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BRIEF COMMUNICATIONS

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A Generalization of the Polia–Szego and Makai Inequalities for Torsional Rigidity

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Abstract—We establish some generalizations of the classical inequalities by Polya–Szego and Makai about torsional rigidity of convex domains. The main idea of the proof is in using an exact isoperimetric inequality for Euclidean moments of domains. This inequality has a wide class of extremal regions and is of independent interest.

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INTRODUCTION

The initial mathematical interest to the torsional rigidity of homogeneous rods with fixed cross sections was related with the problem of measurement accuracy of torsion balance. In addition to physical characteristics of the material of a homogeneous rod, its length and twisting angle, the value of the applied force also depends on the form of its cross section.

One of the first results concerning finding the torsional rigidity P of a homogeneous rod with a circular cross section was the formula $\mathrm{P} = \pi r^4/2$ (here r is the radius of the cross section), suggested by Coulomb in the XIX century [1]. We note that this formula expresses explicitly the dependence of the torsional rigidity from the geometry of domain. However, it turned out that finding exact formulas for torsional rigidity of a rod with a given form of cross section is not a simple problem. On the other side, development of mathematical apparatus allows us to construct new estimations of the torsional rigidity, and these inequalities covered more wide classes of domains. Such estimates include the classical inequalities by Cauchy, St. Venant, Polya, Szego, Payne and many others [1]–[4].

Now we recall some definitions which we will need below. Let G be a simply connected domain on the plane. One of important characteristics of domains in mathematical physics is the functional

$$\mathbf{P}(G) := 2\int_G \mathrm{u}(x, G)\,\mathrm{dA}.$$

In the theory of elasticity it is called torsional rigidity, and in hydrodynamics it is named flow (see, e. g., [1, 5]). Here $u(x, G)$ is the stress function which satisfies the equation $\triangle u = -2$ in G and the boundary condition $u\,|_{\partial G} = 0$; here by $\mathrm{d}A$ we denote the differential area element on the plane.

One of well-known inequalities for torsional rigidity is the St. Venant–Polya inequality

$$\mathbf{P}(G) \le \frac{\mathbf{A}(G)^2}{2\pi} \tag{1}$$

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where $\mathbf{A}(G)$ is the area of G. On one side, it is the simplest inequality and one of the most important in the theory of twisting. On the other side, inequality (1) is an example of one-sided inequality for $\mathbf{P}(G)$, i.e., (1) can not be reversed by multiplying its right-hand side by an absolute constant, even for the class of convex domains.

An essential progress in the problem of obtaining a two-sided estimation of the functional $\mathbf{P}(G)$ was defining and using new characteristics of domains [6]. Denote by $\rho(x, G)$ the function of the distance from point x to the boundary of G. The geometric functional

$$\mathbf{I}_p(G) = \int\limits_G \rho(x, G)^p \, \mathrm{dA}$$

is called the Euclidean moment of the domain with respect to its boundary of order $p > -1$.

F.G. Avkhadiev [7, 8] showed that, in the class of simply connected domains, for $p = 2$ the torsional rigidity and the Euclidean moment are comparable values. Moreover, the two-sided estimation is valid:

$$\mathbf{I}_2(G) \le \mathbf{P}(G) \le 64\, \mathbf{I}_2(G). \tag{2}$$

Then the left inequality in (2) was improved [9]; it was showed that

$$\frac{3}{2}\, \mathbf{I}_2(G) < \mathbf{P}(G). \tag{3}$$

As we know, the constants 64 and 3/2 in (2) and (3) are not sharp.

In [10] it was shown that inequality (3) is a partial case of the inequality

$$\frac{(p+1)\mathbf{I}_p(G)}{2\boldsymbol{\rho}(G)^{p-2}} + \frac{\pi(p-2)\boldsymbol{\rho}(G)}{4(p+2)} \le \mathbf{P}(G), \tag{4}$$

where $p \ge 2$, and $\boldsymbol{\rho}(G)$ is the radius of the biggest disk containing in G. On the other side, in [11] it was established that in the class of convex domains the inequality

$$\mathbf{P}(G) < \frac{4(p+1)}{3}\mathbf{I}_p(G)\boldsymbol{\rho}(G)^{2-p} - \frac{2\pi\boldsymbol{\rho}(G)^4(2-p)}{3(p+2)}, \tag{5}$$

holds, where $-1 \le p \le 2$. Putting $p = 2$ in (5), we obtain Makai's inequality [12]

$$\mathbf{P}(G) < \frac{4}{3}\, \mathbf{A}(G)\boldsymbol{\rho}(G)^2.$$

We note that in (4) and (5) the parameter p changes at different intervals. In the class of convex domains, in [11] two-sided inequalities between Euclidean moments of different orders were established and the following estimation for the torsional rigidity was obtained

$$\frac{(p+1)\mathbf{I}_p(G)}{2\boldsymbol{\rho}(G)^{p-2}} + \frac{p\pi\boldsymbol{\rho}(G)^4}{2(p+2)} \le \mathbf{P}(G) \le \frac{2}{3}\left(\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\, \mathbf{I}_p(G) - p\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3\right),$$

where $p \ge 0$, and $\boldsymbol{l}(\boldsymbol{\rho}(G))$ is defined below, see formula (6). The left inequality for $p = 0$ coincides with the known inequality by Polya and Szego [1] and is its generalization. The right inequality for $p = 0$ belongs to Makai [12].

In this paper, we suggest one more method to construct such estimations in the class of convex domains.

1. MAIN RESULTS

We will call a convex domain G a stretching of a convex domain G_0, if G_0 can be obtained from G by cutting off some its rectangular fragment and then combining the rest parts so that $\boldsymbol{\rho}(G_0) = \boldsymbol{\rho}(G)$. On the other side, it is naturally to call G_0 a contraction of G. For example, a square is a contraction of a rectangle, and square is non-contractible. Stretching and contraction of a domain are defined not by a unique way. We can stretch it up to infinity and contract up to touching its side of the maximal inscribed circle. But not all domains can be stretchable. Actually, it is easy to see that triangles and polygonal domains with odd number of sides are non-stretchable. If G is not stretchable, then we put $G_0 \equiv G$.

Denote by Γ the subset of the set of convex domains which contains polygons, described near circles, and circular polygons obtained from described polygons by replacement of some their sides or parts of sides with arcs of the inscribed circle. At last, we add to Γ domains which are stretching of elements of Γ.

Let $\mu \geq 0$. Denote

$$G(\mu) := \{x \in G \mid \rho(x, G) > \mu\}, \quad \mathbf{a}(\mu) := \mathbf{A}(G(\mu)) := \int\limits_{G(\mu)} \mathrm{dA};$$

they are the level set of the distance function $\rho(x, G)$ and the area of the level set $G(\mu)$.

We will denote by $\mathbf{L}(G)$ the length of the boundary of G. Let

$$\boldsymbol{l}(\mu) := \mathbf{L}(G(\mu)), \quad \boldsymbol{l}(\boldsymbol{\rho}(G)) := \lim_{\mu \to \boldsymbol{\rho}(G)} \boldsymbol{l}(\mu). \tag{6}$$

In [11] the following statements are established.

Lemma 1. *Let G be a convex domain and $0 \leq \mu \leq \boldsymbol{\rho}(G)$. Then $G(\mu)$ is also a convex domain and the set $G(\boldsymbol{\rho}(G)) := \{x \in G : \rho(x, G) = \boldsymbol{\rho}(G)\}$ is either a single-point set or a segment of length $l(\boldsymbol{\rho}(G))/2$.*

Lemma 2. *Let G be a convex domain and $\mathbf{A}(G) < +\infty$. then*

$$\mathbf{a}(\mu) \geq \mathbf{A}(G)\left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right)^2 + \boldsymbol{l}(\boldsymbol{\rho}(G))\, \mu \left(1 - \frac{\mu}{\boldsymbol{\rho}(G)}\right), \tag{7}$$

where $0 \leq \mu \leq \boldsymbol{\rho}(G(\mu))$. The equality for all such μ is attained if and only if $G \in \Gamma$.

We note that in [11] the cases of equality in (7) are not completely described.

Let G be a convex domain. We juxtapose to G any domain $G^{\Diamond} \in \Gamma$ which has the same area, radius of maximal inscribed disc and the functional $\boldsymbol{l}(\boldsymbol{\rho}(G^{\Diamond})) = \boldsymbol{l}(\boldsymbol{\rho}(G))$. We note that such domain $G^{\Diamond}$ is not unique.

Theorem 1. *Let G be a convex domain of finite area. Let $P(t)$ be a nonconstant absolutely continuous function on $(0, \boldsymbol{\rho}(G))$ and $dP(t) \geq 0$. Then*

$$\int\limits_{G} P(\rho(x, G))\mathrm{dA} \geq \int\limits_{G^{\Diamond}} P(\rho(x, G^{\Diamond}))\, \mathrm{dA},$$

where $G^{\Diamond} \in \Gamma$ corresponds to G. The equality is attained for all domains from the class Γ.

Proof is based on the lemmas above and symmetrization methods.

Following [10], for $p > 0$ we define the functional

$$\mathbf{i}_p(\mu) := p \int_{\mu}^{\boldsymbol{\rho}(G)} t^{p-1} \mathbf{a}(t)\, \mathrm{d}t, \tag{8}$$

$0 \leq \mu \leq \boldsymbol{\rho}(G)$. We note that in the class of convex domains there is no need for additional restriction on the parameter p, in contrast to the case considered in [10]. If $\mu = 0$, then (8) coincides with the Euclidean moment of domain G with respect to its boundary, i. e., $\mathbf{i}_p(0) = \mathbf{I}_p(G)$.

The following auxiliary statement is valid; in some sense, it is inverse to the similar lemma from [10].

Lemma 3. *Let G be a convex domain on the plane of finite area. Then for $0 \le \mu \le \boldsymbol{\rho}(G)$ we have*

$$\mathbf{i_q}(\mu) \ge \frac{\mathbf{I}_q(G)\, y_q(\mu)}{2\boldsymbol{\rho}(G)^{q+2}} + \frac{q\, \boldsymbol{l}(\boldsymbol{\rho}(G))\, \mu^q (\boldsymbol{\rho}(G) - \mu)^2}{2\boldsymbol{\rho}(G)}, \tag{9}$$

where $q \ge 0$ and

$$y_q(\mu) = q(q+1)(q+2) \int\limits_{\mu}^{\boldsymbol{\rho}(G)} t^{q-1} (\boldsymbol{\rho}(G) - t)^2 \, \mathrm{dt}.$$

The equality in (9) *for all admissible μ takes place only for domains from the class Γ.*

Theorem 2. *Let G be a convex domain on the plane of finite area and $0 \le q \le 2$. Then for $p \ge q$ the inequality*

$$\mathbf{P}(G) \le \frac{4}{3(q+2)} \left[\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\, \mathbf{I}_p(G) - (p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3 \right] - \frac{2\pi(2-q)\boldsymbol{\rho}(G)^4}{3(q+1)(q+2)} \tag{10}$$

holds. The multiplicative constants of the functionals $\mathbf{I}_p(G)$ and $\boldsymbol{l}(\boldsymbol{\rho}(G))$ on the right-hand side of (10) *are sharp.*

For $q = 2$ the statement of Theorem 2 coincides with a theorem proved in [11].

Theorem 3. *Let G be a convex domain on the plane of finite area and let $q > 0$. Then for $0 \le p \le q$ the inequality*

$$\mathbf{P}(G) \ge \frac{1}{2(q+2)} \left[\frac{(p+1)(p+2)}{\boldsymbol{\rho}(G)^{p-2}}\, \mathbf{I}_p(G) + (q-p)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^3 \right] + \frac{\pi q \boldsymbol{\rho}(G)^4}{2(q+2)} \tag{11}$$

holds where the equality is attained only for a disc.

We note that, in both the theorems, the expressions in the brackets consist of two summands every of which is comparable with the torsional rigidity in the sense of Polya and Szego [1], but the rest of functionals on the right-hand sides of (10) and (11) are values of a higher order of smallness. Therefore, the functionals in the brackets, are examples of more complicated geometric characteristics of domains, comparable with the torsional rigidity.

In proving of the presented statements, the following theorem plays a key role; it is obtained by application of Lemma 3 and has an independent interest.

Theorem 4. *Let G be a convex domain on the plane with finite area. Then for $0 \le q \le p < \infty$ the inequality*

$$\mathbf{I}_p(G) \ge \frac{\boldsymbol{\rho}(G)^{p-q}}{(p+1)(p+2)} \left[(q+1)(q+2)\mathbf{I}_q(G) + (p-q)\boldsymbol{l}(\boldsymbol{\rho}(G))\boldsymbol{\rho}(G)^{q+1} \right]$$

is valid. The equality holds if and only if $G \in \Gamma$.

In conclusion, we note that the variety of inequalities of the considered form for the torsional rigidity is connected with the fact that in the class of convex domains

$$\boldsymbol{L}(G) < +\infty \Leftrightarrow \mathbf{A}(G) < +\infty \Leftrightarrow \mathbf{I}_p(G) < +\infty \ \ (-1 < p < +\infty) \Leftrightarrow \mathbf{P}(G) < +\infty$$

(see [4, 9, 11]).

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