ISSN 1068-7998, Russian Aeronautics, (Iz. VUZ) 2007, Vol. 50, No. 1, pp. 1–9. © Allerton Press, Inc., 2007
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FLIGHT VEHICLE DESIGN

Alternative Statements of Optimal Load-Bearing Structure Design Problems

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Received November 17, 2006

Abstract—We analyze the well-known statements of optimal design problems for load-bearing maximum rigidity structures in which a principle of the minimum of total potential energy of structure deformation is used. A version of the generalized statement of optimization problems for processes described by linear equations with a symmetric operator is proposed. On the basis of the version outlined, we present a mathematical algorithm for obtaining beam structures of maximum rigidity, in which the finite element method is used to solve an analytical problem. We give the numerical results of solving two problems of optimal design of maximum rigidity structures that are simulated by cantilever and statically indeterminate beams.

DOI: 10.3103/S1068799807010011

One of the main requirements for most aircraft structures is to provide rigidity. In this connection, to design optimal load-bearing elements of the structures, use is often made of the condition that maximum rigidity should be provided with the specified bulk of material [1–8]. In particular, it is noted in [1, 3–5, 7, 8] that one of the most efficient methods of optimal design of maximum rigidity structures is the method that is based on minimizing the potential energy of structure deformation. This method was used to solve a number of problems of designing load-bearing elements of aircraft structures.

1. To solve a problem of the structure analysis at constant values of design parameters and specified loading, the functional of total potential energy (or the Lagrange functional) is written in the form:

$$\Psi(\boldsymbol{u},\boldsymbol{r}) = \Pi(\boldsymbol{u},\boldsymbol{r}) - \mathrm{A}(\boldsymbol{u},\boldsymbol{p}), \tag{1}$$

where $\Pi(\boldsymbol{u},\boldsymbol{r})$ is the functional of potential energy of structure deformation; $\mathrm{A}(\boldsymbol{u},\boldsymbol{p})$ is the functional of work of specified external loads; $\boldsymbol{u} = \{u_1, u_2, ..., u_s\}$ is the vector of sought displacement functions of the structure under study; $\boldsymbol{p} = \{p_1, p_2, ..., p_s\}$ is the vector of specified functions of external loads; s is the number of displacement functions; $\boldsymbol{r} = \{r_1, r_2, ..., r_k\}$ is the vector of design parameters; k is the number of design parameters.

To find $\boldsymbol{u}_0$, that is, the solution of the structure analysis problem at the specified or constant values of design parameters and the known external load, subject to the condition that functional (1) is at a minimum, we obtain a system of resolving differential equations, that are nonlinear in the general case.

As the conventional (exact) solutions of the analysis problems are unknown for the most part of practical cases, the approximate methods are used to find the functions of structure displacements $\boldsymbol{u}_0$. At present, the approximate methods based on the application of the generalized solution of the initial problem of structure analysis are the most efficient.

The vector from the domain of functional (1) definition realizing its minimum is called the generalized solution of the structure analysis problem. The generalized solution of the problem coincides with its conventional solution $\boldsymbol{u}_0$; therefore, the problem of finding the solution of the system of differential equations and that of finding the minimum of functional (1) are equivalent. In such a statement the domain of definition in the problem of structure analysis is extended to the domain of functional (1) definition. This mathematical result made it possible to develop efficient approximate methods to solve the problems of structure analysis.

On the basis of the foregoing, the approximated generalized solution of the structure analysis problem is taken as a linear combination of a system of basic functions belonging to the domain of functional (1) definition:

$$\boldsymbol{u}_0 \cong \boldsymbol{u}_1 = \sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i \,, \tag{2}$$

where $\boldsymbol{u}_0$ is the exact generalized solution; $\boldsymbol{u}_0$ is the approximated generalized solution; $a_i = \{a_{1i}, a_{2i}, ..., a_{si}\}$ $i = 1, 2, ..., n$ are the arbitrary coefficients; φ_i, $i = 1, 2, ..., n$ are the basic functions from the domain of functional (1) definition.

After the values $\boldsymbol{u} = \boldsymbol{u}_1$ (2) are substituted into (1) and the conditions of the minimum for the expression obtained with respect to the variables $\boldsymbol{a}_i$, $i = 1, 2, ..., n$ are written, we have a system of algebraic equations with respect to the vector of the coefficients $\boldsymbol{a} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_n\}$:

$$\frac{\partial}{\partial \boldsymbol{a}^T} \Psi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right) \equiv \frac{\partial}{\partial \boldsymbol{a}^T} \Pi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right) - \frac{\partial}{\partial \boldsymbol{a}^T} \boldsymbol{A}\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{p}\right) = 0 \,. \tag{3}$$

Solving system (3), we find the arbitrary coefficients of approximated generalized solution (2). The method for solving the structure analysis problem presented is called the Ritz method.

If the local piecewise-polynomial finite functions are taken as basic functions (2) on the calculation grid, the Ritz method is transformed into the finite element method (FEM).

In the well-known statements a problem of designing optimal structures of maximum rigidity is formulated as a problem of finding the absolute minimum of the modified Lagrange functional (1) of the following structure [1, 7, 8]:

$$\Psi_1(\boldsymbol{u},\boldsymbol{r},\lambda) = \Psi(\boldsymbol{u},\boldsymbol{r}) + \sum_{i=1}^{k} \lambda_i \Phi_i(\boldsymbol{u},\boldsymbol{r}) = \Pi(\boldsymbol{u},\boldsymbol{r}) - A(\boldsymbol{u},\boldsymbol{p}) + \left(\lambda, \Phi(\boldsymbol{u},\boldsymbol{r})\right), \tag{4}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, ..., \lambda_k\}$ is the vector of the Lagrange scalar multipliers; $\Phi_i(\boldsymbol{u}, \boldsymbol{r})$, $i = 1, 2, ..., k$ is the functional determining the i-th condition of designing or the condition of optimality: $\Phi_i(\boldsymbol{u}, \boldsymbol{r}) = 0$, $i = 1, 2, ..., k$ (k is the number of design conditions and conditions of optimality); $\Phi_i(\boldsymbol{u}, \boldsymbol{r}) = \{\Phi_1(\boldsymbol{u}, \boldsymbol{r}), \Phi_2(\boldsymbol{u}, \boldsymbol{r}), ..., \Phi_k(\boldsymbol{u}, \boldsymbol{r})\}$ is the vector of functionals determining the design conditions and conditions of optimality; $(\boldsymbol{u}, \boldsymbol{v})$ is the scalar product of the vectors $\boldsymbol{u}$, $\boldsymbol{v}$.

Then substituting $\boldsymbol{u} = \boldsymbol{u}_1$ into functional (4), according to expression (2), we obtain the value of functional (4): $\Psi_1(\boldsymbol{u}_1, \boldsymbol{r}, \lambda)$. From conditions of the minimum of the functional $\Psi_1(\boldsymbol{u}_1, \boldsymbol{r}, \lambda)$, we obtain a system of nonlinear algebraic equations with respect to the vectors of the sought unknowns $\boldsymbol{a}$, $\boldsymbol{r}$, $\boldsymbol{\lambda}$:

$$\frac{\partial}{\partial \boldsymbol{a}^T} \Psi_1(\boldsymbol{u}_1,\boldsymbol{r},\lambda) \equiv \frac{\partial}{\partial \boldsymbol{a}^T} \Pi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right) - \frac{\partial}{\partial \boldsymbol{a}^T} A\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{p}\right) + \left(\lambda,\ \frac{\partial}{\partial \boldsymbol{a}^T} \Phi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right)\right) = 0; \tag{5}$$

$$\frac{\partial}{\partial \boldsymbol{r}^T} \Psi_1(\boldsymbol{u}_1,\boldsymbol{r},\lambda) \equiv \frac{\partial}{\partial \boldsymbol{r}^T} \Pi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right) + \left(\lambda,\ \frac{\partial}{\partial \boldsymbol{r}^T} \Phi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right)\right) = 0\,; \tag{6}$$

$$\frac{\partial}{\partial \lambda^T} \Psi_1(\boldsymbol{u}_1,\boldsymbol{r},\lambda) \equiv \Phi\left(\sum_{i=1}^{n} \boldsymbol{a}_i \varphi_i, \boldsymbol{r}\right) = 0 \,. \tag{7}$$

The solution of this system of equations gives the values of design parameters and the approximated solution of the problem on the analysis of the structure being designed. It is difficult or impossible to find the solution of the system of equations such as (5)–(7); therefore, the methods for immediate finding of a stationary point of the functional $\Psi_1(\boldsymbol{u}_1, \boldsymbol{r}, \lambda)$ are frequently used.

Let $\tilde{\boldsymbol{u}}_1 = \sum_{i=1}^{n} \tilde{\boldsymbol{a}}_i \varphi_i$ be the solution of the analysis problem previously obtained according to expression (3) or by other method. Then based on this solution for finding (or refining) the vector of design parameters $\boldsymbol{r}$, we can consider only a system of equations (6) and (7) in which $\boldsymbol{u}_1 = \sum_{i=1}^{n} \tilde{\boldsymbol{a}}_i \varphi_i = \tilde{\boldsymbol{u}}_1$ or $\boldsymbol{u}_1 = \tilde{\boldsymbol{u}}_1$ should be taken:

$$\frac{\partial}{\partial \boldsymbol{r}^T} \Pi\left(\tilde{\boldsymbol{u}}_1, \boldsymbol{r}\right) + \left(\lambda, \frac{\partial}{\partial \boldsymbol{r}^T} \Phi\left(\tilde{\boldsymbol{u}}_1, \boldsymbol{r}\right)\right) = 0\,; \quad \Phi\left(\tilde{\boldsymbol{u}}_1, \boldsymbol{r}\right) = 0\,. \tag{8}$$

It is easy to note that system (8) is obtained when the condition of the minimum of the functional

$$\Psi_2\left(\boldsymbol{u}, \boldsymbol{r}, \lambda\right) = \Pi\left(\boldsymbol{u}, \boldsymbol{r}\right) + \left(\lambda, \Phi\left(\boldsymbol{u}, \boldsymbol{r}\right)\right), \tag{9}$$

is considered, if the value $\boldsymbol{u} = \tilde{\boldsymbol{u}}_1$ is previously substituted into it.

Functional (9) is a modified functional of potential energy of structure deformation.

The methods for designing optimal structures of maximum rigidity based on the use of functional such as (9) are given in [3–5].

2. Let us consider the problems of analysis for linear processes that are reduced to the solution of the operator equation of the kind:

$$L\boldsymbol{u} = \boldsymbol{p}\,, \tag{10}$$

where L is the symmetric positive linear operator; $\boldsymbol{u} = \{u_1, u_2, ..., u_s\}$ is the vector of sought functions describing the process under consideration; s is the number of sought functions; $\boldsymbol{p} = \{p_1, p_2, ..., p_s\}$ is the vector of specified functions of the right-hand sides of the system of operator equations; for the operator L at the expense of a number of boundary conditions the following conditions are fulfilled: $(L\boldsymbol{u}, \boldsymbol{v}) = (L\boldsymbol{v}, \boldsymbol{u})$, $(L\boldsymbol{u}, \boldsymbol{u}) > 0\ \forall \boldsymbol{u} \neq 0$; $(\boldsymbol{u}, \boldsymbol{v})$ is the scalar product of the vectors of the functions $\boldsymbol{u}$, $\boldsymbol{v}$.

To find the approximated generalized solution of initial operator equation (10), it is necessary to have the functional, the stationary point of which is realized at the value $\boldsymbol{u} = \boldsymbol{u}_0$, where $\boldsymbol{u}_0$ is the conventional (exact) solution of the operator equation being considered. The problem of obtaining the functional by the initial operator equation is the inverse problem of calculus of variations. The solution of the inverse problem of this type is ambiguous since theoretically there are infinitely many functionals corresponding to Eq. (10). There exist several particular methods for solving the inverse problem presented, and we will consider one of such methods widely used in appendices.

The functional of total energy of the operator L, the stationary point of which is realized with the value $\boldsymbol{u} = \boldsymbol{u}_0$, has the form [9, 10]:

$$F\left(\boldsymbol{u},\ \boldsymbol{r}\right) = \left(L\boldsymbol{u},\ \boldsymbol{u}\right) - 2\left(\boldsymbol{p},\ \boldsymbol{u}\right), \tag{11}$$

where $(L\boldsymbol{u}, \boldsymbol{u})$ is the functional of energy of the operator L that is the functional of the kind $(L\boldsymbol{u}, \boldsymbol{u}) \equiv f(\boldsymbol{u}, \boldsymbol{r})$ in virtue of the fact that $L = L(r)$; $\boldsymbol{r} = \{r_1, r_2, ..., r_k\}$ is the vector of design parameters of the problem under consideration; k is the number of design parameters.

Functional (11) is a quadratic functional that is obtained for a problem of any nature described by operator equation (10). The value $\boldsymbol{u} = \boldsymbol{u}_0$ corresponding to the stationary point of functional (11) is the generalized solution of initial equation (10).

According to the Ritz method or FEM, the approximated generalized solution of the analysis problem is assumed as a linear combination of the system of basic functions belonging to the domain of functional (11) definition in form (2). As a result of substituting $\boldsymbol{u} = \boldsymbol{u}_1$ (2) into functional (11), we obtain a quadratic form depending on the variables $\boldsymbol{a}_i$, $i = 1, 2, ..., n$. After the conditions for finding the stationary point of the quadratic form obtained are written, we have a system of linear algebraic equations with respect to the vector of the sought coefficients $\boldsymbol{a}$ [10]:

$$\boldsymbol{Ka} = \boldsymbol{P}, \tag{12}$$

where $\boldsymbol{k}_{ij} = \{L(\varphi_i, \varphi_j)\}$, $\Phi_i(\boldsymbol{u}, \boldsymbol{r})$, $i = 1, 2, ..., k; j = 1, 2, ..., n$; $\boldsymbol{p}_i = \{(\varphi_i, \boldsymbol{p})\}$, $i = 1, 2, ..., n$; $\boldsymbol{a} = \{\boldsymbol{a}_i\}$, $i = 1, 2, ..., n$.

After functional (11) is obtained, the general statement of optimal design problems based on searching for its stationary point is the possibility. For this purpose, using the Lagrange method of indefinite multipliers, it is easy to obtain the modified functional:

$$F_1(\boldsymbol{u},\boldsymbol{r}) = F(\boldsymbol{u},\boldsymbol{r}) + (\lambda, \Phi(\boldsymbol{u},\boldsymbol{r})) = (L\boldsymbol{u},\boldsymbol{u}) - 2(\boldsymbol{p},\boldsymbol{u}) + (\lambda, \Phi(\boldsymbol{u},\boldsymbol{r})), \tag{13}$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, ..., \lambda_k\}$ is the vector of the Lagrange scalar multipliers; $\Phi_i(\boldsymbol{u}, \boldsymbol{r})$, $i = 1, 2, ..., k$ is the functional determining the ith condition of design of the form $\Phi_i(\boldsymbol{u}, \boldsymbol{r}) = 0$, $i = 1, 2, ..., k$; k is the number of design conditions; $\Phi_i(\boldsymbol{u}, \boldsymbol{r}) = \{\Phi_1(\boldsymbol{u}, \boldsymbol{r}), \Phi_2(\boldsymbol{u}, \boldsymbol{r}), ..., \Phi_k(\boldsymbol{u}, \boldsymbol{r})\}$ is the vector of functionals determining the design conditions for the problem being solved.

Then, by substituting $\boldsymbol{u} = \boldsymbol{u}_1$ into functional (13) according to formula (2), we obtain the approximated value of functional (13): $F_1(\boldsymbol{u}_1, \boldsymbol{r}, \boldsymbol{\lambda})$. To find the stationary point of the functional $F_1(\boldsymbol{u}_1, \boldsymbol{r}, \boldsymbol{\lambda})$, we obtain a system of nonlinear algebraic equations of form (5)–(7) with respect to the vectors of sought unknowns $\boldsymbol{a}, \boldsymbol{r}, \boldsymbol{\lambda}$.

Let $\tilde{\boldsymbol{u}}_1 = \sum_{i=1}^{n} \tilde{\boldsymbol{a}}_i \varphi_i$ be the solution of the analysis problem obtained according to expression (12) or by other method. Then on the basis of this solution to find the vector of design parameter r, we can consider the stationary point of the functional

$$F_2(\boldsymbol{u}_1, \boldsymbol{r}, \boldsymbol{\lambda}) = (L\boldsymbol{u}, \boldsymbol{u}) + (\boldsymbol{\lambda}, \Phi(\boldsymbol{u}, \boldsymbol{r})), \tag{14}$$

where the value $\boldsymbol{u} = \tilde{\boldsymbol{u}}_1$ must be previously substituted.

Functional (14) is a modified functional of energy of the operator L in Eq. (10).

Thus, the method proposed makes it possible to formulate the optimization problems for the processes described by linear operator equations of kind (10) without considering the corresponding physical principles or laws.

3. Let us consider application of the method presented to formulate problems of designing the elements of maximum rigidity structures simulated by a beam. Such elements of load-bearing structures are sufficiently characteristic of aircraft industry [2, 6].

The equation of transverse bending of a variable rigidity beam under the action of lateral external loads has the form:

$$Lv(x) \equiv \left[EI(x)\ v''(x)\right]'' = q(x)\ \forall x \in (a,b), \tag{15}$$

where $v(x)$ is the function of beam axis deflection; E is the modulus of elasticity of the beam material; $I(x)$ is the moment of inertia of the arbitrary beam cross-section; $q(x)$ is the function of the external lateral linear load reduced to the beam axis; Ox is the axis of the Cartesian coordinates made coincident with the line of main central moments of beam cross-sections; a, b are the coordinates of the initial and final beam sections on the Ox axis, respectively.

Materials with different mechanical characteristics are frequently used for elements of compound beam structures. In this case, for each beam section we can calculate the reduced modulus of elasticity and present it as the function $E = E(x)$.

The initial functional of kind (12) corresponding to Eq. (15) will be written as

$$F(v,EI)=\int_a^b\left[E(x)I(x)v''(x)\right]''v(x)dx-2\int_a^b v(x)q(x)dx. \tag{16}$$

Using twice in expression (16) the formula of integration by parts, we will obtain for any combination of main and natural homogeneous boundary conditions for solution of Eq. (15) the following functional:

$$F(v,B)=\int_a^b B(x)\left(v''(x)\right)^2dx-2\int_a^b v(x)q(x)dx, \tag{17}$$

where $B(x)=E(x)I(x)\ \forall x\in(a,\ b)$; $B(x)$ is the function of beam flexural rigidity.

Let us assume that the beam simulates the load-bearing structure element, the outline of which is specified in the plane of bending. In aircraft structures the shape of the load-bearing beam element in the plane of bending is often determined by the external surface of a unit.

The problem of designing a maximum rigidity beam consists in finding the function $B(x)$. Using the function $B(x)$ found, we can always determine design parameters of beam cross-sections without applying functional (17). To solve this problem, we must use one more condition of optimization, for example, the condition of the minimum of the designed structure mass.

In the statement under consideration, we will take as the first design condition the condition of beam flexural rigidity distribution according to the expression

$$\Phi_0(B)\equiv\int_a^b B(x)dx-B_0l=0, \tag{18}$$

where B_0 is the specified averaged value of the beam flexural rigidity function; $l=b-a$ is the beam length.

As the second design condition, we will take the condition of equality of design stresses in all beam cross-sections:

$$\Phi_1(v,\ x,\ \bar{x})\equiv\hat{E}(x)\ \left|\hat{\xi}(x)v''(x)\right|-\hat{E}(\bar{x})\ \left|\hat{\xi}(\bar{x})v''(\bar{x})\right|=0$$
$$\forall(x,\ \bar{x})\in(a,\ b),\ x\neq\bar{x}, \tag{19}$$

where $\hat{\xi}(x)$ is the distance from the neutral axis to the design or the most stressed element in the arbitrary beam cross-section; $\hat{E}(x)$ is the modulus of elasticity of the design (the most stressed) element in the beam cross-section being considered; $\bar{x}$ is the coordinate of the arbitrary beam cross-section taken as a basic section for comparison of design stress values.

According to the Lagrange method, functional (17) and design conditions (18), (19) make it possible to write the functional

$$F_1\left(v,B,\lambda_0,\tilde{\lambda}_1\right)=F(v,B)+\lambda_0\Phi_0(B)+\int_a^b\tilde{\lambda}_1(x)\Phi_1(v,x,\bar{x})dx, \tag{20}$$

where λ_0 is the Lagrange scalar indefinite multiplier; $\tilde{\lambda}_1(x)$ is the indefinite function, that is, the Lagrange multiplier.

To construct a numerical algorithm, we will choose the calculation grid $\Delta: a=x_0<x_1<...<x_n=b$ and present the function $B(x)$ as the piecewise-constant function:

$$B(x)\cong f\left(\bar{\boldsymbol{B}},\ x\right)=B_i\quad\forall x\in\left(x_{i-1},x_i\right),\ i=1,\ 2,\ ...,\ n, \tag{21}$$

where B_i is the flexural rigidity of the ith beam section; $\bar{\boldsymbol{B}}=\{B_1,\ B_2,\ ...,\ B_n\}$ is the vector of design parameters consisting of the flexural rigidity values of the beam sections.

Taking into account functions (21), condition (18) is transformed to the form

$$\Phi_0(B) \cong \Phi_0(\bar{\boldsymbol{B}}) \equiv \sum_{i=1}^{n} B_i l_i - B_0 l = 0, \tag{22}$$

where $l_i = x_i - x_{i-1}$, i = 1, 2, ..., n is the length of the ith beam section; x_j, i = 1, 2, ..., n are the coordinates of the beam section boundaries.

The approximation of the flexural rigidity function by function (21) makes it possible to pass to the finite number of design beam cross-sections. It is natural that the design cross-section should be chosen at the middle of the beam section.

Let us write the conditions of design stress equality for the beam cross-sections chosen in the form:

$$\Phi_1(v, \tilde{x}_m, \tilde{x}_k) \equiv \hat{E}_m \; \left| \; \hat{\xi}_m v''(\tilde{x}_m) \right| - \hat{E}_k \; \left| \; \hat{\xi}_k v''(\tilde{x}_k) \right| = 0,$$

$$m = 1, 2, ..., k, \quad m \neq k \quad \forall k \in [1, \; n], \tag{23}$$

where $\tilde{x}_i = (x_{i-1} + x_i)/2$, i = 1, 2, ..., n are the coordinates of the design cross-sections corresponding to the beam section; $\hat{E}_i = \hat{E}(\tilde{x}_i)$, $\hat{\xi}_i = \hat{\xi}(\tilde{x}_i)$, i = 1, 2, ..., n.

On the basis of conditions (23), we will perform the piecewise-linear approximation of expression (19) according to the formula

$$\Phi_1(v, \; x, \; \bar{x}) \cong \Phi_1(v, \; \tilde{x}_m, \; \tilde{x}_{m+1}) = 0 \quad \forall x \in (x_{m-1}, \; x_m), \; m = 1, 2, ..., n\text{–}1 \tag{24}$$

in which it is assumed that $m \neq n$.

Let us consider the transformation of a term of functional (20) that contains condition (19):

$$\int_a^b \tilde{\lambda}_1(x) \Phi_1(v, x, \bar{x}) dx = \sum_{i=1}^{n} \int_{i-1}^{x_i} \tilde{\lambda}_1(x) \Phi_1(v, x, \bar{x}) dx. \tag{25}$$

Substituting approximating function (24) into expression (25), we will successively obtain

$$\int_a^b \tilde{\lambda}_1(x) \Phi_1(v, x, \bar{x}) dx \cong \sum_{i=1}^{n-1} \Phi_1(v, \tilde{x}_i, \tilde{x}_{i+1}) \int_{x_{i-1}}^{x_i} \tilde{\lambda}_1(x) dx = \sum_{i=1}^{n-1} \lambda_i \Phi_1(v, \tilde{x}_i, \tilde{x}_{i+1}), \tag{26}$$

where $\lambda_i = \int_{x_{i-1}}^{x_i} \tilde{\lambda}_1(x) dx$, i = 1, 2, ..., n–1 are the new Lagrange scalar multipliers.

We must note that the transformations of expression (25) performed lead to the approximation of the function $\tilde{\lambda}_1(x) \;\; \forall x \in (a, \; b)$ corresponding to the approximation in expression (24):

$$\tilde{\lambda}_1(x) \cong (x_i - x_{i-1})^{-1} \int_{x_{i-1}}^{x_i} \tilde{\lambda}_1(x) dx = \frac{\lambda_i}{x_i - x_{i-1}} \quad \forall x \in (x_i, \; x_{i-1}), \; i = 1, 2, ..., n.$$

Taking into account transformations (21)–(26), functional (20) will be written in the form

$$F_1(v, \boldsymbol{B}, \lambda) = \sum_{i=1}^{n} B_i \int_{x_{i-1}}^{x_i} (v''(x))^2 dx - 2 \sum_{i=1}^{n} \int_{i-1}^{x_i} q(x) v(x) dx$$

$$+ \lambda_0 \left(\sum_{i=1}^{n} B_i l_i - B_0 l \right) + \sum_{j=1}^{n-1} \lambda_j \left(\; \hat{E}_j \left| \; \hat{\xi}_j v''(\tilde{x}_j) \right| - \; \hat{E}_{j+1} \left| \; \hat{\xi}_{j+1} v''(\tilde{x}_{j+1}) \right| \right), \tag{27}$$

where $\boldsymbol{\lambda} = \{\lambda_0, \; \lambda_1, \; ..., \; \lambda_{n-1}\}$ is the vector of the Lagrange multipliers; $\boldsymbol{B} = \{B_0, \; B_1, \; ..., \; B_{n-1}\}$.

Then we will present at each beam section the function of deflections in the form of the interpolation algebraic polynomial of the third power with multiple nodes:

$$v(x)=v_{j-1}\left(x_j-x\right)^2\left[2\left(x-x_{j-1}\right)+l_j\right]l_j^3+v'_{j-1}\left(x_j-x\right)^2\left(x-x_{j-1}\right)l_j^2$$
$$+v_j\left(x-x_{j-1}\right)^2\left[2\left(x_j-x\right)+l_j\right]l_j^3-v'_j\left(x-x_{j-1}\right)^2\left(x_j-x\right)l_j^2$$
$$\forall x\in\left(x_{j-1},\ x_j\right),\ j=1,2,\ldots,n, \tag{28}$$

where $v_i=v\left(x_i\right)$, $v'_i=v'\left(x_i\right)$, $i = 0, 1, \ldots, n$ are the nodal variables that are the sought values.

As a result of applying expression (28) to find the approximated solution of Eq. (15), we come to the well-known FEM algorithm that is used to analyze beam structures of variable flexural rigidity with the use of finite elements of constant flexural rigidity.

After function of deflections (28) is substituted into functional (27), it takes the form:

$$F_1\left(v,\boldsymbol{B},\lambda\right)=2\left(\frac{1}{2}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{K}\left(\boldsymbol{B}\right)\boldsymbol{V}-\boldsymbol{V}^{\mathrm{T}}\boldsymbol{P}\right)+\lambda_0\left(\sum_{i=1}^{n}B_il_i-B_0l\right)$$
$$+\sum_{j=1}^{n-1}\lambda_j\left(\hat{E}_j\ \left|\ \hat{\xi}_jv''\left(\tilde{x}_j\right)\right|-\hat{E}_{j+1}\ \left|\ \hat{\xi}_{j+1}v''\left(\tilde{x}_{j+1}\right)\right|\right) \tag{29}$$

where using the well-known FEM algorithm, we can calculate: $\boldsymbol{V}$, that is, the vector of sought nodal variables entering expression (28) taking into account the main boundary conditions of the beam; $\boldsymbol{K}(\boldsymbol{B})$ that is the general matrix of structural rigidity; $\boldsymbol{P}$, that is, the vector of equivalent nodal loads.

After the conditions of the minimum of functional (29) are written, we obtain a system of nonlinear algebraic equations with respect to V, B, and λ that is similar to system (5)–(7). The method of steepest descent is used for the approximated minimization of functional (29). In this case, to obtain the solution of zero approximation, we considered the beam of constant flexural rigidity B_0. The structure of conditions for the minimum of functional (29) makes it possible to construct a simple iteration process to refine $\boldsymbol{V}^{(k-1)}$, that is, the vector of nodal variables. If the vector of design parameters is assumed to be known from the previous approximation $\left(\boldsymbol{B}=\boldsymbol{B}^{(k-1)}\right)$, then the conditions of the functional minimum give a system of linear algebraic equations to find $\boldsymbol{V}^{(k)}$. Then after finding $\boldsymbol{V}^{(k)}$ to obtain the value $\boldsymbol{B}^{(k)}$, taking into account conditions (22), (23), only the vector $\boldsymbol{B}^{(k-1)}$ can be refined.

Based on the structural or technological limitations, the design values of the function of flexural rigidity for each section that are obtained in the process of performing steps of successive minimization of functional (29) must be not less than some specified value: $\bar{B}>0$: $B_i^{(k)}\geq\bar{B}$, $i = 1, 2, \ldots, n$; $k = 0, 1, \ldots, m$, where m is the number of minimization steps of functional (29).

4. Let us consider a homogeneous cantilever beam of a wedge shape in the plane of bending under the action of the constant linear load. It is simple to obtain an exact analytical solution for such a beam.

Let the origin of the coordinate axis Ox be made coincident with the attachment section and the Ox axis is directed to the free beam end. Let the distance from the neutral axis to the most loaded point in the beam cross-sections be changed according to the expression $\hat{\xi}\left(x\right)=\hat{\xi}_0\left(l-x\right)l^{-1}\ \forall x\in\left(0,\ l\right)$, where $\hat{\xi}_0$ is the distance from the neutral axis to the most loaded point in the cross-section of the beam attachment. Then the function of flexural rigidity distribution $B(x) = EI(x)$ satisfying condition (19) is the following:

$$B\left(x\right)=\frac{qE\hat{\xi}_0}{2l[\sigma]}\left(l-x\right)^3\ \forall x\in\left(0,\ l\right), \tag{30}$$

where q is the linear load; E is the modulus of elasticity of the beam material; $I(x)$ is the function, that is, the geometric moment of inertia of the arbitrary beam cross-section; l is the beam length; $[\sigma]$ are the permissible (limiting) stresses depending on the problem statement.

The average value of flexural rigidity for expression (30) is equal to:

$$B_0 = \frac{1}{l}\int_0^l B(x)\,dx = \frac{qE\hat{\xi}_0}{8[\sigma]}l^2 . \tag{31}$$

The relative value of flexural rigidity according to expressions (30), (31) is equal to:

$$\bar{B}(x) = \frac{B(x)}{B_0} = \frac{4(l-x)^3}{l^3} . \tag{32}$$

The numerical initial data of the test problem are the following: l = 10 m; n = 10; B_0 = 35 000 Nm^2; q = 1000 Nm^{-1}; $E\hat{\xi}(x) = E\hat{\xi}_0(l-x)l^{-1}$ $\forall x \in (0, l)$, where $E\hat{\xi}_0$ = 20 000 000 Nm^{-1}. In addition, it is assumed that $E\hat{\xi}(l)$=200 000 Nm^{-1}, since in the numerical algorithm zero values of the function $E\hat{\xi}(x)$ are impermissible.

In Fig. 1, the solid line indicates the graph of the function $f(x) = B_i B_0^{-1} \cong \bar{B}(x)$ $\forall x \in (x_{i-1}, x_i)$, i = 1, 2, ..., n obtained by the numerical algorithm with the number of finite element n = 10; the dashed line shows the graph of the function $\bar{B}(x)$ calculated according to exact solution (32). The discontinuities of the function $f(x)$ on the graph are connected by vertical lines for the vivid representation. We must note, that in the zone of the largest values of the function with the value x = 0.5 m an error of the numerical solution is equal to 2.69 %.

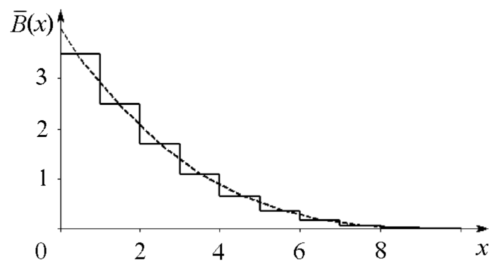

Fig. 1. Optimal variation in the function of flexural rigidity for the cantilever beam.

Let us consider a homogeneous beam fixed in the end sections that has a constant height in the plane of bending under the action of a constant linear load. Let the origin of the coordinate axis Ox be made coincident with the left-hand end section of the beam and the Ox axis is directed to the right-hand end section of the beam (Fig. 2).

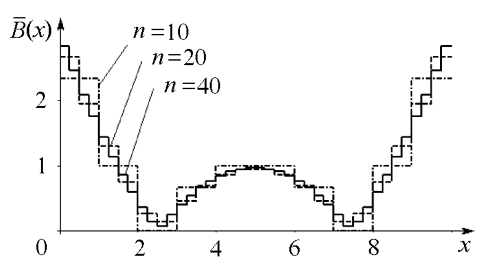

Fig. 2. Optimal variation in the rigidity function for the statically indeterminate beam.

The numerical initial data of the test problem are the following: l = 10 m; B_0 = 35 000 Nm^2; q = 1000 Nm^{-1}; $E\hat{\xi} = E\hat{\xi}_0 = 50\,000\,000$ Nm^{-1}.

The graphs of the function $f(x,\ n) = B_i B_0^{-1} \cong \bar{B}(x)\ \forall x \in (x_{i-1}, x_i)$, i = 1, 2, ..., n presented in Fig. 2 are obtained by the numerical algorithm with the number of finite elements n = 10, 20, 40.

ACKNOWLEDGMENTS

We would like to thank Prof. V.F. Komarov for his participation in discussion of the problem statement and valuable advice.

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