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On τ-essentially Invertibility of τ-measurable Operators

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Abstract

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let I be the unit of the algebra $\mathcal{M}$. A τ-measurable operator A is said to be *τ-essentially right (or left) invertible* if there exists a τ-measurable operator B such that the operator $I-AB$ (or $I-BA$) is τ-compact. A necessary and sufficient condition for an operator A to be τ-essentially left invertible is that A^*A (or, equivalently, $\sqrt{A^*A}$) is τ-essentially invertible. We present a sufficient condition that a τ-measurable operator A not be τ-essentially left invertible. For τ-measurable operators A and $P=P^2$ the following conditions are equivalent: 1. A is τ-essential right inverse for P; 2. A is τ-essential left inverse for P; 3. $I-A$,$I-P$ are τ-compact; 4. PA is τ-essential left inverse for P. For τ-measurable operators $A=A^3$, $B=B^3$ the following conditions are equivalent: 1. B is τ-essential right inverse for A; 2. B is τ-essential left inverse for A. Pairs of faithful normal semifinite traces on $\mathcal{M}$ are considered.

Keywords Hilbert space · Von Neumann algebra · Normal weight · Semifinite trace · Measure topology · τ-measurable operator · τ-compact operator · Rearrangement · τ-essentially invertible operator · Idempotent

1 Introduction

The section of functional analysis, called noncommutative integration theory, is an important part of the theory of operator algebras. This article is devoted to unbounded analogs of the classical results on essential invertibility of linear bounded operators on Hilbert spaces. The beginning of the development of the noncommutative integration theory is related to the names of I. Segal and J. Dixmier, who in the early 1950s created a theory of integration with respect to a trace on a semifinite von Neumann algebra [20]. The results of these investigations found spectacular applications in the duality theory for unimodular groups and stimulated the progress of “noncommutative probability theory”. The theory of algebras of measurable and locally measurable operators is rapidly developing and has interesting applications in various areas of functional analysis, mathematical physics, statistical mechanics,

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and quantum field theory. Hermitian idempotents ($A = A^2$) describe particles that can be found in two states (the spectrum $\sigma(A) = \{0, 1\}$), tripotents ($B = B^3$) relative to three states ($\sigma(B) = \{-1, 0, 1\}$). Every tripotent can be represented as a difference of two idempotents [5]. The differences of idempotents play an important part in the quantum Hall effect [2, 10, 11]. For invertibility of the difference of idempotents see, for example, [16–18].

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space and τ be a faithful normal semifinite trace on $\mathcal{M}$. Let I be the unit of the algebra $\mathcal{M}$. Denote by $\tilde{\mathcal{M}}$ the *-algebra of all τ-measurable operators. Unbounded idempotents and tripotents in $\tilde{\mathcal{M}}$ were studied in [8].

Let $\mu_t(X)$ denote the rearrangement of the operator $X \in \tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_0$ stand for $\{T \in \tilde{\mathcal{M}} : \mu_\infty(T) = \lim\limits_{t\to\infty} \mu_t(T) = 0\}$. An operator $A \in \tilde{\mathcal{M}}$ is said to be *τ-essentially right (or left) invertible* if there exists an operator $B \in \tilde{\mathcal{M}}$ such that the operator $I - AB$ (or $I - BA$) lies in $\tilde{\mathcal{M}}_0$. A necessary and sufficient condition that an operator $A \in \tilde{\mathcal{M}}$ be τ-essentially left invertible is that A^*A (or, equivalently, $\sqrt{A^*A}$) be τ-essentially invertible (Theorem 3.2). In Theorem 3.4 we present a sufficient condition that an operator $A \in \tilde{\mathcal{M}}$ not be τ-essentially left invertible. It is shown in Theorem 3.6 that for τ-measurable operators A and $P = P^2$ the following conditions are equivalent: (i) A is τ-essential right inverse for P; (ii) A is τ-essential left inverse for P; (iii) $I - A, I - P \in \tilde{\mathcal{M}}_0$; (iv) PA is τ-essential left inverse for P. For operators $A = A^3$, $B = B^3$ from $\tilde{\mathcal{M}}$ the following conditions are equivalent: (i) B is τ-essential right inverse for A; (ii) B is τ-essential left inverse for A (Corollary 3.8). Let an operator $T \in \tilde{\mathcal{M}}$ be such that the operator $I - T$ lies in $\tilde{\mathcal{M}}_0$. Then for all operators $X \in \tilde{\mathcal{M}}$ we have $\mu_\infty(TX) = \mu_\infty(XT) = \mu_\infty(X)$ (Theorem 3.11). Let operators $T_1, T_2, \ldots, T_n \in \tilde{\mathcal{M}}$ be such that $I - T_k \in \tilde{\mathcal{M}}_0$ for all $k = 1,2,\ldots,n$. Then $I - |T_1|, I - T_1T_2\cdots T_n \in \tilde{\mathcal{M}}_0$ (Theorem 3.13). Pairs of faithful normal semifinite traces on $\mathcal{M}$ are considered (Corollaries 3.18, 3.19).

2 Notation, Definitions, and Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$, let $\mathcal{M}^{\text{pr}}$ be the lattice of projections in $\mathcal{M}$. Let $\mathcal{M}^+$ be the cone of positive elements from $\mathcal{M}$ and let I be the unit of the algebra $\mathcal{M}$. A mapping $\varphi : \mathcal{M}^+ \to [0, +\infty]$ is called a *weight* if $\varphi(X + Y) = \varphi(X) + \varphi(Y)$, $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X, Y \in \mathcal{M}^+$, $\lambda \geq 0$ (here $0 \cdot (+\infty) \equiv 0$). Let $\mathcal{M}^{\text{pr}}_\varphi = \{P \in \mathcal{M}^{\text{pr}} : \varphi(P) < +\infty\}$. A weight φ is said to be *faithful* if $\varphi(X) > 0$ for all $X \in \mathcal{M}^+, X \neq 0$; *normal* if $X_i \nearrow X (X_i, X \in \mathcal{M}^+) \Rightarrow \varphi(X) = sup\varphi(X_i)$; a *trace* if $\varphi(Z^*Z) = \varphi(ZZ^*)$ for all $Z \in \mathcal{M}$. A trace is called *semifinite* if $\varphi(X) = sup\{\varphi(Y) : Y \in \mathcal{M}^+, Y \leq X, \varphi(Y) < +\infty\}$ for each $X \in \mathcal{M}^+$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A closed operator X, affiliated to $\mathcal{M}$ and possesing a domain $\mathcal{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be *τ-measurable* if, for any $\varepsilon > 0$, there exists a $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathcal{D}(X)$ and $\tau(I - P) < \varepsilon$. The set $\tilde{\mathcal{M}}$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [19, 20]. If an operator X belongs to $\tilde{\mathcal{M}}$ then its real and imaginary components $RX = (X + X^*)/2, IX = (X - X^*)/(2i)$ lie in the set $\tilde{\mathcal{M}}^{\text{sa}}$ of all self-adjoint τ-measurable operators, and $|X| = \sqrt{X^*X}$ lies in the cone $\tilde{\mathcal{M}}^+$ of all positive τ-measurable operators.

Let $\mu_t(X)$ denote the *rearrangement* of the operator $X \in \tilde{\mathcal{M}}$, i.e., nonincreasing right-continuous function $\mu(X)$: $(0,\infty) \to [0,\infty)$, given by the formula

$$\mu_t(X) = inf\{\| XP \|: P \in \mathcal{M}^{\mathrm{pr}}, \quad \tau(I-P) \le t\}, \quad t > 0.$$

The sets $U(\varepsilon, \delta) = \{X \in \tilde{\mathcal{M}} : (\| XP \| \le \varepsilon$ and $\tau(I-P) \le \delta$ for some $P \in \mathcal{M}^{\mathrm{pr}})\}$, where $\varepsilon > 0$, $\delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $\tilde{\mathcal{M}}$, called *the measure topology* ([19, 22, p. 18]). Equipped with this topology, $\tilde{\mathcal{M}}$ is a complete topological *-algebra in which $\mathcal{M}$ is dense. We will write $X_n \tau \to X$ if a sequence $\{X_n\}_{n=1}^{\infty}$ converges to $X \in \tilde{\mathcal{M}}$ in the measure topology on $\tilde{\mathcal{M}}$. A sequence $\{X_n\}_{n=1}^{\infty}$ is said to be converge τ-locally to $X \in \tilde{\mathcal{M}}$ (notation: $X_n \tau l \to X$) if $X_n P \tau \to XP$ for all $P \in \mathcal{M}_\tau^{\mathrm{pr}}$, cf. [12, p. 114].

The set of τ-compact operators $\tilde{\mathcal{M}}_0 = \{X \in \tilde{\mathcal{M}} : \mu_\infty(X) \equiv \lim_{t\to\infty} \mu_t(X) = 0\}$ is an ideal in $\tilde{\mathcal{M}}$ [23]. The set of τ-elementary operators $\mathcal{F}(\mathcal{M}) = \{X \in \mathcal{M} : \mu_t(X) = 0$ for some $t > 0\}$ is an ideal in $\mathcal{M}$. Let m be a linear Lebesgue measure on $\mathbb{R}$. A non-commutative L_p-Lebesgue space $(0 < p < +\infty)$ affiliated with $(\mathcal{M}, \tau)$ can be defined as $L_p(\mathcal{M}, \tau) = \{X \in \tilde{\mathcal{M}} : \mu(X) \in L_p(\mathbb{R}^+, m)\}$ with the F-norm (the norm for $1 \le p < +\infty$) $\|X\|_p = \|\mu(X)\|_p$, $X \in L_p(\mathcal{M}, \tau)$. We have $\mathcal{F}(\mathcal{M}) \subset L_p(\mathcal{M}, \tau) \subset \tilde{\mathcal{M}}_0$ for all $0 < p < +\infty$.

An operator $A \in \tilde{\mathcal{M}}$ is said to be τ-*essentially right (or left) invertible* if there exists an operator $B \in \tilde{\mathcal{M}}$ such that the operator $I - AB$ (or $I - BA$) is τ-compact. An operator $A \in \tilde{\mathcal{M}}$ is said to be τ-*essentially invertible* if there exists an operator $B \in \tilde{\mathcal{M}}$ such that the operators $I - AB$ and $I - BA$ are simultaneously τ-compact. On τ-compactness of products of τ-measurable operators see [9].

Lemma 2.1 *(see* [13, 23]*). We have* $\mu_{s+t}(X+Y) \le \mu_s(X) + \mu_t(Y)$ *for all* $X, Y \in \tilde{\mathcal{M}}$ *and* $s, t > 0$.

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$ is the *-algebra of all bounded linear operators on $\mathcal{H}$ and $\tau = \mathrm{tr}$ is the canonical trace then $\tilde{\mathcal{M}}$ coincides with $\mathcal{B}(\mathcal{H})$, and $\tilde{\mathcal{M}}_0$ coincides with the ideal of compact operators on $\mathcal{H}$; the τ-local convergence coincides with the strong-operator convergence. We have $\mu_t(X) = \sum_{n=1}^{\infty} s_n(X)\chi_{[n-1,n)}(t)$, $t > 0$, where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X [14, Chap. II, §2] and χ_A is the indicator of the set $A \subset \mathbb{R}$. Then the space $L_p(\mathcal{M}, \tau)$ is a Shatten–von Neumann ideal $\mathfrak{S}_p$, $0 < p < +\infty$.

Let (Ω,ν) be a measure space and $\mathcal{M}$ be the von Neumann algebra of multiplicator operators by functions from $L_\infty(\Omega,\nu)$ on the space $L_2(\Omega,\nu)$. The algebra $\mathcal{M}$ containes no compact operators $\Longleftrightarrow$ the measure ν has no atoms [1, Theorem 8.4].

3 On τ-essentially Invertible τ-measurable Operators

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$ and $\tau(I) = \infty$.

Proposition 3.1 *For every τ-essentially left (or right) invertible operator $A \in \tilde{\mathcal{M}}$ its τ-essential left (or right) inverse may be chosen in $\mathcal{M}$. The set $L_A = \{B \in \tilde{\mathcal{M}} : I - \mathrm{BA} \in \tilde{\mathcal{M}}_0\}$ (or $\{B \in \tilde{\mathcal{M}} : I - AB \in \tilde{\mathcal{M}}_0\}$) is convex and t_τ-closed in $\tilde{\mathcal{M}}$. If operators $A, T \in \tilde{\mathcal{M}}$ have τ-essential left inverses B,S, respectively, then* TA *has a τ-essential left inverse* TSBS.

Proof By [21, p. 75] we have $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_0 + \mathcal{M}$ (i.e. every operator $X \in \tilde{\mathcal{M}}$ has the form $X_1 + X_2$ with $X_1 \in \tilde{\mathcal{M}}_0$ and $X_2 \in \mathcal{M}$). We also have $I - (tB_1 + (1-t)B_2)A = t(I - B_1A) + (1-t)(I - B_2A) \in \tilde{\mathcal{M}}_0$ for all $t \in \mathbb{C}$ and $B_1, B_2 \in L_A$. Since the multiplication operation $Z \mapsto \mathrm{ZY}(\tilde{\mathcal{M}} \to \tilde{\mathcal{M}})$ is t_τ-continuous and $\tilde{\mathcal{M}}_0$ is t_τ-closed in $\tilde{\mathcal{M}}$, the set L_A is t_τ-closed in $\tilde{\mathcal{M}}$. Finally, we have $C = I - \mathrm{ST}, \mathrm{TACBS} \in \tilde{\mathcal{M}}_0$ and $T(I - AB)S = (-I + TS) + (I - \mathrm{TA}(TS + C)\mathrm{BS}) = (-I + TS) + (I - \mathrm{TA} \cdot \mathrm{TSBS}) - \mathrm{TACBS} \in \tilde{\mathcal{M}}_0$. If an operator $T \in \tilde{\mathcal{M}}$ is invertible in $\tilde{\mathcal{M}}$ then TAT^{-1} is τ-essentially left (or right) invertible. The proposition is proved. □

Theorem 3.2 *A necessary and sufficient condition for an operator $A \in \tilde{\mathcal{M}}$ to be τ-essentially left invertible is τ-essential invertibility of A^*A (or, equivalently, $|A|$).*

Proof The proof is similar to the proof of Theorem 14.4 [15]. □

Corollary 3.3 *A necessary and sufficient condition for an operator $A \in \tilde{\mathcal{M}}$ to be τ-essentially right invertible is τ-essential invertibility of AA^* (or, equivalently, $|A^*|$).*

Theorem 3.4 *A sufficient condition for an operator $A \in \tilde{\mathcal{M}}$ not to be τ-essentially left invertible is the existence of a t_τ-bounded sequence $\{X_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{M}}$ suchthat*

$$X_n^* \xrightarrow{\tau l} 0, \quad X_n \overset{\tau}{\not\longrightarrow} 0, \quad AX_n \xrightarrow{\tau} 0 \quad (n \to \infty).$$

Proof If $AX_n\tau \to 0 (n \to \infty)$, and $I - \mathrm{BA} = C \in \tilde{\mathcal{M}}_0$, then

$$X_n - CX_n = X_n - (I - \mathrm{BA})X_n = \mathrm{BA}X_n\tau \to 0 \quad (n \to \infty),$$

since the multiplication operation $Z \mapsto YZ(\tilde{\mathcal{M}} \to \tilde{\mathcal{M}})$ is t_τ-continuous. The mapping $Z \mapsto Z^*(\tilde{\mathcal{M}} \to \tilde{\mathcal{M}})$ is t_τ-continuous, hence

$$(X_n - CX_n)^* = X_n^* - X_n^*C^*\tau \to 0 \quad (n \to \infty). \tag{1}$$

Since $C^* \in \tilde{\mathcal{M}}_0$ and $X_n^*\tau l \to 0 (n \to \infty)$, by Theorem 2 of [3] we have

$$X_n^*C^*\tau \to 0 \quad (n \to \infty).$$

Now via (1) we have $X_n^*\tau \to 0 (n \to \infty)$. The mapping $Z \mapsto Z^*(\tilde{\mathcal{M}} \to \tilde{\mathcal{M}})$ is t_τ-continuous, hence $X_n\tau \to 0 (n \to \infty)$ – a contradiction. Theorem is proved. □

Corollary 3.5 *A sufficient condition for an operator $A \in \tilde{\mathcal{M}}$ not to be τ-essentially right invertible is the existence of a t_τ-bounded sequence $\{X_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{M}}$ such that*

$$X_n^* \xrightarrow{\tau l} 0, \quad X_n \overset{\tau}{\not\longrightarrow} 0, \quad A^*X_n \xrightarrow{\tau} 0 \quad (n \to \infty).$$

Theorem 3.6 *For τ-measurable operators A and $P = P^2$ the following conditions are equivalent:*

(i) *A is τ-essentialright inverse for P;*
(ii) *A is τ-essentialleft inverse for P;*
(iii) *$I - A, I - P$ are τ-compact;*
(iv) *PA is τ-essentialleft inverse for P.*

Proof

(i)⇒(iii). We have $I - P = (I - P)(I - PA) \in \tilde{\mathcal{M}}_0$ and $I - A = I - PA - (I - P)A \in \tilde{\mathcal{M}}_0$.

(iii)⇒(i), (ii). We have $I - PA = I - A + (I - P)A \in \tilde{\mathcal{M}}_0$ and $I - AP = I - A + A(I - P) \in \tilde{\mathcal{M}}_0$.

(ii)⇒(iii). Since $I - P = (I - AP)(I - P) \in \tilde{\mathcal{M}}_0$ the inclusion $I - A = I - AP - A(I - P) \in \tilde{\mathcal{M}}_0$ holds.

(iv)⇒(i). Note that $I - P = (I - P)(I - PAP) \in \tilde{\mathcal{M}}_0$ and $AP - PA = (I - P)A - A(I - P) \in \tilde{\mathcal{M}}_0$. So $(I - PA) - (I - PAP) = (AP - PA)(I - P) \in \tilde{\mathcal{M}}_0$ and $I - PA \in \tilde{\mathcal{M}}_0$.

(i)⇒(iv). We have $I - P = (I - P)(I - PA) \in \tilde{\mathcal{M}}_0$ and $-(I - P) + (I - PAP) = (I - PA)P \in \tilde{\mathcal{M}}_0$. Thus $I - PAP \in \tilde{\mathcal{M}}_0$. Theorem is proved. □

Example 3.7 (Example 1 in [7]). The idempotence does not imply boundedness. Assume that $0 < p,q < \infty$ and $a_n = 2^{n+1}n^{-q}$, $n \in \mathbb{N}$. Let us equip the von Neumann algebra $\mathcal{M} = \oplus_{n=1}^{\infty}\mathbb{M}_2(\mathbb{C})$ with the faithful normal finite trace $\varphi = \oplus_{n=1}^{\infty}2^{-n}\mathrm{tr}_2$ and put $A = \oplus_{n=1}^{\infty}\begin{pmatrix} 1 & a_n \\ 0 & 0 \end{pmatrix}$. We have $A = A^2$, the operator A lies in $L_p(\mathcal{M}, \varphi)$ if $pq > 1$, and $A \notin L_p(\mathcal{M}, \varphi)$ if $pq \le 1$.

Corollary 3.8 *For operators $A = A^3$, $B = B^3$ from $\tilde{\mathcal{M}}$ the following conditions are equivalent:*

(i) B *is τ-essential right inverse for* A*;*
(ii) B *is τ-essential left inverse for* A.

Proof We show (i)⇒(ii). Let $P = P^2$, $Q = Q^2$, $S = S^2$, $T = T^2$ in $\tilde{\mathcal{M}}$ be such that

$$\mathrm{PQ} = \mathrm{QP} = 0, \mathrm{ST} = TS = 0 \text{ and } A = P - Q, B = S - T,$$

see Proposition 1 in [5]. Since $I - AB \in \tilde{\mathcal{M}}_0$, we conclude that

$$I - A^2B^2 = A(I - AB)B + (I - AB) \in \tilde{\mathcal{M}}_0.$$

Since $A^2 = P + Q = (A^2)^2$ and $B^2 = S + T = (B^2)^2$, by Theorem 3.6 we have

$$I - B^2A^2 \in \tilde{\mathcal{M}}_0. \tag{2}$$

Since $V = I - \mathrm{PS} - \mathrm{QT} = 2^{-1}((I - AB) + (I - A^2B^2)) \in \tilde{\mathcal{M}}_0$, the operators $P - S = PV - VS$, $Q - T = QV - VT$ lie in $\tilde{\mathcal{M}}_0$. Therefore, there exist $K, C \in \tilde{\mathcal{M}}_0$ such that $P = S + K$, $Q = T + C$. Now by (2) we have

$$\begin{aligned} I - \mathrm{BA} &= I - B^2A^2 + 2\mathrm{TP} + 2\mathrm{SQ} = I - B^2A^2 + 2T(S + K) + 2S(T + C) \\ &= I - B^2A^2 + 2\mathrm{TK} + 2\mathrm{SC} \in \tilde{\mathcal{M}}_0. \end{aligned}$$

The implication (ii)⇒(i) can be handled similarly. The assertion is proved. □

Theorem 3.9 *If an operator $A \in \tilde{\mathcal{M}}$ is τ-essentially right and left invertible then it is τ-essentially invertible.*

Proof We show that the difference $B_1 - B_2$ of any τ-essential right B_1 and left B_2 inverses is τ-compact and both B_1 and B_2 are τ-essential inverse for A. Let

$$I - AB_1 = X_1, I - B_2A = X_2.$$

Then $X_1, X_2 \in \tilde{\mathcal{M}}_0$ and $(I - X_2)B_1 = B_2AB_1 = B_2(I - X_1)$, hence $B_1 = B_2 + X$ with $X = X_2B_1 - B_2X_1 \in \tilde{\mathcal{M}}_0$. Therefore

$$B_1A = B_2A + XA = I - X_2 + XA.$$

This completes the proof. □

Corollary 3.10 *For an operator $A \in \tilde{\mathcal{M}}^{\mathrm{sa}}$ the following conditions are equivalent:*

(i) A *is τ-essentially right invertible;*
(ii) A *is τ-essentially left invertible;*
(iii) A *is τ-essentially invertible.*

In this case the operator A *also possesses a self-adjoint τ-essential inverse.*

Proof An operator $X \in \tilde{\mathcal{M}}$ is τ-essentially right invertible if and only if the adjoint operator X^* is τ-essentially left invertible. Let an operator $A \in \tilde{\mathcal{M}}^{\mathrm{sa}}$ and an operator $B \in \tilde{\mathcal{M}}$ be such that $I - AB, I - \mathrm{BA} \in \tilde{\mathcal{M}}_0$. Then $2\left(I - A\frac{B+B^*}{2}\right) = 2I - AB - AB^* = I - AB + (I - \mathrm{BA})^* \in \tilde{\mathcal{M}}_0$. This completes the proof. □

Theorem 3.11 *Let an operator $T \in \tilde{\mathcal{M}}$ be such that the operator $I - T$ is τ-compact. Then for all operators $X \in \tilde{\mathcal{M}}$ we have $\mu_\infty(TX) = \mu_\infty(XT) = \mu_\infty(X)$.*

Proof For $X \in \tilde{\mathcal{M}}_0$ the assertion is obvious. Assume that $X \notin \tilde{\mathcal{M}}_0$. Since $(I - T)X \in \tilde{\mathcal{M}}_0$, we have

$$\forall\varepsilon > 0 \exists t_1 > 0 \forall t > t_1\, (\mu_{t/2}((I - T)X)) < \varepsilon).$$

The function $\mu(X)$ is nonincreasing, so

$$\forall\varepsilon > 0 \exists t_2 > 0 \forall t > t_2\, (\mu_\infty(X) \le \mu_t(X) < \mu_\infty(X) + \varepsilon).$$

Let $\varepsilon > 0$ be arbitrary and $t_0 = \max\{t_1, t_2\}$. We have for all $t > t_0$ via Lemma 2.1 the following estimates:

$$\begin{aligned}\mu_\infty(X) &\le \mu_t(X) = \mu_t((I - T)X + \mathrm{TX}) \le \mu_{t/2}((I - T)X) + \mu_{t/2}(\mathrm{TX}) \\ &\le \varepsilon + \mu_{t/2}(\mathrm{TX}),\end{aligned}$$

i. e. $\mu_t(TX) \ge \mu_\infty(X) - \varepsilon$ for all $t > 2t_0$. On the other hand via Lemma 2.1 for all $t > 2t_0$ we have

$$\begin{aligned}\mu_t(\mathrm{TX}) &= \mu_t(X - (I - T)X) \le \mu_{t/2}(X) + \mu_{t/2}((I - T)X) \\ &\le \mu_\infty(X) + 2\varepsilon.\end{aligned}$$

The rest is obvious. The theorem is proved. □

Corollary 3.12 *Let an operator $A \in \tilde{\mathcal{M}}$ have a τ-essential right (or left) inverse $B \in \tilde{\mathcal{M}}$. Then for all operators $X \in \tilde{\mathcal{M}}$ we have $\mu_\infty(ABX) = \mu_\infty(XAB) = \mu_\infty(X)$ (or $\mu_\infty(BAX) = \mu_\infty(XBA) = \mu_\infty(X)$).*

Theorem 3.13

(i) *s Let operators* $T_1, T_2, \ldots, T_n \in \tilde{\mathcal{M}}$ *be such that* $I - T_k \in \tilde{\mathcal{M}}_0$ *for all* $k = 1,2,\ldots,n$. *Then* $I - |T_1|, I - T_1T_2\cdots T_n \in \tilde{\mathcal{M}}_0$.
(ii) *Let operators* $T_1, T_2, \ldots, T_n \in \mathcal{M}$ *be such that* $I - T_k \in \mathcal{F}(\mathcal{M})$ *for all* $k = 1,2,\ldots,n$. *Then* $I - |T_1|, I - T_1T_2\cdots T_n \in \mathcal{F}(\mathcal{M})$.

Proof

(i). We have $I - |T_1|^2 = I - T_1^*T_1 = I - T_1 + (I - T_1)^*T_1 \in \tilde{\mathcal{M}}_0$. Since the operator $I + |T_1|$ is invertible with $(I + |T_1|)^{-1} \in \mathcal{M}^+$, the operator $I - |T_1| = (I - |T_1|^2)(I + |T_1|)^{-1}$ belongs to $\tilde{\mathcal{M}}_0$. Consider $A = T_1T_2\cdots T_n$. We have

$$\begin{aligned} I - A &= (I - T_1) + (T_1 - T_1T_2) + (T_1T_2 - T_1T_2T_3) + \cdots + (T_1T_2\cdots T_{n-1} - A) \\ &= (I - T_1) + T_1(I - T_2) + T_1T_2(I - T_3) + \cdots + T_1T_2\cdots T_{n-1}(I - T_n) \in \tilde{\mathcal{M}}_0. \end{aligned}$$

The item (ii) can be handled similarly. The theorem is proved. □

Example 3.14 Let us equip the von Neumann algebra $\mathcal{M} = \oplus_{n=1}^{\infty}\mathbb{M}_2(\mathbb{C})$ with the faithful normal semifinite trace $\tau = \oplus_{n=1}^{\infty}\mathrm{tr}_2$ and set

$$X = \oplus_{n=1}^{\infty}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \quad Z = \oplus_{n=1}^{\infty}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $X \in \mathcal{M}^{\mathrm{pr}}$, $Z \in \mathcal{M}^{\mathrm{sa}}$ and $\mathrm{XZX} = 0 \in \tilde{\mathcal{M}}_0$, but the operator $\mathrm{XZ} \notin \tilde{\mathcal{M}}_0$.

Proposition 3.15 *If an operator* $A \in \tilde{\mathcal{M}}$ *is such that* $A^n \in \tilde{\mathcal{M}}_0$ *for some* $n \in \mathbb{N}$ *then A is not* τ*-essentially left (or right) invertible.*

Proof An operator $A \in \tilde{\mathcal{M}}$ is τ-essentially left invertible if and only if A^{2^k} is τ-essentially left invertible for all $k \in \mathbb{N}$. Indeed, if $I - BA^2 = C \in \tilde{\mathcal{M}}_0$, then BA is a τ-essential left inverse of A; if, on the other hand, $I - \mathrm{BA} = C \in \tilde{\mathcal{M}}_0$, then

$$I - B^2A^2 = I - B(I - C)A = I - \mathrm{BA} + \mathrm{BCA} = C + \mathrm{BCA} \in \tilde{\mathcal{M}}_0,$$

so B^2 is a τ-essential left inverse of A^2. Thus, for a τ-essentially left invertible A and $k \in \mathbb{N}$ with $2^k \geq n$ we have a τ-essentially left invertible τ-compact operator A^{2^k} – a contradiction. Since $A^{*n} = A^{n*} \in \tilde{\mathcal{M}}_0$, an operator A is also not τ-essentially right invertible. By Corollary 3.12 we have $\mu_\infty(B^{2^m}A^{2^m}X) = \mu_\infty(XB^{2^m}A^{2^m}) = \mu_\infty(X)$ for all $X \in \tilde{\mathcal{M}}$ and $m \in \mathbb{N}$. The assertion is proved. □

Remark 3.16 Corollary 3.8, Theorems 3.4, 3.6, 3.11 and 3.13 are new even for the algebra $\mathcal{M} = \mathcal{B}(\mathcal{H})$ endowed with the canonical trace $\tau = \mathrm{tr}$.

Theorem 3.17 *Let* τ *be a faithful normal semifinite trace on a von Neumann algebra* $\mathcal{M}$ *and* φ *be a normal weight on* $\mathcal{M}$. *If* $\mathcal{M}_\tau^{\mathrm{pr}} \subset \mathcal{M}_\varphi^{\mathrm{pr}}$, *then*

$$\forall \varepsilon > 0 \exists \delta > 0\, (P \in \mathcal{M}^{\mathrm{pr}}, \tau(P) < \delta \Rightarrow \varphi(P) < \varepsilon).$$

Proof Assume the contrary. Then

$$\exists \varepsilon > 0 \forall \delta > 0 \exists P_\delta \in \mathcal{M}^{\mathrm{pr}}\, (\tau(P_\delta) < \delta \text{ and } \varphi(P_\delta) \geq \varepsilon).$$

Choose $\delta_n = 2^{-n}$ for all $n \in \mathbb{N}$ and put $P = \vee_{n=1}^{\infty} P_{\delta_n}$, the least upper bound of $\{P_{\delta_n}\}_{n=1}^{\infty}$. So

$$\tau(P) \leq \sum_{n=1}^{\infty} \tau(P_{\delta_n}) < \sum_{n=1}^{\infty} 2^{-n} = 1 < +\infty$$

and $\varphi(P) < +\infty$. The restrictions $\tau_1 = \tau|_{\mathcal{M}_P}$, $\varphi_1 = \varphi|_{\mathcal{M}_P}$ are normal functionals on the reduced von Neumann algebra $\mathcal{M}_P (= P\mathcal{M}P)$, moreover, $\varphi_1 = \varphi(P \cdot P)$ and $\tau_1 = \tau(P \cdot P)$ is faithful and tracial. Let $T = \frac{d\varphi_1}{d\tau_1} \in L_1(\mathcal{M}_P, \tau_1)$ be the Radon–Nicodym derivative [20, Theorem 14]. Then by absolute continuity of the Segal's integral (see also [6, Theorem 2.10]) we have $\varphi(P_{\delta_n}) = \varphi_1(P_{\delta_n}) = \tau_1(TP_{\delta_n}) = \tau_1(P_{\delta_n} T P_{\delta_n}) \to 0$ as $n \to \infty$. A contradiction. Theorem is proved. □

Corollary 3.18 *Let τ, φ be faithful normal semifinite traces on a von Neumann algebra $\mathcal{M}$ and $\mathcal{M}_\tau^{\mathrm{pr}} \subset \mathcal{M}_\varphi^{\mathrm{pr}}$. Then*

(i) *every τ-measurable operator is φ-measurable operator;*
(ii) *every τ-compact operator is φ-compact operator;*
(iii) *every τ-elementary operator is φ-elementary operator;*
(iv) *every τ-essentially left invertible operator is φ-essentially left invertible operator;*
(v) *every t_τ-convergent sequence of τ-measurable operators is t_φ-convergent.*

Proof

(i). Let X be a closed densely defined linear operator affiliated to the von Neumann algebra $\mathcal{M}$ and $|X| = \int_0^\infty \lambda P^{|X|}(\mathrm{d}\lambda)$ be the spectral decomposition. Then X is τ-measurable if and only if there exists $\lambda \in \mathbb{R}$ such that $\tau(P^{|X|}((\lambda, +\infty))) < +\infty$.
(ii). We have $\tilde{\mathcal{M}}_0 = \{X \in \tilde{\mathcal{M}} : \tau(P^{|X|}(\lambda, \infty)) < +\infty \forall \lambda > 0\}$.
(iii). An operator $X \in \mathcal{M}$ is τ-elementary if and only if there exists $P \in \mathcal{M}_\tau^{\mathrm{pr}}$ such that $XP = 0$.
(iv). Follows by (ii).
(v). The topology t_τ can be determined by the metric

$$\rho_\tau(X, Y) = \inf_{P \in \mathcal{M}^{\mathrm{pr}}} \max\{\| (X - Y)P \|, \tau(I - P)\}$$

for all τ-measurable operators X, Y. □

Corollary 3.19 ([4, Corollary 1]). *For a von Neumann algebra $\mathcal{M}$ with faithful normal finite trace, the topology of convergence in measure on $\tilde{\mathcal{M}}$ is independent of the choice of such a trace.*

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