie algebra are given in the table:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	$-gX_4$	gX_5
X_2	0	0	0	0	0
X_3	0	0	0	0	0
X_4	gX_4	0	0	0	X
X_5	$-gX_5$	0	0	$-X$	0

where we denoted

$$X = 2gX_1 + \Big(\frac{g^2\rho k}{\beta} + 2r\Big)X_2 - \frac{1}{2}g(g-4r)X_3.$$

1) Algebra $\{X_1 + aX_3, X_2 + bX_3\}$. The invariant solution has the form $u(x,y,t) = e^{at}x^b w(y)$, where $w(y)$ satisfies the second order ODE

$$\beta^2(y-m)^2 w'' + (2\beta\rho bk - g(y-m)^2)(y-m)w' + \big(2(a + r(b-1))(y-m)^2 + k^2 b(b-1)\big)w = 0.$$

Its solution can be expressed via Whittaker functions $M_{\kappa\nu}, W_{\kappa\nu}$

$$w(y) = \exp\left(\frac{g(y-m)^2}{4\beta^2}\right)(y-m)^\gamma\left(M_{\kappa\nu}\left(\frac{g(y-m)^2}{2\beta^2}\right) + W_{\kappa\nu}\left(\frac{g(y-m)^2}{2\beta^2}\right)\right), \quad \gamma = -\frac{1}{2} + \frac{kb\rho}{\beta},$$

and

$$\kappa = \frac{(\beta + 2kb\rho)g + 4\beta(a + r(b-1))}{4g\beta}, \quad \nu = \frac{1}{4\beta}\sqrt{\beta^2 + 4k(k-\rho\beta)b - 4k^2b^2(1-\rho^2)}.$$

2) Algebra $\{X_1 + aX_3, X_4\}$. The invariant solution has the form

$$u(x,y,t) = e^{at}x^{1-a/r}w(h(t,y)), \quad h(t,y) = x^g(y-m)^{2r}.$$

The function $w(h)$ satisfies the second order Euler equation

$$r^2(4\beta^2r^2 + g^2k^2 + 4rkg\rho\beta)h^2w''$$
$$- r(k^2g(2a - g(r+1)) + 4r\rho\beta(a - g(r+1))k - 2r^2\beta^2(2r-1))hw' + ak^2(a-r)w = 0.$$

3) Algebra $\{X_2 + bX_3, X_4\}$. The invariant solution has the form

$$u(x,y,t) = x^b(y-m)^\gamma w(h(t,y)), \quad h(t,y) = e^{gt}(y-m)^2, \quad \gamma = \frac{2r(b-1)}{g}.$$

The function $w(h)$ satisfies the second order Euler equation

$$4\beta^2g^2h^2w'' + 2g\beta(g(2bk\rho + \beta) + 4r\beta(b-1))hw'$$
$$+ (b(k^2g^2 + 4r^2\beta^2 + 4kgr\rho\beta) - 2r\beta^2(2r+g))w = 0.$$

4) Algebra $\{X_2 + bX_3, X_5 + pX_3\}$. The invariant solution has the form

$$u(x,y,t) = x^b(y-m)^\gamma\exp(\varphi(t,y))w(h(t,y)),$$

where we denoted

$$h(t,y) = (y-m)^2e^{-gt}, \quad \varphi(t,y) = -\frac{g^2y(2m-y) + 2p\beta^2e^{-gt}}{2g\beta^2}; \quad \gamma = -\frac{(g - 2r + 2rb)\beta + 2b\rho kg}{g\beta}.$$

The function $w(h)$ satisfies the second order ODE

$$4g^2\beta^2h^2w'' - 2\beta g(g\beta + 2b\rho kg + 4\beta r(b-1))hw' + (2pg^2h + 6g\beta^2rb - 8\beta^2r^2b + 4\beta^2r^2b^2 - 6g\beta^2r$$
$$+ k^2b^2g^2 - k^2bg^2 + 4b^2\rho\beta kgr + 2g^2\beta^2 + 4\beta^2r^2 + 4g^2\beta b\rho k - 4b\rho\beta rkg)w = 0.$$

Its solution is

$$w(h) = h^{\gamma}\left(C_1 J_n\left(\frac{\sqrt{2ph}}{\beta}\right) + C_2 Y_n\left(\frac{\sqrt{2ph}}{\beta}\right)\right),$$

where

$$\gamma = \frac{(4r(b-1)+3g)\beta + 2bkg\rho}{4g\beta}, \quad n = \frac{1}{2\beta}\sqrt{(4(\rho^2-1)k^2b^2 - 4k(4\beta\rho - k)b + \beta^2}$$

and J_n, Y_n are the Bessel functions.

5) Algebra $\{X_2 + aX_1, X_4\}$. The invariant solution has the form

$$u(x,y,t) = (y-m)^{-2r/g} w(h(x,y,t)), \quad h(x,y,t) = e^{gt}x^{-ag}(y-m)^{2(1-ar)}.$$

The function $w(h)$ satisfies the second order Euler equation

$$g^2((4\beta^2r^2 + 4kgr\rho\beta + k^2g^2)a^2 - 4\beta(gk\rho + 2\beta r)a + 4\beta^2)h^2w'' + g(g(4\beta^2r^2 + 4kgr\rho\beta + k^2g^2)a^2 + (k^2g^2 + 8\beta^2r^2 + 4kgr\rho\beta - 6gr\beta^2 - 4kg^2\rho\beta)a + 2g\beta^2)hw' + 2r\beta^2(2r+g)w = 0.$$

6) Algebra $\{X_2 + aX_1, X_5\}$. The invariant solution has the form

$$u(x,y,t) = (y-m)^{-1+2r/g}\exp\left(\frac{gy(y-2m)}{2\beta^2}\right)w(h(x,y,t)),$$

where

$$h(x,y,t) = \frac{(y-m)^{\gamma}e^{\beta gt}}{x^{a\beta g}}, \quad \gamma = 2(agk\rho + ar\beta - \beta).$$

The function $w(h)$ satisfies the second order Euler equation

$$g^4\beta^2(a^2(4r^2\beta^2 + 4rgk\beta\rho + k^2g^2) - 4a\beta(gk\rho + 2\beta r) + 4\beta^2)h^2w'' + g^3\beta(4g(ar-1)^2\beta^3 + 2(ar-1)(2ag^2k\rho + 4r - 3g)\beta^2 + agk(akg^2 + 4\rho(r-g))\beta + k^2g^2a)hw' + (g-r)(g-2r)w = 0.$$

2.3. The Case $f = \dfrac{k}{y-m}, g = 0$

The Lie Brackets of the Lie algebra are given in the table:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	$X_1 + rX_2 + rX_3$	$-\frac{1}{2}X_3 + 2X_4$
X_2	0	0	0	0	0
X_3	0	0	0	0	0
X_4	$-X_1 - rX_2 - rX_3$	0	0	0	X_5
X_5	$\frac{1}{2}X_3 - 2X_4$	0	0	$-X_5$	0

1) Algebra $\{X_1 + aX_3, X_2 + bX_3\}$. The invariant solution has the form $u(x,y,t) = e^{at}x^bw(y)$, where $w(y)$ satisfies the second order ODE

$$\beta^2(y-m)^2w'' + 2\beta\rho bk(y-m)w' + \big(2(a + r(b-1))(y-m)^2 + k^2b(b-1)\big)w = 0.$$

Its solution can be expressed via Whittaker functions

$$w(y) = (y-m)^{-bk\rho/\beta}(C_1M_{0,\nu}(z) + C_2W_{0,\nu}(z)),$$

where

$$z = \frac{2\sqrt{(2-2b)r-2a}(y-m)}{\beta}, \quad \nu = \frac{\sqrt{4b(1-b+b\rho^2)k^2+\beta(\beta-4bk\rho)}}{2\beta}.$$

2) Algebra $\{X_2 + bX_3, X_4\}$. The invariant solution has the form

$$u(x,y,t) = x^b(y-m)^\gamma \exp(\varphi(t,y))w(h(t,y)),$$

where we denoted

$$h(t,y) = \frac{t}{(y-m)^2}, \quad \varphi(t,y) = \frac{r(b-1)(2m-y)ty}{(y-m)^2}, \quad \gamma = -\frac{b\rho k}{\beta}.$$

The function $w(h)$ satisfies the second order ODE

$$4\beta^2h^2w'' + 2(4rm^2\beta^2(b-1)h^2 + 3\beta^2h + 1)w'$$
$$+ \big(4m^4r^2\beta^2(b-1)^2h^2 + 6rm^2\beta^2(b-1)h + (k^2b + 2m^2r)(b-1) + bk\rho(\beta - bk\rho)\big)w = 0.$$

Its solution can be expressed via Kummer functions

$$w(h) = e^{-m^2hr(b-1)}h^{-\gamma}\left(C_1M\left(\gamma, 2\gamma + \frac{1}{2}, \frac{1}{2\beta^2h}\right) + C_1U\left(\gamma, 2\gamma + \frac{1}{2}, \frac{1}{2\beta^2h}\right)\right),$$

where

$$\gamma = \frac{\beta + \sqrt{4(\rho^2-1)k^2b^2 + 4k(k-4\rho\beta)b + \beta^2}}{4\beta}.$$

3) Algebra $\{X_2 + bX_3, X_5\}$. The invariant solution has the form

$$u(x,y,t) = x^b(y-m)^\gamma \exp(\varphi(t,y))w(h(t,y)),$$

where we denoted

$$h(t,y) = \frac{t}{y-m}, \quad \varphi(t,y) = \frac{y(y-m)}{2\beta t} - \frac{r(b-1)ty}{y-m}, \quad \gamma = -\frac{1}{2} - \frac{b\rho k}{\beta}.$$

The function $w(h)$ satisfies the second order ODE

$$4\beta^4h^4w'' - 4\beta^2h^2(2\beta^2rm(b-1)h^2 - 3\beta^2h + m)w' + \big(4\beta^4r^2m^2(b-1)^2h^4 - 12\beta^4rm(b-1)h^3$$
$$+ \beta^2(4kb(k(b-1) - kb\rho^2 - \rho\beta) + 4m^2r(b-1) + 3\beta^2)h^2 - 2\beta^2mh + m^2\big)w = 0.$$

Its solution is

$$w(h) = (C_1h^{\lambda_1} + C_2h^{\lambda_2})\exp\left(\frac{m(2r\beta^2(b-1)h^2-1)}{\beta^2h}\right),$$

where λ_1, λ_2 are the roots of the quadratic equation

$$\lambda^2 + 2\lambda + \frac{1}{4\beta^2}\big(4(1-\rho^2)b^2k^2 + 3\beta^2 - 4k^2b + 4bk\beta\rho\big) = 0.$$

4) Algebra $\{X_4, X_5\}$. The invariant solution has the form

$$u = x\sqrt{rt}\exp\left(\frac{(y-m)^2}{2\beta^2t}\right)(y-m)^{-1-k\rho/\beta}w(h),$$

where $h(t,x,y) = k\rho\ln(y-m) + \beta(rt - \ln x)$, and the function $w(h)$ satisfies the second order ODE with the constant coefficients

$$k^2\beta^2(1-\rho^2)w'' + k\beta(k + \beta\rho - 2k\rho^2)w' + (\beta + k\rho)(2\beta - k\rho)w = 0.$$

REFERENCES

1. F. Black and M. Scholes, "The valuation of option contracts and a test of market efficiency," J. Financ. **27**, 399–417 (1972).
2. F. Black and M. Scholes, "The pricing of options and corporate liabilities," J. Polit. Econ. **81**, 637–659 (1973).
3. Y. Bozhkov and S. Dimas, "Group classification of a generalized Black–Scholes–Merton equation," Commun. Nonlin. Sci. Numer. Simul. **19**, 2200–2211 (2014). https://doi.org/10.1016/j.cnsns.2013.12.016
4. R. K. Gazizov and N. H. Ibragimov, "Lie symmetry analysis of differential equations in finance," Nonlin. Dyn. **17**, 387–407 (1998).
5. R. C. Merton, "On the pricing of corporate data: The risk structure of interest rates," J. Financ. **29**, 449–470 (1974).
6. A. Paliathanasis, K. Krishnakumar, K. M. Tamizhmani, and P. G. L. Leach, "Lie symmetry analysis of the Black-Scholes-Merton model for European options with stochastic volatility," Mathematics (Spec. Iss.: Math. Finance) **4** (2), 28 (2016). https://doi.org/10.3390/math4020028.
7. O. Sinkala, P. Leach, and J. O'Hara, "Invariance properties of a general bond-pricing equation," J. Diff. Equat. **244**, 2820–2835 (2008).