On the best constants for the Brezis-Marcus inequalities in balls

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Abstract

We study the best possible constants $c(n)$ in the Brezis-Marcus inequalities

$$\int_{B_n} |\nabla u|^2 \, dx \;\geq\; \frac{1}{4} \int_{B_n} \frac{|u|^2}{(\rho - |x - x_0|)^2} \, dx + \frac{c(n)}{\rho^2} \int_{B_n} |u|^2 \, dx$$

for $u \in H_0^1(B_n)$ in balls $B_n = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. The quantity $c(1)$ is known by our paper in ZAMM **87** (2007), no.8-9, p. 632–642. In the present paper we prove the estimate $c(2) \geq 2$ and the assertion

$$\lim_{n \to \infty} \frac{c(n)}{n^2} = \frac{1}{4},$$

which gives that the known lower estimates by G. Barbatis, S. Filippas, and A. Tertikas in Comm. Cont. Math. **5** (2003), no. 6, 869–881 for $c(n)$, $n \geq 3$, are asymptotically sharp as $n \to \infty$. Also, for the 3-dimensional ball $B_3^0 = \{x \in \mathbb{R}^3 : |x| < 1\}$ we obtain a new Brezis-Marcus type inequality which contains two parameters $m \in (0, \infty)$, $\nu \in (0, 1/m)$ and has the following form

$$\int_{B_3^0} |\nabla u(x)|^2 \, dx \;\geq\; \frac{1}{4} \int_{B_3^0} \left\{ \frac{1 - \nu^2 m^2}{(1 - |x|)^2} \;+\; \frac{m^2 j_\nu^2}{(1 - |x|)^{2-m}} \right\} |u(x)|^2 \, dx,$$

where j_ν is the first zero of the Bessel function J_ν of order ν and the constants

$$\frac{1 - \nu^2 m^2}{4} \qquad \text{and} \qquad \frac{m^2 j_\nu^2}{4}$$

are sharp for all admissible values of parameters m and ν.

Keywords. Hardy type inequality, Bessel functions, Brezis-Marcus inequalities.

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