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Research Article

Oscillating Cosmological Solutions in the Modified Theory of Induced Gravity

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This work is the extension of author's research, where the modified theory of induced gravity (MTIG) is proposed. In the framework of the MTIG, the mechanism of phase transitions and the description of multiphase behavior of the cosmological scenario are proposed. The theory describes two systems (stages): Einstein (ES) and "restructuring" (RS). This process resembles the phenomenon of a phase transition, where different phases (Einstein's gravitational systems, but with different constants) pass into each other. The hypothesis that such transitions are random and lead to stochastic behavior of cosmological parameters is considered. In our model, effective gravitational and cosmological "constants" arise, which are defined by the "mean square" of the scalar fields. These parameters can be compared with observations related to the phenomenon of dark energy. The aim of the work is to solve equations of MTIG for the case of a quadratic potential and compare them with observational cosmology data. The interaction of fundamental scalar fields and matter in the form of an ideal fluid is introduced and investigated. For the case of Friedmann-Robertson-Walker space-time, numerical solutions of nonlinear MTIG equations are obtained using the qualitative theory of dynamical systems and mathematical computer programs. For the case of a linear potential, examples joining of solutions, the ES and RS stages, of the evolution of the cosmological model are given. It is shown that the values of such parameters as "Hubble parameter" and gravitational and cosmological "constants" in the RS stage contain solutions oscillating near monotonically developing averages or have stochastic behavior due to random transitions to different stages (RS or ES). Such a stochastic behavior might be at the origin of the tension between CMB measurements of the value of the Hubble parameter today and its local measurements.

1. Introduction

This work is related to research in the field of the theory of gravity and cosmology in connection with existing problems given below.

(1) The difference in the values of the cosmological constant obtained from astrophysical observations and predictions of the general relativity theory (GRT), taking into account the quantum effects of vacuum polarization, is known in science as the "problem of the cosmological constant" (see [1]). The acuity of this problem reinforces the fact that this difference is huge - 10^{120}.

(2) There is a problem of "accuracy of measurement of the gravitational constant" G. For example, in the International System of Units (SI), for 2008: $G = 6,67428{\times}10^{-11} m^3 c^{-2} kg^{-1}$; the value of the gravitational constant was obtained in 2000 (Cavendish Experiment): $G = 6,67390 \times 10^{-11}$; in 2010, the value of G was corrected: $G = 6,67384 \times 10^{-11}$; in 2013 a group of scientists from the International Bureau of Weights and Measures: $G = 6,67545 \times 10^{-11}$; in 2014, the value of the gravitational constant recommended by CODATA became as follows: $G = 6,67408{\times}10^{-11}$. In fact, G is not determined even with an accuracy of the fourth decimal place.

The Λ Cold Dark Matter model ΛCDM represents the current standard model in cosmology. Within this, there is a tension between the value of the Hubble constant, H0, inferred from local distance indicators (the predicted value given in article [2] is $H_{local} = 73.48{\pm}1.66\ km{\cdot}s^{-1}Mpc^{-1}$), and the angular scale of fluctuations in the Cosmic Microwave Background (CMB) (as follows from [3] $H_{CMB} = 67.0 \pm 1.2\ km \cdot s^{-1} Mpc^{-1}$). These two independent measurements give a discrepancy of approximately 9% and tension with

Planck+ΛCDM increases to 3.7 sigma [2–4]. It also follows from the above that the measurement accuracy of H_{local} is about 4.5%.

In our work we present a model where, due to the oscillatory regime in the solutions of equations, the Hubble parameter also fluctuates with respect to the mean value, which is also a function of time. We investigate a mechanism associated with the nonlocal behavior of a gravitational system with scalar fields in the classical approximation.

(3) P. A. M. Dirac put forward the principle that all large numbers are determined by the lifetime of the Universe and change with it [5, 6]. The Dirac hypothesis gave impetus to the development of a program for studying the possible dependence of physical constants on cosmological time. Dirac developed his hypothesis based on the modernization of Einstein's theory by introducing an additional scalar field. In the 1970s he managed to shape his ideas into a conformally invariant scalar-tensor theory of gravity. Earlier, in the 1950-60s, other variants of scalar-tensor theories were created, for example, the Brans-Dicke scalar-tensor theory [7–9], based on the use of the Mach principle to explain the force of gravitational interaction as caused by the mass and size of the Universe. P. Jordan [10], J. Narlikar [11], and a number of other physicists worked in this direction. New versions of this kind of theories appear until recently, for example, modified scalar-tensor theories of gravity. The main idea of the modified theory of gravity is to eliminate the cosmological constant and the exact solution to the problem of dark matter. The theory alters the laws of the very large (on a cosmological scale) distances and lengths of time. The main idea of the modified theory of gravity is to eliminate the cosmological constant and to solve the problem of dark matter. The theory alters the very laws of large-scale (at cosmological scales) distances and lengths of time. In the MTIG, we additionally study the evolution of the gravitational constant, since it is not initially introduced into the theory but is obtained due to a fixed solution of the scalar field. Thus, MTIG is directly related to the problems of the so-called "dark energy" (DE) and "dark matter" (DM).

The theoretical challenge posed by these problems triggered many attempts to directly change Einstein's gravity at large distances [12]. An example of such an infrared (IR) modification is the *DGP* brane-world model [13–15]. In [12–16], authors put forward the idea that if gravity is sufficiently weakened in the infrared, then vacuum energy could effectively decouple from gravity or degravitate over time. As shown in [17, 18], the current Planck data used is best suited to the model in a nonplanar ΛDGP. The author considers closed and open cosmological models.

Our theory is a phenomenological model used for comparison with observational data DE and DM. Within the framework of modified theory of induced gravity (MTIG), proposed in the works of [19–21], we attempted to solve the above problems based on the idea of the existence of macroscopic parameter of the theory $(X, X) = X^A X^B \eta_{AB} \equiv Y$, which generates both gravitational and cosmological "constants":

$$\begin{aligned} k_{eff} &= \pm \frac{wc^3}{16\pi\xi(X,X)\hbar} \equiv G_{eff}\frac{c^3}{8\pi\hbar}, \\ \Lambda_{eff} &= \frac{1}{2\xi Y}\left(-B + U_{eff}\right), \\ n &= 4, \end{aligned} \tag{1}$$

where $\hbar$ is Planck's constant, c is speed of light, $B = B_0(n-2)/2 - w\varepsilon_v$, ε_v is vacuum energy B_0, w, ξ are constants of the theory, and $U_{eff} = U_{eff}(Y)$ is effective potential of the theory.

We are going to compare the experimental value of the gravitational constant G_m with the effective "gravitational constant."

$$G_m \equiv \frac{8\pi k_n \hbar}{c^3} \equiv 6.565362 \cdot 10^{-65} cm^2 = \pm\frac{w}{2\xi C_m}, \tag{2}$$

where C_m is the current value of the function $Y = Y(t_m)$, $k_n = G$ is the gravitational constant of Newton, the value of which is $6.674286 \times 10^{-8} cm^3 c^{-2} g^{-1}$, and t_m is a time parameter corresponding to the current value (approximately 13.8 billion years). Similarly, in accordance with astrophysical observational data, the modern value of the cosmological constant is assumed to be equal to $\Lambda_m = 1.27143 \cdot 10^{-56}/cm^2$.

Functions $X^A = X^A(\sigma^\mu)$, where $A, B = 1, 2, \ldots, D$, $\mu, \nu = 0, 1, \ldots, n-1$, represent n-dimensional Riemannian manifold Π described by the metric $g_{\mu\nu}$, into D-dimensional flat space-time M with the metric η_{AB} [19]. It is convenient to leave the signature of the plane space M arbitrary. For further calculations we set $n = 4$.

For a cosmological model (a similar model is constructed for a centrally symmetric space as well), the mechanism proposed by us reduces to the fact that the differential equations describing the evolution of the functions $Y(t)$ and $a(t)$ have the form

$$\begin{aligned} \dot{Y} \cdot \left(\Phi_1(Y, a)\right) &= 0; \\ \Phi_2(Y, a) &= 0, \end{aligned} \tag{3}$$

where $\Phi_1(Y, a)$ $\Phi_2(Y, a)$ are some expressions of functions $Y(t)$ and the cosmological scale factor $a(t)$ and their derivatives up to the second order. For $Y(t) = const$, the second equation goes to the equation matching with the equation in general relativity, and the first equation disappears. Thus, there are solutions that can both match and not match with the solutions of the standard theory of gravity. Then the fundamental "constants" of theory, such as gravitational and cosmological ones, can evolve in time and also depend on coordinates. In a fairly general case, the theory describes two systems (stages): Einsteinian (ES stage), when $Y = const$ and equation (3) disappears, and "restructuring" (RS stage), when $\Phi_1(Y, a) = 0$. This process resembles the phenomenon of a phase transition, where different phases (Einstein's gravitational systems, but with different constants) pass into each other. So far we cannot present the final mathematic mechanism of a computable description of such transitions. We can only indicate the "favorable" points at which such transitions are possible. These are the moments of time when the second derivative of the scale factor $a(t)$ or the first derivative of $Y(t)$ equals zero. In this paper we show that the values of the observed characteristics of the gravitational field are affected not only by the values of the gravitational constants, but also, for the most part, by their derivatives.

In [20] to solve Problem (1), we considered two mechanisms for reducing the constant part of the vacuum energy ε_{vac}. In the first variant, the value ε_{vac} is compensated by other terms $(-B_0 + U_{eff})/(2\xi Y)$ from Λ_{eff}. The reduction of two values imposes requirements on the constants of the theory $(w, \xi, C_0,)$ to the accuracy of high orders. The second mechanism for reducing the constant part of the vacuum energy reduces to the multiplicative reduction. Its principle is simple and is based on the law of conservation of energy in phase transitions corresponding to different stages of the evolution of the universe and the structure of the theory [20].

Note that this mechanism for reducing the vacuum energy is analogical to the mechanism for reducing the divergences in the quantum renormalization theory, despite the fact that the theory under consideration is classic. In [20] the influence of matter in the form of perfect fluid on the behavior of $Y(t)$ was studied. We can say that, in the cosmology based on the MTIG, the principle of the "whole" Universe (slightly analogous to Mach's principle) is realized to some extent, which reduces to the existence of a certain parameter Y, which in turn depends on all material fields and generates physical "constants."

In this theory we consider the influence of the quadratic, standard potential on the solutions of the RS stage. In our opinion, solutions containing anharmonic oscillations caused by random initial and boundary conditions are of special interest. Unlike the solutions of the Einstein equation with the asymptotics of flat space-time, the presence of a variable "cosmological term" leads to a nonlocal self-interaction of the field Y. Fluctuations with a complex spectrum impose monotonically varying solutions (e.g., cited in [20]. Such behavior leads to fluctuations in the parameters relative to their mean classical values. Thus, we propose the hypothesis that the value of such parameter as the gravitational "constant" G, apart from the slow evolution in the RS stage, can fluctuate near the classical value. For example, Figures 1, 2(a), and 2(b) show the normalized numerical solutions $Z(t) \equiv Y(t)/Y_0$, $b(t) \equiv a(t)/a_0$, and $H(t) \equiv \dot{b}/b$, where $x \equiv t/a_0$, of equations (cited in this article).

First of all, we require the consistency of the results of the theory (after comparison with observational data) and then look for the predictive possibilities of the theory. Proceeding from this, the choice of the parameters of the theory should lead to fluctuations affecting the fourth (maximum to third) order of the solution $Y(t)/Y_0 \simeq 1,000m_1m_2..$, interpreted as the modern value of this parameter. Of course, back in time, the values of the parameters could have been substantially different and these parameters implemented qualitatively different stages of evolution (Figures 1 and 2). The graph of the scale factor shown in Figure 1 can be reconciled with the results of observational data (e.g., for calculating the Hubble constant), if the 9 percent difference obtained by measuring the CMB (corresponding to the early Universe) and local measurements is attributed to the average (for fluctuations) graph and 4.5 percent of measurement errors are attributed to the local fluctuations themselves.

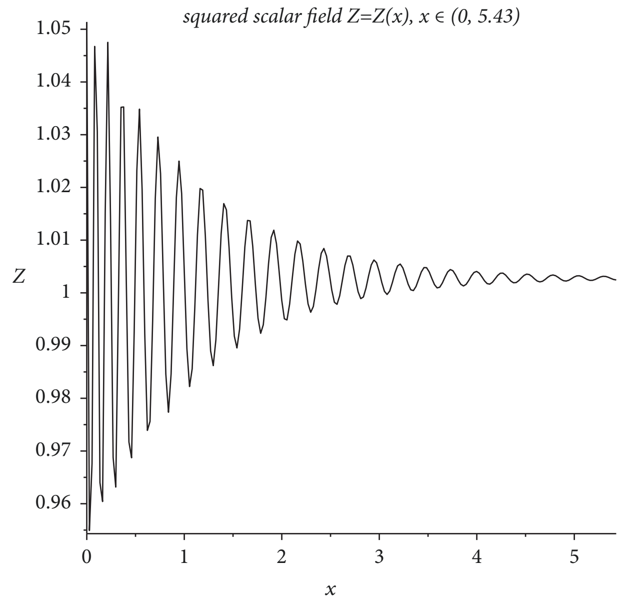


Figure 1: Relation of $Z = Y/Y_0$, $Z(1) = 1.0041022218$; $x = t/t_m$ is current age of the Universe. $\widetilde{B}_n = 144.517022939$, $k_2 = -0.4$.

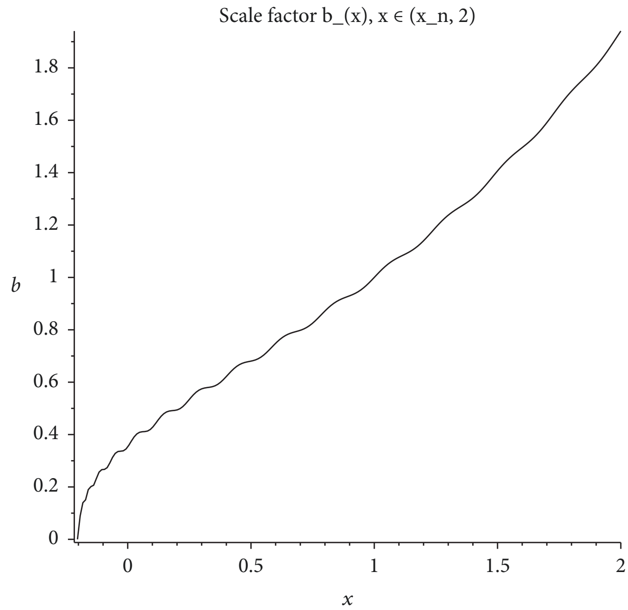
Research on numerical solutions for the case of a centrally symmetric space has been made in [22]. On the one hand, found solutions agree with the observational data; on the other hand, at far distances from the center (more than 0.01 parsecs from the Sun and more than one kiloparsec from the center of the galaxy) they can lead to a significantly different astronomical picture. We are going to inform about these decisions in the next article. Comparison of the solutions of the cosmological model and the model in the case of a centrally symmetric space leads to the fact that DM and DE can be described in a single concept.

2. Description of the Phenomenological Model MTIG

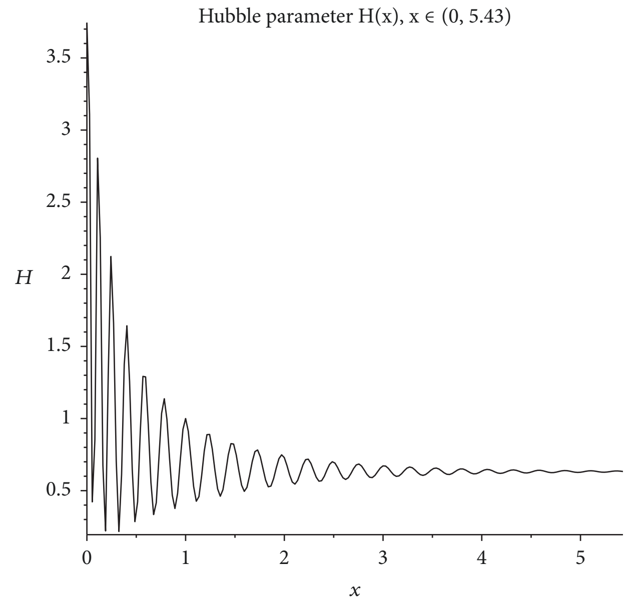
2.1. Equations of MTIG. Below are the main ideas of the theory described in our next works [19, 20].

There are arguments in favor of the theory of physical fields that it must have the property of conformal invariance, at least at the classical level, up to the time when this symmetry is not broken. The action for the membrane ($n-1$-branes) admits no conformal transformations. To avoid this difficulty while remaining within the ideology of the string theory, in previous papers we proposed the following generalization of string theory [19].

$$S_0 = \frac{1}{w} \cdot \int \left\{ -\frac{1}{2} \left(\nabla_\nu X, \nabla^\nu X\right) + \xi R(X, X) + U + L_m(X, S) \right\} \cdot \sqrt{-g} \hat{d}^n \sigma. \quad (4)$$



(a) Relation of $b(x) = a(x)/a_m$, $Z(1) = 1.0041022218$; $x = t/t_m$ is current age of the Universe. $\widetilde{B}_n = 144.517022939$, $k_2 = -0.4$.



(b) Relation of $H(x) = \dot{b}(x)/b(x)$, $Z(1) = 1.0041022218$; $H(1) = 1$ is the current value of the Hubble constant. $\widetilde{B}_n = 144.517022939$, $k_2 = -0.4$.

Figure 2

In (4) the following notation is used: $Y \equiv (X,X) = X^A X^B \eta_{AB}$, $(\nabla_\nu X, \nabla^\nu X) = \nabla_\nu X^A \nabla_\mu X^B g^{\nu\mu}\eta_{AB}$, and $U = U(X^A)$ is the potential depending on the fields X^Ap; R is scalar curvature of manifold Π. The operator ∇_ν means covariant derivative in a manifold Π, where the Christoffel symbols are connected with the metric in a standard way. The variables w, ξ are arbitrary constants.

For simplicity, in this paper $U(X^A) = U(Y(X^A))$. $L_m(X,S)$ characterizes all possible interactions X^A with other fields of matter.

In the context of our paper, many modified scalar-tensor theories of gravitation can be transformed to the form (4), without taking into account the Einstein term absent in (4). Note that in [23–27], some classes of modified gravity, considered as a gravitational alternative to dark energy, were presented.

The "induced gravity" means that the initial action of the Einstein's term $R/(2\kappa)$ is not explicitly introduced. The introduction of such term initially at first violates the conformal invariance of the theory and secondly leads to the instability of the known solutions because of the emergence of the effective gravitational "constant"

$$\frac{1}{\kappa_{eff}} = \frac{1}{\kappa} + 2\xi(X,X) \Longrightarrow 0. \tag{5}$$

At the point when it tends to zero, additional singularities arise. For example, in the case of one scalar field (φ), instability arises for the conformally invariant case ($\xi = \xi_0$) [28]. In connection with what has been said, I want to point out the problem connected with the sign of (42) and (2). Unlike single scalar field φ (where analogy of (X,X) is $\varphi^2 > 0$), the sign of (X,X) is undefined. Also, there is not enough data to compare with the proposed theory and to define all the parameters of the theory.

For a system with matter, the following self-consistent equations were obtained [20, 29]:

$$\begin{aligned} G_{\alpha\beta} = {} & \frac{1}{2\xi Y}\left[-\frac{n-2}{2}B + U\right] g_{\alpha\beta} + \frac{1}{Y}\left[\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box\right] Y \\ & - \frac{w}{2\xi Y} T_{(e)\alpha\beta}, \end{aligned} \tag{6}$$

where $G_{\alpha\beta}$ is the Einstein tensor; $T_{(e)\alpha\beta}$ is the Energy-Momentum Tensor (EMT) of matter fields (e.g., perfect fluid).

The consequence of these equations is the law of conservation of energy, which has the form

$$-\frac{n-2}{2}\nabla_\beta B + \nabla_\beta Y \cdot \left(\xi R + \frac{dU}{dY}\right) - w\nabla_\alpha T^\alpha_{(e)\beta} = 0, \tag{7}$$

and the equation on the field Y

$$\begin{aligned} \Box Y = {} & \frac{n-2}{4(n-1)\xi}\left[-nB + 2\xi R Y + \frac{2n}{n-2}U\right] \\ & - \frac{w}{2\xi(n-1)} T^\alpha_{(e)\alpha} \end{aligned} \tag{8}$$

Equation (6) is an analogue of Einstein's equations for a macroscopic medium.

In the derivation of macroscopic equations by varying the fields g, X, the following assumptions were made.

(1) The induced metric (mapping) $(\nabla_\mu X, \nabla_\nu X)$ is related to the metric of the manifold M by means of formula

$$B_0 g_{\mu\nu} = \left(\nabla_\mu X, \nabla_\nu X\right) \quad \mu,\nu = \overline{0, n-1}, \tag{9}$$

so, this model allows us to interpret the development of Universe as development of $n = 4$ dimensional objects embedded in the multidimensional flat space-time M. Self-consistency of the equations is ensured by introducing additional currents S^A.

(2) Due to interaction with vector fields, the equation for scalar fields acquires an additional term S^A. Then these equations have the form

$$\Box X^A + 2\xi R X^A + 2\frac{dU}{dY} X^A = S^A. \tag{10}$$

The specific form of this term depends on the model. Thus, from a mathematical point of view, the solution of the inverse problem is assumed. Solving the macroscopic equation (6), we find the metric $g_{\mu\nu}$ and the field Y. Then, solving equations (9) and (10) we find X^A, S^A. Our approach is similar to the method of finding the unknown potentials given in [30, 31].

The condition of "embedding" (9), which was used in deriving equations, does not greatly restrict the proposed theory. Indeed, in the most general case, we can make the substitution $(\nabla_\mu X, \nabla_\nu X) = B_0 g_{\mu\nu} + k_{\mu\nu}$, where $k_{\mu\nu}$ are some tensor functions. If you do the calculations described above, it can be shown that the resulting equations have the same form as in (6)-(8); in these equations redefine EMT matter.

The scalar field in this theory is $\Phi = 16\pi\xi Y/w$. Equation (6), except for the first term on the right side of the equation, is the same as a special case of Brans-Dicke.

In order to take into account the effect of vacuum polarization energy into gravity, we highlighted from EMT matter a part related to this energy, which satisfies the equation of state $\varepsilon_{vac} + p_v = 0$, where ε_{vac} and p_v are interpreted as the energy density and vacuum polarization pressure. Therefore, in the equations (except for (9)) to made a substitution$B_0 \Longrightarrow B$,

$$B = \frac{n-2}{2} B_0 - w\varepsilon_{vac}. \tag{11}$$

The action (4) has the property of conformal invariance if $\xi = \xi_0$, $(\delta_\xi = 0)$, $U(X^A) = U_0 \equiv \Lambda(X,X)^2$, where

$$\begin{aligned} \xi_0 &= -\frac{n-2}{8(n-1)}, \\ \xi &\equiv \xi_0 - \frac{\delta_\xi}{4(n-1)}. \end{aligned} \tag{12}$$

For dimension $n = 4$: $\delta_\xi = -12\xi - 1$.

Also note that (6) can be obtained by varying the action of

$$S_1 = \frac{1}{w}\int \left\{\xi R Y + \widetilde{U}(Y) + L_m(Y,S)\right\}\sqrt{-g} d^n\sigma, \tag{13}$$

by the metric g, where $\widetilde{U}(Y) = U(Y) - ((n-2)/2)B$ is the potential.

This invariance is expressed in the fact that the equations obtained by varying the action (4) with respect to the fields $\widehat{g}$ and $\widehat{X}$ are invariant under the local Weyl scale changes

$$\begin{aligned} g_{\mu\nu} &\Longrightarrow \exp(2\phi)\, g_{\mu\nu}, \\ X^A &\Longrightarrow \exp(4\xi_0(n-1)\phi)\, X^A, \end{aligned} \tag{14}$$

for an arbitrary function $\phi = \phi(\sigma^\mu)$.

The condition (9) for $B_0 = const$ limits the conformal invariance to scale transformation. Thus, fixing the function B_0 (in general, this parameter can be a function of coordinates) results in fixing the scale of the theory. In our notation, B_0 is a dimensionless quantity. The fields X^A, interpreted as the coordinates of the space M, have the dimension of a centimeter, which implies $[Y] = cm^2$ and $[w] = cm^4$. Action (4) is a dimensionless quantity ($\hbar = 1$, $c = 1$).

The authors of [32] investigated the possibility that probably observed additional dark radiation has an origin associated with the scale invariance.

In general case, we get the systems of "macroscopic" equations (6)-(8), "microscopic" equations (10), and constraint equations (9). The study of the complete system of equations requires the definition of the model, that is, definition of the functions S^A. The fixed sector of the fields $\{X^1, X^2, \ldots X^k\}$, $k < D$ can play the role of Higgs scalar fields. It is proper to consider the function Y as the averaged field (the vacuum mean in the tree approximation) by analogy with the mechanism of spontaneous symmetry breaking (the Higgs mechanism). So the previous formulas should be understood in the following sense: $Y = \langle 0|(X,X)|0\rangle \simeq (\langle 0|X|0\rangle, \langle 0|X|0\rangle)$,

$$B_0 g_{\mu\nu} \simeq \left(\nabla_\mu \langle 0|X|0\rangle, \nabla_\nu \langle 0|X|0\rangle\right) \quad \mu,\nu = \overline{0, n-1}. \tag{15}$$

The latter can be interpreted in the sense that "geometry" is created by vacuum averages.

We note the works of Claudia de Rham and his colleagues [23, 24], in which cosmological models with scalar fields, including branes, were studied, taking into account quantum effects.

2.2. Different Types of Solutions. From (7) it follows that three types of solutions are possible as follows:

(I) $Y = C = const$, $\nabla_\beta T^\beta_{(e)\alpha} = 0$. Note also that $Y = C = const, B = const. \Longrightarrow \nabla_\beta T^\beta_{(e)\alpha} = 0$. In this case, we obtain equations that match with the Einstein equations, with the gravitational constant $G_{eff} = const$ and with the cosmological constant $\Lambda_{eff} = const$. Equations (10) can be rewritten as

$$\Box X^A + \left(4\frac{B-U}{C} + 2\frac{dU}{dY} + \frac{w}{C} T^\alpha_{(e)\alpha}\right) X^A = S^A. \tag{16}$$

For the cosmological model with the EMT of perfect fluid and the potential $U = U_0 \equiv \Lambda(X,X)^2$, free fields $(S^A = 0)$ X^A acquire mass μ, when $\mu^2 = -4(B/C) + (w/C)(\varepsilon - 3P)$, where ε, P is the density of energy and pressure.

(II) $Y \neq const$, *and a separate conservation law for matter is fulfilled*: $\nabla_\beta T^\beta_{(e)\alpha} = 0$. In this case, from (7) it follows that

$$\xi R + \frac{dU}{dY} = 0. \tag{17}$$

Equations (10) can be rewritten as

$$\square X^A = S^A \tag{18}$$

Free fields ($S^A = 0$) X^A have zero mass.

(III) When $Y \neq const, B = const$ *separate law of conservation of matter is not necessarily fulfilled*. This case is a generalization of the previous case. The law of conservation takes the form

$$\nabla_\beta Y \cdot \left(\xi R + \frac{dU}{dY}\right) = w \nabla_\beta T^\beta_{(e)\alpha}. \tag{19}$$

3. Cosmological Solutions

3.1. Cosmological Vacuum Solutions: Y=const. Let us consider the above equations under potential

$$U = \Lambda_X (X, X)^2 + f_w (X, X) \equiv \Lambda_X Y^2 + f_w Y, \tag{20}$$

$n = 4$; $\rho = 2$; $B_0 = const$, for the case of a homogeneous, isotropic cosmological model.

The metric form of the manifold Π has the form

$$ds^2 = -dt^2 + a^2(t)\left((d\chi)^2 + K(\chi)\, d\Omega^2\right), \tag{21}$$

when $K(\chi) = \{\sinh^2\chi; \sin^2\chi; \chi^2\}$, respectively, for the models of open, closed, and flat types. $d\Omega^2$ is the metric form of a sphere, with a unit radius, expressed in spherical coordinates.

$Y = C = const$. For the case of vacuum ($S^A = 0$, $T_{(e)\alpha\beta} = \varepsilon_{vac} g_{\alpha\beta}$, and $B = B_0 - w\varepsilon_{vac}$.) (6), (7), and (9) can be analytically solved.

The equations for the scale factor have the form

$$\begin{gathered} \dot a^2(t) = -k + h_0^2 a^2(t), \\ h_0^2 = -\frac{\Lambda C^2 - B + f_w C}{6\xi C} = \frac{\Lambda_{eff}}{3}, \end{gathered} \tag{22}$$

$k = -1, 1, 0$ are open, closed, and flat types of spaces, respectively.

The equations for the fields X^A take the following form:

$$\ddot X + 3\frac{\dot a}{a}\dot X + \frac{(3 + l)\,k}{a^2} X - \left(4\frac{B}{C} - 2f_w\right) X^A = 0, \quad l \in N. \tag{23}$$

Here, l eigenvalues for the three-dimensional Laplacian $\triangle_3 X = -(3 + l)k$.

Particular solutions ($l = 0$) of these equations are found in [19] that satisfy the conditions of "immersion" (9). For the closed model, these solutions have the form

$$a(t) = \frac{\cosh(th_0)}{h_0}. \tag{24}$$

$$X^5 = \frac{\sinh(th_0)}{h_0}, \quad X^a = a(t)\, k^a, \tag{25}$$

where k^a is immersion function of 3-dimensional sphere:

$$\begin{aligned} k^1 &= \sin\chi \sin\theta \cos\phi, \\ k^2 &= \sin\chi \sin\theta \sin\phi, \\ k^3 &= \sin\chi \cos\theta, \\ k^4 &= \cos\chi. \end{aligned} \tag{26}$$

And for the open type of space

$$a(t) = \frac{\sinh(th_0)}{h_0}, \tag{27}$$

$$X^5 = \frac{\cosh(th_0)}{h_0}, \quad X^a = a(t)\,\tilde k^a, \tag{28}$$

where $\tilde k^a$ follows from k^a by replacing $\sin\chi, \cos\chi$ with $\sinh\chi, \cosh\chi$.

For the closed model manifold Π forms "one-sheeted hyperboloid" in a five-dimensional subspace of flat space M and is described by

$$\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 + \left(X^4\right)^2 - \left(X^5\right)^2 = \frac{q}{h_0^2}. \tag{29}$$

For the case of an open model surface equation has the form

$$\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 - \left(X^4\right)^2 + \left(X^5\right)^2 = \frac{q}{h_0^2}. \tag{30}$$

In the derivation of (27)-(30) we assumed that the matrix η_{AB} (the metric of the space M) dimension $D > 5$ is diagonal and this diagonal for the closed and open type of space has the form $(q, q, q, q, -q, q_1, .., q_{D-5})$, $(q, q, q, -q, q, q_1, .., q_{D-5})$.

For the given solutions, the conditions (9), (23), and $(X, X) = C$ define the relationship between the constants:

$$\Lambda_X C = \frac{B\left(3 + \delta_\xi\right)}{2C} - \frac{f_w\left(5 + \delta_\xi\right)}{4}, \tag{31}$$

which follows from the requirement of (23) for the functions (28).

Note that when $f_w = 0 \Longrightarrow \Lambda_{eff} = 3B/C$ does not depend on ξ; and when $\delta_\xi = -3 \Longrightarrow \Lambda_X = 0$.

From the requirement (9) $\Longrightarrow q = B_0$.

The condition $(X, X) = C$ for (28) leads to the relation

$$C = \frac{q}{h_0^2} = \frac{3(B_0)}{\Lambda_{eff}}. \tag{32}$$

From (22), (31), and (32) it follows that

$$\Lambda_{eff} = \frac{3B}{C} - \frac{3f_w}{2} = \frac{3B_0}{C}. \tag{33}$$

From (34) and $B = (B_0 - w\varepsilon_{vac})$ we get

$$\begin{aligned} \Lambda_{eff} &= \frac{3B_0}{C}, \\ w\varepsilon_{vac} &= -\frac{f_w C}{2}, \\ \Lambda_X C &= \frac{B_0\left(3 + \delta_\xi\right)}{2C} + \frac{w\varepsilon_{vac}}{C}. \end{aligned} \tag{34}$$

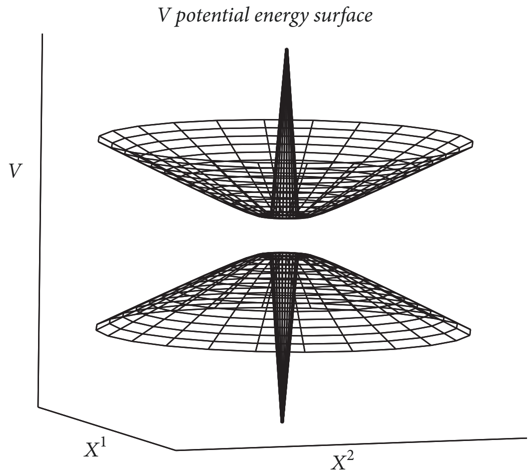


Figure 3: Schematic images of the potential energy V_1 in a relation to Y and $(X^1, X^2, ..)$ are shown.

Thus, if all embedding conditions of the Friedmann world into multidimensional flat space-time M are met, the cosmological constant does not depend on the polarization energy of the vacuum for the model constructed by us. The result obtained can be used to investigate the case of perturbations.

The above solutions correspond to the special case $S^A = 0$ without taking into account the interaction of the fields X^A with other fields. Then, the case (34) corresponds to the minimum of the potential $V_1 = (\Lambda_X Y + f_w - B/Y)/(6\xi)$ (Figure 3), included in (23) and (6) (in the particular case (22)).

There is a problem of defining the numerical values of the parameters of the theory. The number of essential parameters can be reduced to three. For them we use the following notation:

$$f_n = \frac{f_w}{6\xi}, \quad L_n = \frac{\Lambda_X C}{6\xi}, \quad B_n = \frac{B}{6C\xi}. \tag{35}$$

So that

$$L_n + f_n - B_n = -\frac{\Lambda_{eff}}{3}. \tag{36}$$

In order to generalize the case and for $S^A \neq 0$ and to reduce the number of parameters of the theory, we will not be limited to the model described by the solutions (22)-(34). For this we adopt the following arguments.

For minimum potential energy: $\dot{V}_1 = 0 \Longrightarrow \Lambda_X Y_{min} = -B/Y_{min} \Longrightarrow V_{1min} = -(2B/Y_{min} - f_w)/(6\xi)$. To agree with observational facts and taking into account (22), let us assume $V_{1min} = -\Lambda_{eff}/3 \Longrightarrow$

$$f_n = 2B_n - \frac{\Lambda_{eff}}{3}, \quad L_n = -B_n, \quad C \equiv Y_{min}. \tag{37}$$

Instead of selecting the ansatz (37) due to the small value of the observed cosmological constant $\Lambda_{mod} = \Lambda_{eff}$ at $t = t_0$, for $|\Lambda_X|C^2 \ll |B|$ two other ansatzes were considered as well:

$$f_n = 2B_n - 2\frac{\Lambda_{eff}}{3}, \quad L_n = -B_n + \frac{\Lambda_{eff}}{3}, \quad C \equiv Y_{min} - \tag{38}$$

corresponding to the minimum potential $V_2 = (\Lambda_X Y^2 + f_w Y - B)/(6\xi)$;

$$f_n = 2B_n, \quad \Lambda_X = 0 \tag{39}$$

This case is interesting because it is possible to obtain analytical solutions of differential equations, even if the substance is present in the form of a perfect fluid.

3.2. Cosmological Solutions with Matter. In [21] a phenomenological model was proposed. The model of the interaction of the field Y and matter in the form of a perfect fluid, with the density of energy and pressure,

$$\varepsilon = \frac{\varepsilon_{r0}}{a^4} + \frac{\varepsilon_{p0}}{a^3} + \left[\frac{\left(Yf_{r1} + Y^2 f_{r2}\right)}{a^4} + \frac{\left(Yf_{p1} + Y^2 f_{p2}\right)}{a^3}\right]. \tag{40}$$

$$p = \frac{\varepsilon_{r0}}{(3a^4)} + \left[\frac{\left(Yf_{r1} + Y^2 f_{r2}\right)}{(3a^4)}\right]. \tag{41}$$

Equations (6) and (7) take the form

$$\lambda^2 = -\frac{\dot{Z}}{Z}\lambda - \frac{k}{b^2} - Z\left(\widetilde{L}_n + \widetilde{F}_2\right) - \widetilde{f}_n - \widetilde{F}_1 + \frac{\widetilde{B}_n}{Z} - \frac{\widetilde{E}}{Z}, \tag{42}$$

$$\dot{Z}\left\{\dot{\lambda} + 2\lambda^2 + \frac{k}{b^2} + 2Z\left(\widetilde{L}_n + \widetilde{F}_2\right) + \widetilde{f}_n + \widetilde{F}_1\right\} = 0. \tag{43}$$

For convenience in computer modeling, dimensionless variables are introduced

$$x = \frac{t}{t_m}; \quad b = b(x) = \frac{a(x)}{a_m};$$

$$[5pt]Z = Z(x) = \frac{Y(x)}{C_0},$$
$$\lambda = \lambda(x) = \frac{\dot{b}}{b} \tag{44}$$

where t is proper time; dot denotes the derivative by x; $C_0 = Y(t_1)$ is some value of the field Y, which we associate with a constant solution $Y = const$, discussed above in particular ($C = C_0$); a_m is dimension value cm^2; it is convenient to correspond to the modern value of the scale factor or the age of the universe. In the first case $b(t_m) = 1$ corresponds to the modern value of the scale factor and t_m to the age of the universe. Such a scale is convenient if the required functions are expressed through the scale factor. However, we do not know the modern value of the scale factor, but we assume $t_m \sim 13.7 \cdot 10^9$ years. Therefore, when the desired functions are expressed in terms of time, we select t_m for the parameter a_m. Then $x = 1, b(1) = b_m$ correspond to the modern values of the parameters.

The last equation (taking into account the previous one) for $\dot{Z} \neq 0$ can be rewritten as

$$\dot{\lambda} = \frac{k}{b^2} + \frac{2\dot{Z}}{Z}\lambda + \tilde{f}_n + \tilde{F}_1 - 2\frac{\tilde{B}_n}{Z} + 2\frac{\tilde{E}}{Z}. \tag{45}$$

Here we have introduced the following notation:

$$\begin{aligned}
\tilde{L}_n &= L_n t_m^2,\\
\tilde{f}_n &= f_n t_m^2,\\
\tilde{B}_n &= B_n t_m^2;\\
\tilde{E} &= -\frac{\rho_p}{b^3} - \frac{\rho_r}{b^4};\\
\tilde{F}_1 &= -\frac{\mu_{p1}}{b^3} - \frac{\mu_{r1}}{b^4},\\
\tilde{F}_2 &= -\frac{\mu_{p2}}{b^3} - \frac{\mu_{r2}}{b^4},
\end{aligned} \tag{46}$$

and we also reparameterized constants taking into account their dimensions

$$\begin{aligned}
\rho_p &= -\frac{\varepsilon_{p0} w t_m^2}{(6\xi C_0 a_m^3)},\\
\rho_r &= -\frac{\varepsilon_{r0} w t_m^2}{(6\xi C_0 a_m^4)};\\
\mu_{p1} &= -\frac{f_{p1} w t_m^2}{(6\xi a_m^3)},\\
\mu_{r1} &= -\frac{f_{r1} w t_m^2}{(6\xi a_m^4)};\\
\mu_{p2} &= -\frac{f_{p2} C_0 w t_m^2}{(6\xi a_m^3)},\\
\mu_{r2} &= -\frac{f_{r2} C_0 w t_m^2}{(6\xi a_m^4)}.
\end{aligned} \tag{47}$$

It is interesting to compare the equations obtained from (42) and (45); in these equations we substitute $Z = const = Z_0$, with Einstein's equations with the same EMT, having the form

$$\lambda^2 + \frac{k}{b^2} = \gamma + \frac{8\pi}{3}G_{eff}\left|\tilde{\varepsilon}\right|, \tag{48}$$

$$\dot{\lambda} = \frac{k}{b^2} - 4\pi G_{eff}\left|\tilde{\varepsilon} + \tilde{p}\right|, \tag{49}$$

where

$$\begin{aligned}
G_{eff} &= \left|\frac{w}{16\pi\xi C_0 Z_0}\right|,\\
\gamma &= \frac{\tilde{B}_n}{Z_0} - \tilde{L}_n Z_0 - \tilde{f}_n;\\
\tilde{\varepsilon} &= -\frac{\varepsilon}{a_m},\\
\tilde{p} &= -\frac{p}{a_m}.
\end{aligned} \tag{50}$$

The first of these equations ((42) and (48)) will match, and (43) disappears. Equivalent (at $\dot{Y} \neq 0$) to (43), (45) does not match (49). We recall that in the case of Einstein's equations the second one is a differential consequence of the first.

Thus, as already indicated in previous works, in the proposed model, the evolution of the universe contains two stages that were named as "Einstein" (ES - stage) when $\dot{Y} = 0$ and "restructuring" (RS - stage) when $\dot{Y} \neq 0$. This process resembles the phenomenon of a phase transition, where different phases (Einstein's gravitational systems, but with different constants) pass into each other.

From a mathematical point of view, at any time the solutions of (42) and (43), describing the ES and RS stages, can pass into each other. To describe such solutions it is necessary to join functions of the scale factor $b(t)$ and the field $Z(t)$ and their first derivatives at the point t_1 corresponding to the moment of transition. These transitions are similar to the first-order phase transitions and apparently can be used to describe transition from the inflationary phase to the next phase [20]:

$$\begin{aligned}
a_e(t_1) &= a_r(t_1);\\
\dot{a}_e(t_1) &= \dot{a}_r(t_1);\\
Y(t_1) &= Y_0;\\
\dot{Y}(t_1) &= 0,
\end{aligned} \tag{51}$$

where the index "e" denotes the solutions $\dot{Y}(t) = 0$ or the corresponding ES stages and the index "r"- RS stages.

Transitions similar to the second-order phase transitions are described by the system, if the conditions (51) are supplemented by the condition of equality of second derivatives at the transition point:

$$\begin{aligned} a_e(t_1) &= a_r(t_1); \\ \dot{a}_e(t_1) &= \dot{a}_r(t_1); \\ \ddot{a}_e(t_1) &= \ddot{a}_r(t_1); \\ Y(t_1) &= Y_0; \\ \dot{Y}(t_1) &= 0; \\ \ddot{Y}(t_1) &= 0. \end{aligned} \tag{52}$$

A necessary condition for the existence of solution $Z(t) = Z_0 = const$ (if $Y_0 = C_0$, then $Z_0 = 1$), equations describing both ES and RS stages, is the fulfillment of the following conditions on the model parameters:

$$\begin{aligned} Z_0 \widetilde{f}_n &= 2\widetilde{B}_n; \\ \mu_{p1} Z_0 - \rho_p + 3\mu_{p2} Z_0^2 &= 0; \\ \mu_{r1} + 2\mu_{r2} Z_0 &= 0. \end{aligned} \tag{53}$$

3.3. The Case without Quadratic Terms. In [20, 21] in order to obtain analytical solutions, we consider a linear approximation of Y, so that

$$\widetilde{L}_n + \widetilde{F}_2 = 0. \tag{54}$$

In this paper we want to focus on the existence of nonstandard solutions related to the branching effect of solving equations.

Equation (43) is integrated and reduced to the form

$$\dot{Z}\left(\frac{\dot{b}^2}{b^2} + \frac{k}{b^2} + \frac{\widetilde{f}_n}{2} - \frac{2\mu_{p1}}{b^3} - \frac{F_0 + \mu_{r1}\ln\left((b/b_0)^2\right)}{b^4}\right) = 0, \tag{55}$$

where F_0, b_0 are integration constants.

We can prove that for the case of $\dot{Z} \neq 0$ the scale factor is the solution of (43), taking into account (54), and where $Z = Z(t)$ is found by the formula

$$Z = \dot{b}\left(c_2 + \int \frac{b}{\dot{b}^3}\left[\widetilde{B}_n - \widetilde{E}\right] db\right), \tag{56}$$

which follows from (42) and (43).

Let us consider the so-called "equilibrium state" in more detail. This state is obtained by applying the conditions (53) and the additional condition on the constant F_0: $F_0 \cdot Z_0 = \rho_r$. From (53), taking into account (54), it follows $\mu_{r1} = 0$. These conditions are obtained from the requirement of existence and matching solutions $Z = const$ of (55) and (42).

After substituting these values of the parameters, besides the condition $\mu_{r1} = 0$, the equations (for $\dot{Z} \neq 0$) can be reduced to the form

$$\dot{Z}\left(\frac{\dot{b}^2}{b^2} + \frac{k}{b^2} + \frac{1}{Z_0}\left(\widetilde{B}_n - \frac{2\rho_p}{b^3} - \frac{\rho_r}{b^4}\right) - \mu_{r1}\frac{\ln\left((b/b_0)^2\right)}{b^4}\right) = 0, \tag{57}$$

$$\dot{Z}\left(\frac{\dot{Z}\dot{b}}{b} + \frac{Z - Z_0}{Z_0}\left(\widetilde{B}_n + \frac{\rho_p}{b^3} + \frac{\rho_r}{b^4}\right) + Z\mu_{r1}\frac{\ln\left((b/b_0)^2\right) - 1}{b^4}\right) = 0. \tag{58}$$

It is of interest to study the influence of the logarithmic term in (57) and (58) on their solutions, so we left this term as some perturbation violating the solutions of the "equilibrium state."

The solution of (57) and (58), in the case $\mu_{r1} = 0$, is found by the formula

$$Z(x) = c_2 \cdot \dot{b} + Z_0, \quad c_2 = const. \tag{59}$$

The function $b = b(x)$ is defined as the solution of (57) (for $c_2 \neq 0$). Equation (58), taking into account (59) and (57), becomes their differential consequence.

Surprisingly, the solution for the scale factor does not depend on the constant c_2. This equation has the form

$$\frac{\dot{b}^2}{b^2} = -\frac{k}{b^2} + \gamma_0 + \frac{2\rho_p}{Z_0 b^3} + \frac{\rho_r}{Z_0 b^4}, \tag{60}$$

where $\gamma_0 = -\widetilde{B}_n/Z_0$ defines the cosmological constant (from which, presumably, $\widetilde{B}_n/Z_0 < 0$ follows). As a consequence, it follows from (60), (6), and the solution (59) that there are two "gravitational constants": $|w/2\xi Y_0|$, cosmological gravitational constant, and $|w/2\xi Y(t)|$, time-dependent function, (possibly) contributing to the gravitational interaction between massive bodies. In addition, the solution (59) is noteworthy by the fact that the transition between ES and RS stages takes place at the point when the first derivative $Z(t)$ and the second derivative of the scale factor $\dot{Z}(t_1) = c_2\ddot{b}(t_1) = 0$ equal zero. Thus, the transitions between the stages will be located in the vicinity of the special points ($\ddot{b}(t_1) = 0$) for the scale factor function. From this point of view, it is interesting to study all special points, including the equilibrium points $\dot{b}(t_1) = 0, \ddot{b}(t_1) = 0$. In the next section, we present a model of a quasistatic universe, where the scale factor fluctuates with respect to a constant value.

At $Z = Z_{cr} = const$, (57) and (58) vanish, and (42) takes form

$$\frac{\dot{b}^2}{b^2} + \frac{k}{b^2} + \widetilde{B}_n\left(\frac{2}{Z_0} - \frac{1}{Z_{cr}}\right) - \frac{\rho_p}{b^3}\left(\frac{1}{Z_0} + \frac{1}{Z_{cr}}\right) - \frac{\rho_r}{Z_{cr} b^4} - \frac{\mu_{r1}}{b^4} = 0. \tag{61}$$

Joining of solution (61) with (57) and (58) at some point $t = t_c r$ is performed at the equality of functions $b(x)$, $Z(x)$, and their first derivatives. As for the continuity of the first derivative of a function this function is not directly related to the four-dimensional geometry, although in Section 3 we were able to interpret this parameter as the "radius" of the four-dimensional hyperboloid (embedded in the five-dimensional space-time). In the vicinity of the transition point, we require the continuity of the function $b(x)$, as well as its first derivative. This requirement is associated with the requirement of energy conservation. We can separately consider the question of the continuity of the second derivatives of these functions. Then we can prove the following relations at the transition point $x = x_{cr}$:

$$\frac{\dot{Z}_r \cdot \dot{b}_r}{Z \cdot b} = \frac{\dot{b}_e^2 - \dot{b}_r^2}{b^2};$$
$$\frac{\ddot{Z}_r}{Z} = 2\frac{\ddot{b}_e - \ddot{b}_r}{b} - \frac{\dot{b}_e^2 - \dot{b}_r^2}{b^2}. \quad (62)$$

For greater clarity, (58) is reduced to the form

$$\dot{Z} \cdot \dot{b} = (Z - Z_0) \cdot \ddot{b} - \frac{Z_0 \cdot \mu_{r1}}{b^3}\left(\ln\left(\frac{b^2}{b_0^2}\right) - 1\right) \quad (63)$$

Let us denote by $F(x)$ the solution (58).

Let us consider the following conditions for joining of solutions at the transition point $x = x_{cr}$:

$$Z(x_{cr}) \equiv F(x_{cr}) = Z_{cr};$$
$$\dot{Z} \cdot \dot{b}_r \equiv (Z_{cr} - Z_0) \cdot \ddot{b}_r - \frac{Z_0 \cdot \mu_{r1}}{b_{cr}^3}\left(\ln\left(\frac{b_{cr}^2}{b_0^2}\right) - 1\right) \quad (64)$$
$$= 0,$$

where $b(x_{cr}) = b_{cr}$.

In spite of the fact that the values of parameters including the value $\mu_{r1} = 0$ were called the "equilibrium state," it is interesting to consider a more general case $\mu_{r1} \neq 0$. Moreover, there is a free parameter b_0, and we can demand that the term in (63) associated with the parameter μ_{r1} at the critical point $x = x_{cr}$ to equal zero. To do this, we must select $b_0 = b_{cr}/\sqrt{e}$.

It can be shown that transitions from the RS stage into ES stage are possible in the following form. For points $x < x_{cr}$: $Z(x) = F(x)$, where $b(x)$ is defined as the solution of (57); at the point $x = x_{cr}$: $\ddot{b}_r = 0$ and $Z_{cr} = Z(x_{cr})$; further for points $x > x_{cr}$:$Z(x) = Z_{cr} = const$, where $b(x)$ is defined as the solution of (61). If $Z_{cr} - Z_0 \neq 0$, then an inverse transition from ES stage into RS is possible, for example, at the point where the second derivative of the scale factor (varying by (61)) $\ddot{b}_e = 0$.

As an example, Figures 4 and 5 show the joining of solutions for the function $Z = Z(b)$ describing the transition from RS into ES (Figure 4) and with the possibility of double transition (Figure 5). The first point (A) is defined by the moment of time $x_{cr} \simeq 0.512625142$, $b(x_{cr}) \simeq 0.5665163348$, where $\ddot{b}_r(x_{cr}) = 0$, $Z_{cr} = Z(x_{cr})$; from the point (A) the system can evolve along two trajectories: d is transition

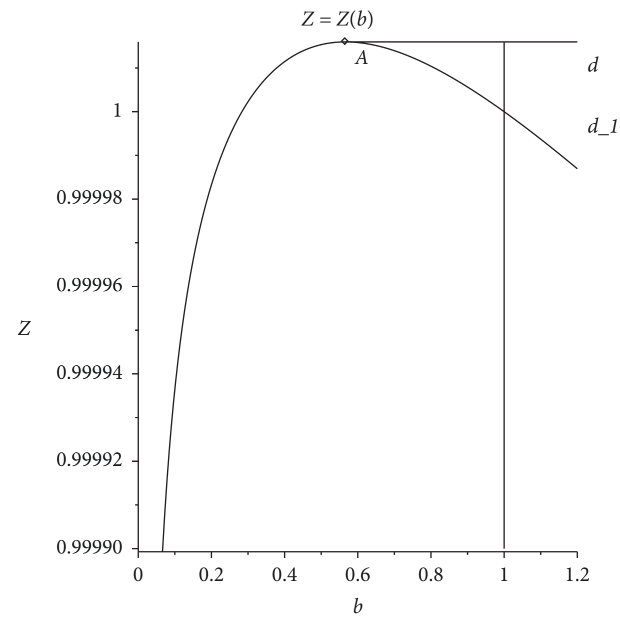


FIGURE 4: Plot of $Z = Y/Y_0$ and $b(t) = a(t)/a_0$: $b = 1$ is current age of the universe. After reaching the point A, the function Y (t) branches. It may evolve on the straight line d or the curve d_1.

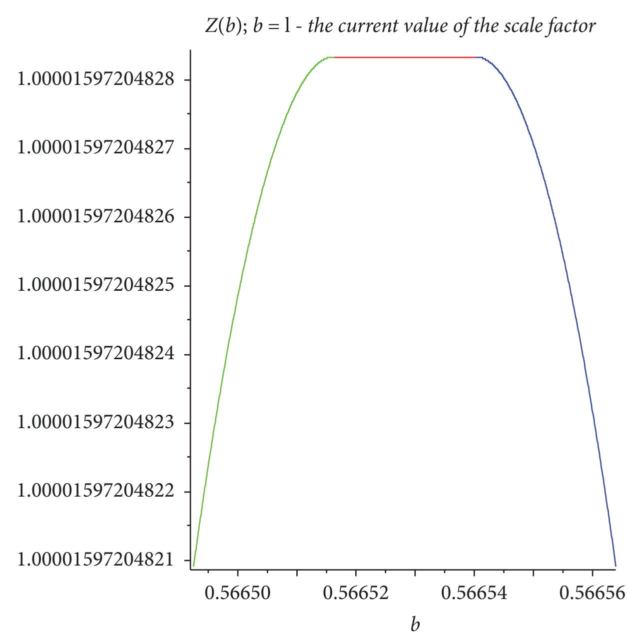


FIGURE 5: Plot of $Z = Y/Y_0$ and $b(t) = a(t)/a_0$: $b = 1$ is current age of the universe. Double transition from RS into ES and back from ES into RS.

into the ES stage or it continues to move along the same curve d_1 (remaining in RS). When the system transits and evolves according to (61), under certain initial conditions, the transition back into RS stage is possible, as shown in Figure 6. In the second transition in Figure 5, we associate it with the point x_2, which is defined by the condition $\ddot{b}_e(x_2) = 0$. The

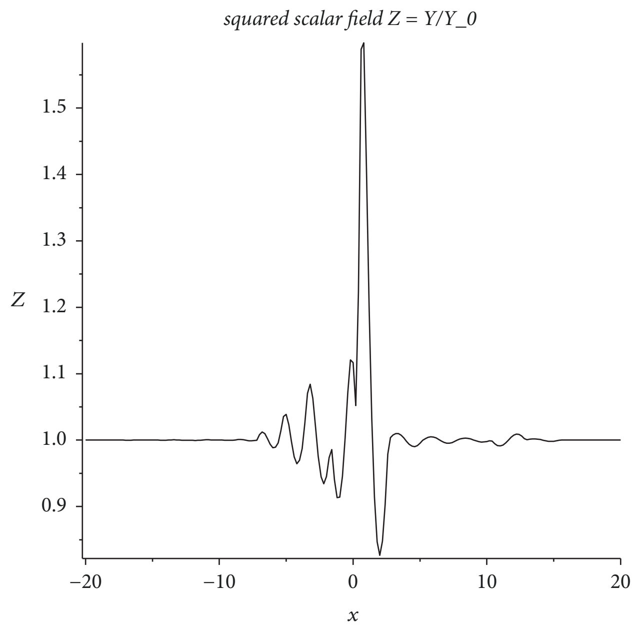


FIGURE 6: Relation of $Z = Z(x)$, where $Z(1) = Z_m = 1.40908675530186$: $x = t/t_m$ is current age of the Universe; $\widetilde{B}_n = 1.44517022939$, $k_2 = 0$.

lifetime of the ES stage: $\delta_x = x_2 - x_{cr} = 0.0000238$. If we assume that the scale factor is similar to the lifetime of the universe, then δ_x is about 400000 years. Assuming that the RS stage is currently continuing, the estimate of the rate change of Z (for the model considered in Figure 5) is $-2.4 \cdot 10^{-15}$ per year, and the "gravitational constant" increases with the same speed.

Figures 4 and 5 correspond to the following parameters: $k = 0$, $Z_0 = 1.0001$, $\widetilde{B}_n/Z_0 = -0.7333$, $\mu_{r1} = 0$, $2\rho_p/Z_0 = 0.2666$, and $\rho_r = 0$; and the following initial conditions: $b(0) = 10^{-20}$, $Z = 1$, for $b = 1$.

However, if the function $Z(x)$ is treated as a macroscopic parameter and only the function $Z(x)$ (not including its derivatives) requires the continuity, then transitions are possible at the time when $\dot{b}_r(x_{cr}) = 0$ and the second derivative of scale factor is not necessarily zero. From (64) follows that in the latter case $Z_{cr} = Z_0$.

As a result, we have possible transitions from RS stage into ES stage of the following form. For points $x < x_{cr}$: $Z(x) = F(x)$, $b(x)$ is defined as the solution of (57); at the point $x = x_{cr}$: $\ddot{b}_r = 0$ or $\dot{b}_r = \dot{b}_e = 0$ and $Z_{cr} = Z_0$; for points $x > x_{cr}$:$Z(x) = Z_0 = const$, $b(x)$ is defined as the solution of (61). Similarly, a solution describing the transition from ES into RS can be found.

For the solutions analysis of obtained equations we used qualitative research methods for special solutions of differential equations. From this analysis it follows that the case of the transition from ES stage into RS and back is more likely near the special points, where the first or second derivative (or both) of the scale factor equals zero or infinity.

4. Oscillating Solutions: Influence of Quadratic Terms

Let us consider (42) and (43) under the conditions (53). Recall that these conditions are the requirement for existence of common (for both stages) static solutions $Z(x) = Z_0 = const$. In doing so, we must understand that the fulfillment of these conditions implements a "strongly" nonlinear model with potential (20). In this article, we do not claim to develop the final realistic cosmological model but want to identify the effects associated with nonlinear terms. From the conditions (53) let us express μ_{p2}, μ_{r2} in terms of $\rho_p, \mu_{p1}, \mu_{r1}, Z_0$ and substitute them in the equations under study that can be reduced to the form

$$\frac{db}{dt} \equiv \dot{b} = p,$$

$$\begin{aligned}\frac{\dot{p}}{b} &= -2ZL_1 - f_s - \frac{p^2}{b^2} - \frac{k}{b^2} \\ &\quad + \frac{1}{b^3}\left(m_{p1} + \frac{2Z\rho_p}{3{Z_0}^2} - \frac{2m_{p1}Z}{3Z_0}\right) \\ &\quad + \frac{m_{r1}}{b^4}\left(1 - \frac{Z}{Z_0}\right),\end{aligned} \tag{65}$$

$$\begin{aligned}\frac{\dot{Z}}{Z}\frac{p}{b} &= -\frac{k}{b^2} - \frac{p^2}{b^2} \\ &\quad + \frac{1}{b^3}\left(\left(-\frac{Z}{3Z_0} + 1\right)m_{p1} + \left(\frac{Z}{3{Z_0}^2} + \frac{1}{Z}\right)\rho_p\right) + \\ &\quad + \frac{1}{b^4}\left(\left(1 - \frac{Z}{2Z_0}\right)m_{r1} + \frac{\rho_r}{Z}\right) - L_1 Z - f_s + \frac{B_1}{Z}\end{aligned} \tag{66}$$

To simplify and reduce the number of parameters, consider the following additional relations that arise from the requirement of the existence of solutions $a = const$ at $Z = Z_0$:

$$\begin{aligned}m_{p1} &= -\frac{2}{Z_0}\rho_p; \\ m_{r1} &= -\frac{2}{Z_0}\rho_r.\end{aligned} \tag{67}$$

Also consider two different types of relations, corresponding to the ansatzes examined earlier in (37) and (38):

$$\begin{aligned}f_s &= 2\frac{B_1}{Z_0} + k_2; \\ L_1 &= -\frac{B_1}{Z_0^2}.\end{aligned} \tag{68}$$

$$\begin{aligned}f_s &= 2\frac{B_1}{Z_0} + 2k_2; \\ L_1 &= -\frac{B_1}{Z_0^2} - \frac{k_2}{Z_0}.\end{aligned} \tag{69}$$

Then, (65) and (66) are reduced to the form

$$\dot{Z}\left(\frac{\dot{p}}{b}+\frac{p^2}{b^2}+\frac{k}{b^2}-\frac{2}{Z_0}\left(\frac{Z}{Z_0}-1\right)\left(B_1+\frac{\rho_p}{b^3}+\frac{\rho_r}{b^4}\right)+K_{rv}\right)=0; \quad (70)$$

$$\frac{\dot{Z}}{Z}\frac{p}{b}+\frac{p^2}{b^2}+\frac{k}{b^2}-\frac{1}{Z}\left(\frac{Z}{Z_0}-1\right)^2\left(B_1+\frac{\rho_p}{b^3}+\frac{\rho_r}{b^4}\right)+K_{ev}=0. \quad (71)$$

Where K_{rv} and K_{ev} for the case (68) equals $K_{rv} = K_{ev} = k_2$ and for the case (69)

$$K_{rv}=2k_2\left(1-\frac{Z}{Z_0}\right),$$
$$K_{ev}=k_2\left(2-\frac{Z}{Z_0}\right). \quad (72)$$

The parameter k_2 has the value of the cosmological constant at the point $Z(t_0) = Z_0$ with the minus sign.

Let us consider some numerical solutions of the obtained equations (70) and (71), for the case of a flat space $k = 0$ and ansatz (72). To define the unknown parameters, we use the following reasoning. If in the modern era $t = t_m$ the value of the field $Z(t_m) = Z_m$, then the fractions consistent with observations (similarly to ΛCDM) of the effective "cosmological constant," dust-like and ultra-relativistic matter, as follows from (70), for RS stage equal

$$\Omega_{\Lambda_R}=\frac{2B_1}{Z_0H_m^2(1+q_m)}\left(\frac{Z_m}{Z_0}-1-K_{rv}\frac{Z_0}{2B_1}\right)\simeq 0.739; \quad (73)$$

$$\Omega_{p_R}=\frac{2\rho_p}{b_m^3Z_0H_m^2(1+q_m)}\left(\frac{Z_m}{Z_0}-1\right)\simeq 0.26; \quad (74)$$

$$\Omega_{r_R}=\frac{2\rho_r}{b_m^4Z_0H_m^2(1+q_m)}\left(\frac{Z_m}{Z_0}-1\right)\simeq 0.001. \quad (75)$$

By means of $H_m = p/b$ and q_m we denoted the normalized Hubble constant $H_m = \dot{b}_x/b = t_m(\dot{a}_t/a) \simeq 1$ and the deceleration parameter (with the minus sign) $q_m = (\ddot{b}/b)/H_m^2 \simeq 0.6$, calculated to the moment $t = t_m$ ($x = 1$). Without loss of generality we can choose $Z_0 = 1$. Setting B_1, k_2, b_m, from the relations (73)-(75), we can express Z_m, ρ_p, ρ_r and substitute them in (70)-(71). The relations (73)-(75) are obtained by analogy with the ΛCDM model, but our model differs from ΛCDM, by the time dependence of the function simulating the Λ term and the presence in the equations of the derivatives of Z. If $\dot{Z} \neq 0$, as follows from (71), there is a fraction of the energy associated with the term $\dot{Z}p/(Zb)$, which can have a larger contribution than a term interpreted as a Λ term. Also we want to note that in this section we do not investigate the joining of solutions issues of equations describing ES and RS stages; we will study only the solutions of (71) and (70) without the multiplier $\dot{Z}$. Although from the results of the previous section it follows that random transitions between the stages are possible, at the vicinity of $\dot{Z} = 0$.

Here are some numerical solutions, with the initial conditions

$$b(1)=1,$$
$$Z(1)=Z_m, \quad (76)$$
$$p(1)\equiv\dot{b}=1.$$

The equations under study are invariant with respect to time shifts $x \longrightarrow x + const$. That means that the variable x can also take negative values $x \ni (-\infty, \infty)$. For the initial point we will take a point with a singular solution (if it exists). As it turned out, the parameters $\widetilde{B}_n$ and k_2 strongly influence the solutions. Taking into account quadratic terms, at $\widetilde{B}_n > 0$, leads to solutions with anharmonic oscillations, where the "mean" oscillation frequency depends on the value $\widetilde{B}_n$.

The case: $k_2 = 0$. The peculiarity of this case (in our opinion it is not contrary to the observational data) associated with the fact that (70)-(71) for $Z = Z_0$ allows any constant solutions $b(x) = const$. The initial conditions $\gamma_0 : Z_m = Z(x_0), b_m = b(x_0), P_{bm} \equiv \dot{b}_m = \dot{b}(x_0)$ violating this initial state ($Z = \forall Z_0$, $b(x) = \forall const$) initiate solutions $\gamma : Z = Z(x, Z_m), b = b(x, b_m)$ which are related to the initial conditions as some perturbations. The resulting solutions are of stochastic (random) character. The stochasticity is as follows: if from the resulting solution γ we choose different point $\gamma_1 : Z_n = Z(x_1), b_n = b(x_1), p_{Zn} = \dot{Z}(x_1), p_{bn} = \dot{b}(x_1)$ (corresponding to the moment of time x_1) as new initial conditions for the same equations, then the new solutions γ_1 will not match with γ. This assertion follows from our computer studies and means that the uniqueness condition for the Cauchy problem for the equations under study is violated.

Interesting features of these solutions are the absence of singularity and infinite extension, as in the model ΛCDM. The evolution continues with the transition of the scale factor $b(x)$ (and the function $Z(x)$) to a "quasi-constant" value, which is not defined in advance, but rather of random character (which depends on the choice of the initial data).

To characterize possible solutions, let us see graphs of numerical solutions, with the values of the parameters indicated in Figures 6–8. A spatially flat case is considered, $k = 0$.

Consider, for example, the solution corresponding to Figure 7. For large values of time (e.g., for > 2.3·) the modern accelerated expansion gradually transforms into fluctuating solution in the vicinity of $a \simeq 2 \cdot a_m$. In the past, the scale factor fluctuated around the value $a \simeq 0.62 \cdot a_m$. The amplitudes of the oscillations are small from 10^{-3} to 10^{-4} and vary with time.

Consideration of the case $k_2 \neq 0$ leads to the models similar to ΛCDM in the general scenario of evolution, which are additionally accompanied by fluctuations. Figures 1, 2(a), and 2(b) show graphs of numerical solutions of equations with initial conditions (76), for the case with parameters: $k_0 =$

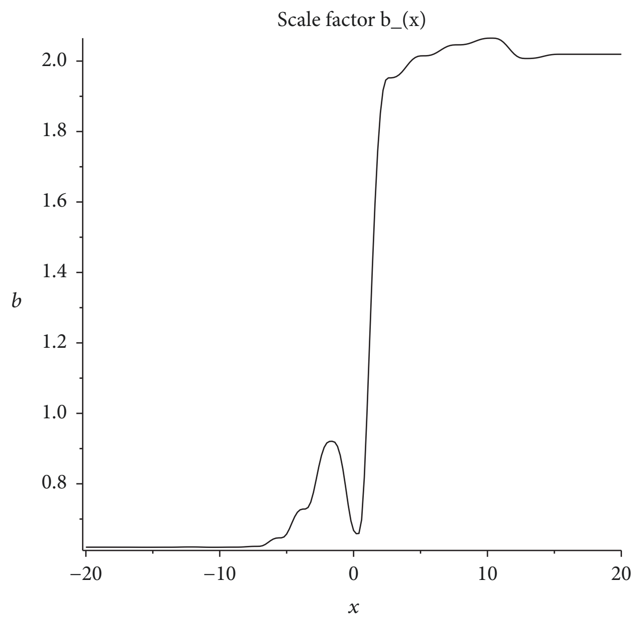


FIGURE 7: Relation of $b = b(x) = a(x)/a_m$; $b(1) = 1$ is the current value of the scale factor. $\widetilde{B}_n = 1.44517022939$, $k_2 = 0$.

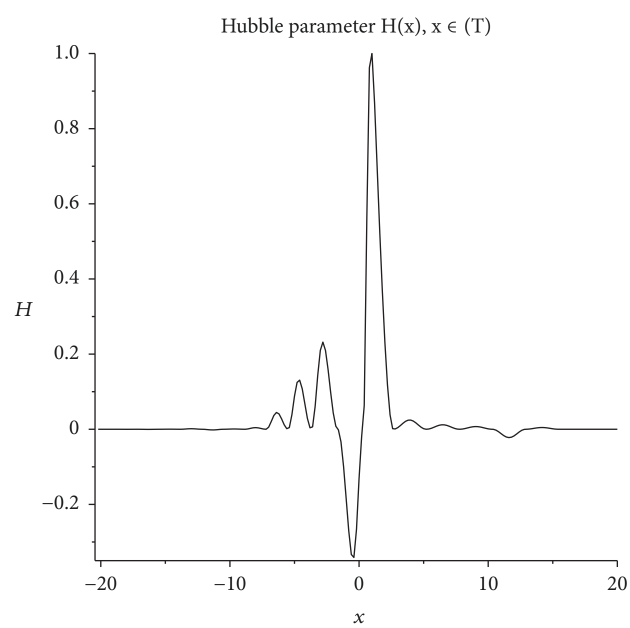


FIGURE 8: Relation of $H(x) = \dot{b}(x)/b(x)$; $H_m(1) = 1$ is the current value of the Hubble constant. $\widetilde{B}_n = 1.44517022939$, $k_2 = 0$.

0, $k_1 = -0.4$, $B_1 = 144.517022939$, $Z_m = 1.0041022218..$, $\rho_p = 50.7042299920..$, $\rho_r = 0.1950162692..$ (dots after numbers mean that we give only the first ten numbers after the decimal point). The main feature of these solutions is the presence of fluctuations that lead to alternation of accelerated with decelerated expansion passing into a contraction state (at certain values of the parameters).

For the given model, there is an initial state at certain moment of time $x_{in} \simeq -0.203793929714828...$ The scale factor $b(x) \longrightarrow b_{in} \sim 0$ for $x \longrightarrow x_{in}$. It is remarkable that the model has the characteristic properties of the inflationary scenario [33], perhaps without the scenario of transition from initial *ES* stage to *RS* stage proposed in the author's work in [20]. As follows from computer calculations, for $x \longrightarrow x_{in}$, the Hubble function tends to a huge value $H(x) \longrightarrow H_{in} > 0$ and $Z(x) \longrightarrow Z_{in} < 0$. For example, at $(x) = 2.93904.. \cdot 10^{-10} \Longrightarrow Z(x) = -4.03515.. \cdot 10^9$, $H(x) = 5.62489.. \cdot 10^{23}$. We have not been able to fully explore the model for $b(x) \longrightarrow b_{in} \sim 0$ yet. This topic needs further research. In addition, we must remember that the existence of "quasi-static" periods of evolution in our model initiates a new perspective on solving the problem of large-scale homogeneity and isotropy of the universe. It can be assumed that during the "quasi-static" period the Universe manages to pass into equilibrium state.

Nonsingular cosmological solutions: $k_2 \neq 0$.

There are also nonsingular cosmological solutions under certain conditions on the parameters of the theory and boundary conditions. Below (Figures 9–12) are graphs of the numerical solution of the equations under consideration for the parameters: $k = 0$, $\widetilde{B}_n = 535.364179666666..$, and $k_2 = -0.84$, as well as the following boundary conditions: $Z(1) = 0.999329006493718..$, $P(1) = 1$, and $b(1) = 1$.

Solutions still oscillate. The scale factor has a minimum of $a_{min} \simeq 0.0002225 \cdot a_m$, and t_m a_m are the modern values of age and scale factor. The solution, unlike the De Sitter space, contains an initial rapid expansion, resembling an inflationary stage, however, with much less acceleration, which in turn depends on the values of the parameters. This is followed by a slow stage turning into a stage of secondary expansion. The figures show plots of the field $Z(t)$, scale factor $b(t)$, and the Hubble function $H(t)$, for the time interval $\Delta t \simeq 2.2575 \cdot t_m$.

With the same parameters and boundary conditions, the scale factor near the minimum (for the interval $2.8858 \cdot 10^{-10} \cdot t_m$) has the form shown in Figure 12.

5. Results and Discussion

We present here the main results of our research.

The MTIG model is proposed for a macroscopic description of gravity and cosmology, which (possibly) is capable of solving problems (1)-(3), given at the beginning of the article, and motivating to do further experiments. We propose the working hypothesis according to which the physical parameters, associated with gravitation, such as the gravitational and cosmological "constants" G and Λ_{eff}, and the Hubble "constant" H, in addition to monotonic evolution, fluctuate about their mean values. Because of the implementation of the two branches of solutions, these fluctuations can contain elements of stochasticity. It is shown that the cosmological model under consideration also contains nonsingular solutions.

As a discussion, we present the following idea. If the observations related to CMB are attributed to the early stages of the Universe expansion ($H_{CMB} = 67.0 \pm 1.2\ km \cdot s^{-1} Mpc^{-1}$) and local measurements at later moments ($H_{local} = 73.48 \pm 1.66\ km \cdot s^{-1} Mpc^{-1}$), then it follows that at later moments

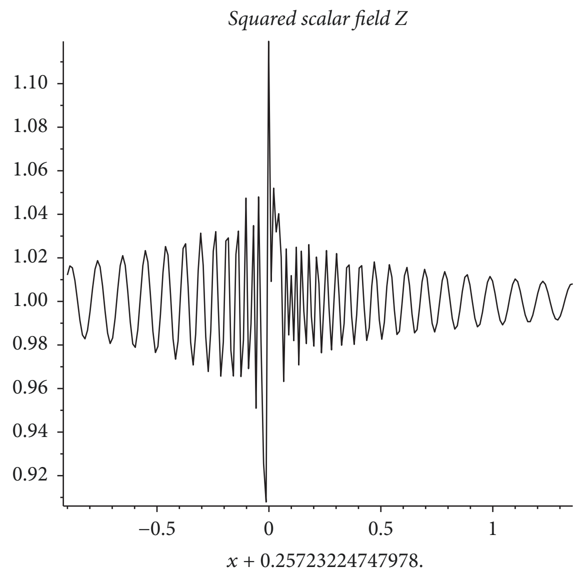


Figure 9: Relation of $Z = Y/Y_0$, $Z(1) = 0.999329006493718$; $\widetilde{B}_n = 535.364179666666$, $k_2 = -0.84$.

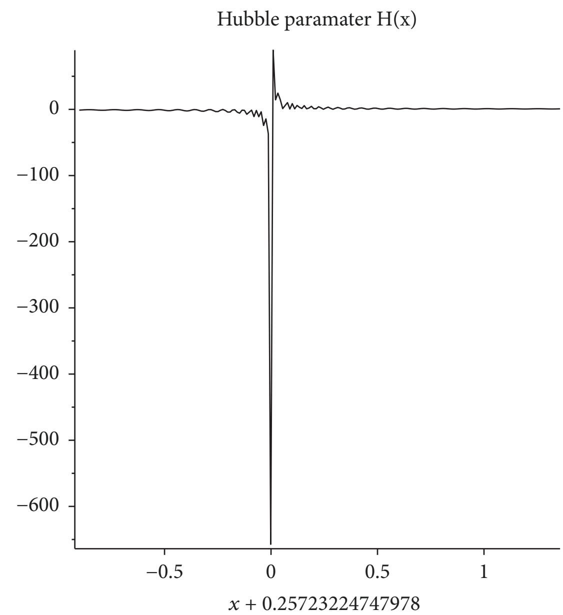


Figure 11: Relation of $H(x) = \dot{b}(x)/b(x)$: $H_m(1) = 1$ is the current value of the Hubble constant. $Z(1) = 0.999329006493718$; $\widetilde{B}_n = 535.364179666666$, $k_2 = -0.84$.

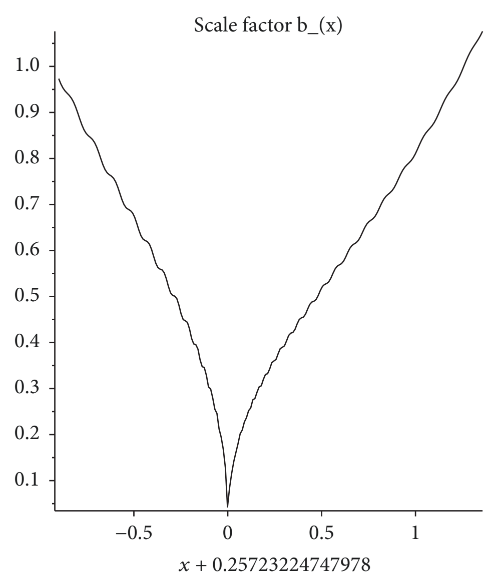


Figure 10: Relation of $b(x) = a(x)/a_m$, $Z(1) = 0.999329006493718..$; $\widetilde{B}_n = 535.364179666666$, $k_2 = -0.84$.

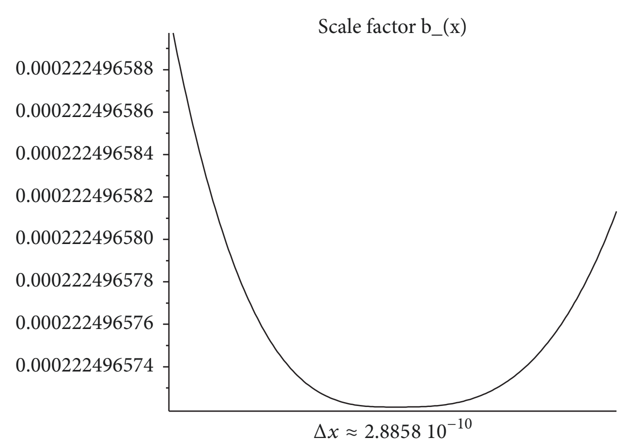


Figure 12: Relation of $b(x) = a(x)/a_m$, $Z(1) = 0.999329006493718..$: Δx is relative time interval. $\widetilde{B}_n = 535.364179666666$, $k_2 = -0.84$.

the amplitudes of the Hubble constant oscillations should increase compared to the previous moments. From the initial comparison with our models, it follows that this paradox is best explained by stochastic models. This follows from the fact that the graphs in the distant past (to the left from the present time value) should contain a mode of oscillations with lower amplitudes. For example, CMB data may contain information about the moments to the left of the minimum point of the graph in Figure 8.

The consideration of the solutions of the MTIG equations in homogeneously isotropic space-time (and in centrally symmetric space-time) leads to a model claiming a single description of DM and DE. We have not finished the study of solving equations in a centrally symmetric space when the field Z is not static: $Z = Z(r, t)$. These studies will be continued.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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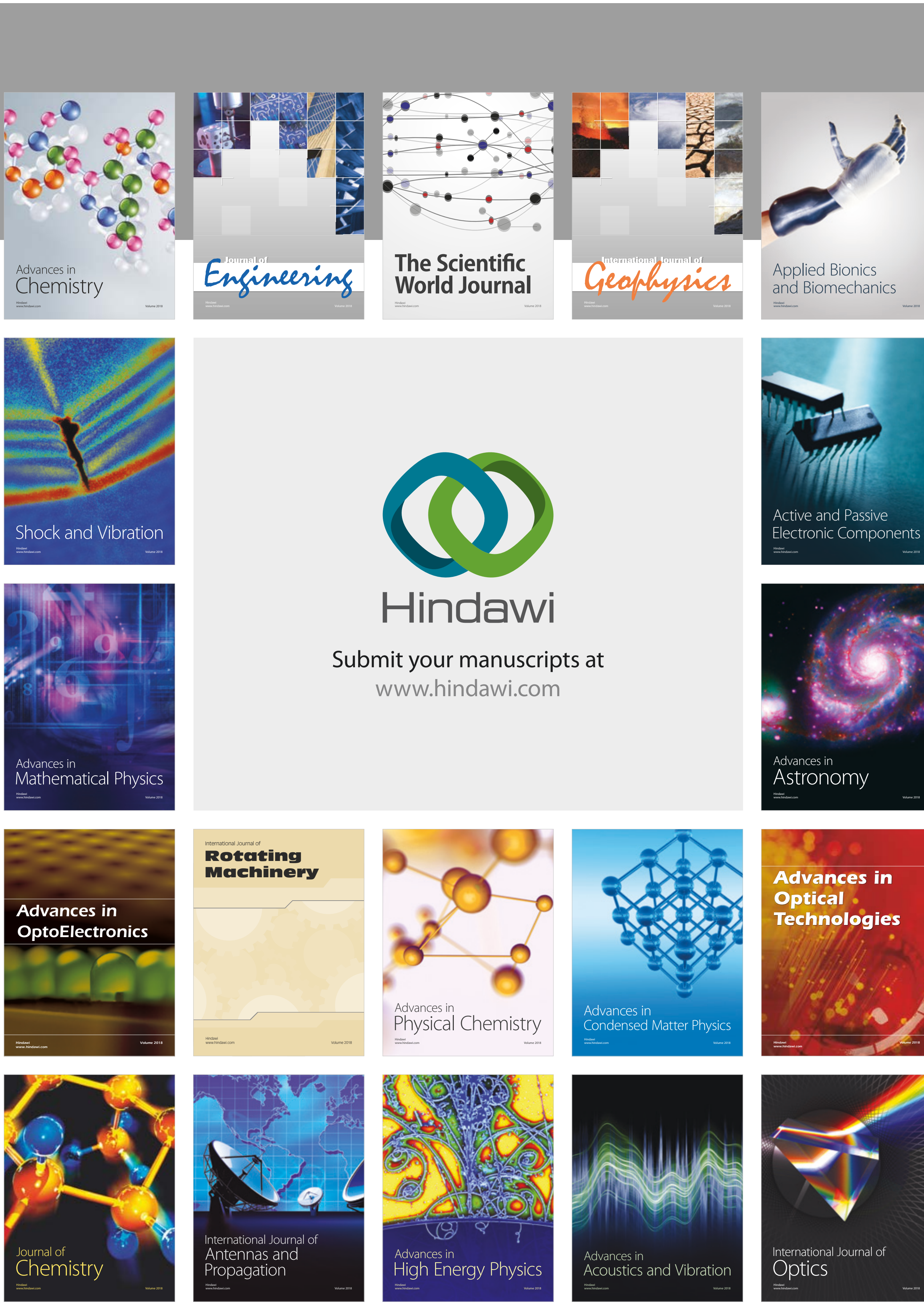
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