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# Mathematics of Planet Earth

## Proceedings of the 15th Annual Conference of the International Association for Mathematical Geosciences

Springer

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ISSN 2193-8571 ISSN 2193-858X (electronic)
ISBN 978-3-642-32407-9 ISBN 978-3-642-32408-6 (eBook)
DOI 10.1007/978-3-642-32408-6
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013943567



Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

# New Analytical Solutions for Phreatic Darcian Flows Over Non-Planar Bedrocks

**Anvar Kacimov, Yurii Obnosov and Osman Abdalla**

## 1 Introduction

Groundwater flow in unconfined aquifers is characterized by a free (phreatic) surface and nonlinear boundary conditions there [1, 2]. Common catchment-scale reconnaissance models or regular annual assessment of aquifers' resources utilize a hydraulic Dupuit–Forchheimer (DF) approximation, which in steady regimes requires solving a boundary-value problem for a second-order ordinary differential equation. A more general, potential theory (PT), solves Laplace's equation, provided the aquifer is homogeneous. In the arid climate of Northern Oman, with a periodic occasional rainfalls of 200–300 mm/year in mountains (2–3 km high) and 100 m/year in the valley zones of catchments, which are several tens of kilometers long, recharge from the vadose zone to the water table can be neglected everywhere but the fractured rock in upper reaches of North Oman Mountains (NOM). The main factor controlling the shape and locus of the phreatic surface is the subjacent bedrock whose geometry is commonly inferred from geological data.

In the study area (Northern Oman), for which our model is developed, the geology ranges from the Precambrian basement rocks, mainly phyllites and slates, at the bottom of the succession occupying the core of NOM to karstified carbonate rocks (Hajar Supergroup, HSG) at the elevated areas to fractured ophiolitic sequence overlain by porous medium of Tertiary limestones and Quaternary alluvium gravel at the top of the geologic section. Recent monitoring of the water table, whose slope is steep in

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E. Pardo-Igúzquiza et al. (eds.), *Mathematics of Planet Earth*,
Lecture Notes in Earth System Sciences, DOI: 10.1007/978-3-642-32408-6_91,

the mountains and relatively mild in the valley part of the catchment, revealed a puzzling spatial variability, detected in direct borehole observations and reconstructed geophysical (mostly TDEM) surveys. The degree of this steepness, position of the water table and other aquifer characteristics are vital because groundwater is the main resource for agriculture and other sectors of Omani economy.

In standard DF or PT models the bedrock boundary of an unconfined aquifer is assumed to be planar [2]. In Refs. [3, 4] steep slopes of the free surface were attributed to a "groundwater fall" geometry of the bedrock, i.e. a non-planar aquifuge boundary making a step-down. In hillslope hydrology, both the DF and PT models are used but explicit closed-form solutions (like ours below) to phreatic-surface flow problems are rare. Here we extend the model of Ref. [4] and consider the following bedrock "anomalies": (a) an aquifer with an underlying aquifuge whose inclination changes abruptly from aquifer's upflow to downflow (Fig. 1) and (b) an aquifuge with a continuously varying slope. Correspondingly, we apply two different techniques: the hodograph method [4] and boundary-value problem method [5]. We assume a Darcian flow, ignore the capillary fringe, accretion or evapotranspiration to/from the vadose zone and any sinks-sources (e.g. pumping wells) in the flow domain.

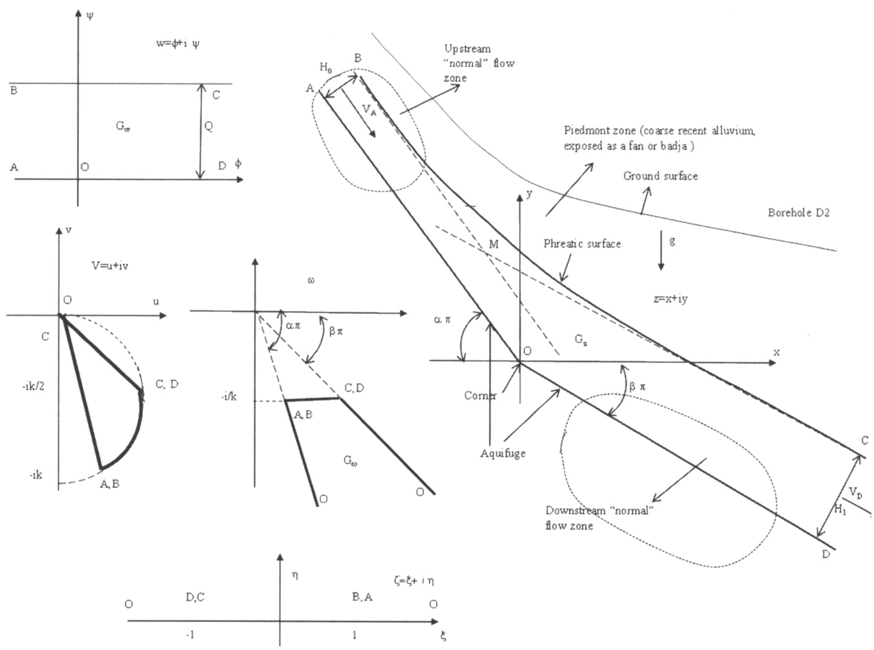


**Fig. 1** Phreatic flow over a corner-shaped aquifuge; physical, complex potential, hodograph, inverted and auxiliary domains

## 2 Flow Over Non-Planar Aquifuge

The bedrock AOD makes a corner (Fig. 1). The origin of a Cartesian coordinate system coincides with the vertex O. The flanks of the wedge, OD and AO, dip at angles $\alpha\pi$ (counted from Ox positive clockwise) and $\beta\pi$ (positive counterclockwise), correspondingly. Without any loss of generality we consider here the "hillslope" case of $0 < \alpha = \text{const} < 1/2$, $0 < \beta = \text{const} < 1/2$. If $\alpha > \beta$, flow decelerates downstream of the transition zone near point O, otherwise it accelerates. The flow rate (per unit length perpendicular to the plane of Fig. 1) is Q. A PT-based solution was obtained in Ref. [1] for $\alpha = -1/2$, $\beta = 1/2$; in Refs. [3, 4] the case of $\alpha = 0, \beta = 1/2$ was studied, in Ref. [2] winding seepage in domains with sharp-edged impermeabilities was considered.

If $\alpha > 0$ and $\beta < 1/2$, then BC far upflow and downflow of O is parallel to the bedrock i.e. a 1-D unidirectional flow is "normal" of saturated thicknesses $H_0$ and $H_1$ far above and below point O, respectively. The corresponding zones are schematically demarcated by dotted lines in Fig. 1. In these zones flow is aligned with the bedrock, the 1-D DF approximation works well and gives exactly the same solution as PT. In the conjugation zone of Fig. 1, the free surface BC is essentially non-straight. Dashed lines in Fig. 1 represent the "primitive" phreatic surface corresponding to two "normal" flows at constant slopes $\alpha\pi$ and $\beta\pi$, i.e. the straight lines $y = -\tan\alpha\pi x + H_0/\cos\alpha\pi$ and $y = -\tan\beta\pi x + H_1/\cos\beta\pi$. The "primitive" lines intersect at the point M and the corresponding "phreatic corner" BMC would be a simplistic Dupuit replica of AOD, translated. The angularity of AOD affects the shape of BMC in PT.

We introduce a complex physical coordinate $z = x+iy$, hydraulic head $h(x, y)$, Darcian velocity vector $\mathbf{V} = -k\nabla h$, velocity potential $\phi = -kh$, stream function $\psi$, complex potential $w = \phi + i\psi$ and complexified Darcian velocity $V = u + iv$. $\phi$, $\psi$ and h are harmonic functions. $\phi + ky = 0$ along BC. In the w-plane we have a strip $G_w$ (Fig. 1) corresponding to the flow domain $G_z$. In the hodograph plane, we have a circular triangle $G_V$. In Fig. 1 the case of $\alpha > \beta$ is illustrated with O being a stagnation point. If $\alpha < \beta$ then $V_O = \infty$ i.e. the hodograph trigon is infinite. The magnitudes of velocities in the "normal" flow zones of Fig. 1 are $|V_A| = |V_B| = k\sin\alpha\pi$ and $|V_C| = |V_D| = k\sin\beta\pi$. From the mass balance, $Q = H_0|V_A| = H_1|V_D|$. We use the method of inversion [2] and invert $G_V$ into a trigon $G_\omega$ where $\omega = dz/dw$ (Fig. 1). If $\alpha < \beta$ then $G_\omega$ in Fig. 1 is a standard triangle. We map conformally $G_w$ onto $G_\omega$ via an auxiliary plane $\zeta = \xi + i\eta$ using the Schwarz–Christoffel formula twice. After some algebra we obtain equations of BC. Fig. 2 represents the results of computations for $\alpha = 0.35, \beta = 0.1$. This and computations for other tilt angles illustrate the accuracy of the DF approximation as compared with a full 2-D model.

The case of AOD in Fig. 1 as an arbitrary curve is tackled by the method from

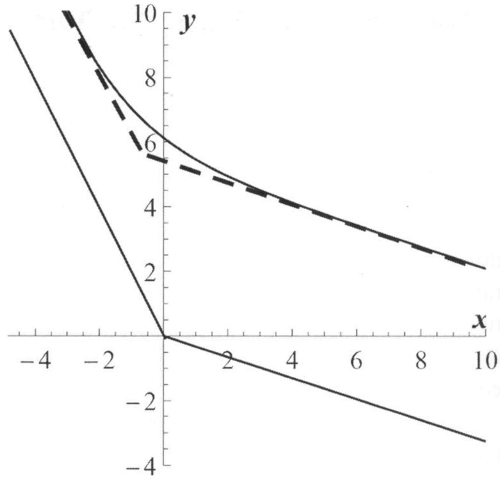


**Fig. 2** Phreatic surface for $\alpha = 0.35, \beta = 0.1$ (*solid curve*) computed by PT, *dashed lines* are DF asymptotics

singular (Cauchy-type) integrals. Like on Fig. 2 the phreatic surfaces are plotted but the confining boundary also emerging as a part of solution.

Similarly to Ref. [6] our groundwater system if gravity-controlled. However, contrary to a humid climates (e.g. Canada), the water table in arid climates and catchments with a tick vadose zone is a "hydraulic" replica of the bedrock rather than of land topography.

**Acknowledgments** Support by the grant SR/SCI/ETHS/11/01, His Majesty Research Trust Fund (Oman) and by Russian Foundation of Basic Research grant N 12-01-97015-r_povolgh'e_a is appreciated.

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