INTERACTION OF *N*-VORTEX STRUCTURES IN A CONTINUUM, INCLUDING ATMOSPHERE, HYDROSPHERE AND PLASMA

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Abstract

The results of analysis and numerical simulation of evolution and interaction of the *N*-vortex structures of various configuration and different vorticities in the continuum including atmosphere, hydrosphere and plasma are presented. It is found that in dependence on initial conditions the regimes of weak interaction with quasi-stationary evolution and active interaction with the "phase intermixing", when the evolution can lead to formation of complex forms of vorticity regions, are realized in the *N*-vortex systems. For the 2-vortex interaction the generalized critical parameter determining qualitative character of interaction of vortices is introduced. It is shown that for given initial conditions its value divides modes of active interaction and quasi-stationary evolution. The results of simulation of evolution and interaction of the two-dimensional and three-dimensional vortex structures, including such phenomena as dynamics of the atmospheric synoptic vortices of cyclonic types and tornado, hydrodynamic 4-vortex interaction and also interaction in the systems of a type of “hydrodynamic vortex – dust particles” are presented. The applications of undertaken approach to the problems of such plasma systems as streams of charged particles in a uniform magnetic field **B** and plasma clouds in the ionosphere are considered. It is shown that the results obtained have obvious applications in studies of the dynamics of the vortex structures dynamics in atmosphere, hydrosphere and plasma.

Keywords: vortices; interaction; fluids; atmosphere; cyclonic vortices; tornado; hydrosphere; plasma; streams of particles; plasma clouds; computer simulation.

1. Introduction. Basic equations

In this paper we study numerically the interaction of the vortex structures [so-called FAVRs, see (Zabusky et al., 1979)] in the continuum, and, specifically, in fluids (such as atmosphere and hydrosphere) and plasmas in two-dimensional (2D) approximation, when the Euler-type equations are applicable. The Euler equation for the inviscid incompressible fluid $\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \boldsymbol{F} - \frac{1}{\rho}\operatorname{grad} p$ in the 2D case takes form of the following set:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= F_y - \frac{1}{\rho}\frac{\partial p}{\partial y}. \end{aligned} \tag{1}$$

Add here the equation of continuity:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) = 0$$

where for ideal incompressible fluid $\mathrm{d}\rho/\mathrm{d}t = 0$ and, hence,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0. \tag{2}$$

Introduce further the flow function

$$\psi = \int |\boldsymbol{u}| \sin\alpha \, \mathrm{d}s$$

where $\boldsymbol{u}$ is a fluid velocity, $\boldsymbol{s}$ is a displacement, α is an angle between $\boldsymbol{u}$ and $\boldsymbol{s}$ (function ψ is positive when the streamlines are directed clockwise). It is easy to show, that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{3}$$

Function

$$\operatorname{rot}\boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ u & v & 0 \end{vmatrix} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \boldsymbol{k}$$

is the vector of vortex, and for flat motion

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{4}$$

is a vorticity.

Now present the Euler equations in new variables – vorticity and flow function, making differentiation the equations in (1) on *y* and *x* accordingly. Then, in absence of external forces, after elementary transformations we obtain:

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \zeta\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) = 0\,. \tag{5}$$

As, according to Eq. (2) for a flat motion $\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$, that from Eq. (5) we have

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = 0\,. \tag{6}$$

Equation (6) in variables “vorticity – the flow function” is the equation of carry of a vortex and is nonlinear, as *u* and *v* are the functions of ζ. Last two terms in (6) are convective ones, and the convection in this case means that the vortex is carried on a current.

With due account of (3) we can rewrite (4) in form $\zeta = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)$, that is the Poisson equation for flow function

$$\Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\zeta\,. \tag{7}$$

Thus, the dynamics of vortical structures in their flat movement for the case of an inviscid incompressible fluid is described by the set of equations of carry of a vortex (6) and the Poisson equation (7) for flow function.

Simple model of 2D magnetized plasma (Taylor and McNamara, 1971) is the quasi-particles (or the charged filaments aligned in a uniform magnetic field $\boldsymbol{B}$) which move with the central-directed velocity $\boldsymbol{E}\times\boldsymbol{B}/B^2$. The equations of motion of these quasi-particles (filaments) have form

$$e_i\frac{\mathrm{d}x_i}{\mathrm{d}t}=\frac{1}{B}\frac{\partial H}{\partial y_i},\qquad e_i\frac{\mathrm{d}y_i}{\mathrm{d}t}=-\frac{\partial H}{\partial x_i}\tag{8}$$

where e_i is the charge per unit length of the filament and

$$H=\sum -e_i e_j\ln\left(\left|\mathbf{r}_i-\mathbf{r}_j\right|\right)\tag{9}$$

is the Hamiltonian which has the sense of energy of Coulomb interaction.

In a continuous limit this 2D plasma satisfies to the equations:

$$\frac{\partial\rho}{\partial t}+\boldsymbol{v}\cdot\nabla\rho=0,\quad \boldsymbol{v}=-\frac{\hat{\boldsymbol{z}}\times\nabla\psi}{B},\tag{10}$$

$$\nabla^2\psi=-\rho,\tag{11}$$

where ρ is the charge density, $\boldsymbol{v}=(v_x,v_y)$, ψ is the potential of an electric field and $\nabla=(\partial/\partial x,\ \partial/\partial y)$. Independently on the scale of coefficient B these continuous equations are formally identical to the equations for the 2D movement of an inviscid incompressible fluid (6), (7) where ρ is the z-component of vorticity ζ, and ψ is the flow function. If the vorticity is presented by discrete vortexes (with circulations e_i) then the motion of a fluid is described by the Hamilton equations with B=1.Note, that the equations of motion of clouds of ideal ionospheric plasma have a similar form.

Another 2D continuous models can be represent by vortexes or the filaments (quasi-particles) with the Coulomb interaction (Taylor, 1977) and include the Debye radius of shielding in the Poisson equation (11). At this, it is necessary proposed that the ions move with the guiding-centre velocity, and electrons (for example, moving along a field $\boldsymbol{B}$) have the Boltzmann distribution. Then ionic current is still described by (10), and the Eq. (11) is rewritten in the form of

$$\nabla^2\psi - k^2\psi = -\rho, \tag{12}$$

where $k^2\psi$ is the Debye shielding. This model is also presented by charged filaments (quasi-particles) satisfying (8), but, unlike (9), with Hamiltonian

$$H = \sum -e_i e_j k_0(k\left|\boldsymbol{r}_i - \boldsymbol{r}_j\right|), \tag{13}$$

which describes the shielded Coulomb interaction between filaments.

One more model of plasma which can be expressed in the similar form, has been introduced in (Hasegawa and Mima, 1978). Its distinctive feature is inclusion of the ionic-polarized current through the equation of motion of ions

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{e}{M}(-\nabla\varphi + \boldsymbol{v}\times\boldsymbol{B}). \tag{14}$$

The electrons have also the Boltzmann distribution here but the Debye length is supposed tending zero so, that full charging neutrality is conserved. In this case k^{-1} is not Debye length, but it is the ion Larmour radius (electron temperature), and shielding is an indirect effect of the ion-polarized current. Let us note here that in space plasmas, in addition to vortices with dimensions of the order of the ion Larmor radius calculated at the electron temperature the vortical structures with spatial scales of the order of the Larmor radius calculated at the ion temperature can exist (Aburdzhaniya et al., 1984), however “classic” model of Hasegawa-Mima (Hasegawa and Mima, 1978) does not take into account them. In this case the general structure of the equations is the same, but it is necessary to consider k^{-1} as some generalized ion Larmour radius.

The hydrodynamical model of a rotating fluid (Charney, 1948) describing a motion of the Earth atmosphere also corresponds to shielded interaction. Atmospheric currents in a horizontal plane are described by the equation:

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -g\nabla h + \boldsymbol{R}v \times \hat{\boldsymbol{z}}\,, \tag{15}$$

where h is atmospheric depth, and $\boldsymbol{R}$ is the Coriolis force. Small change of h satisfies to the equations identical to model of shielded guiding-centre plasma and a role of length of shielding plays the Rossby radius, $\sqrt{gH_0}/R$.

There are also some other examples of vortex motion in plasmas and rotating fluids which were discussed in detail, for example, in (Mikhailovskii et al., 1984; Petviashvili and Pokhotelov, 1992). They also use hydrodynamic description and can be reduced to the equations similar to presented above.

Thus, write a set of equations describing a motion of a fluid, gas or plasma in the generalized variables:

$$\begin{gathered} \frac{\partial\rho}{\partial t} + \boldsymbol{v}\cdot\nabla\rho = 0, \qquad \boldsymbol{v} = -\frac{\hat{\boldsymbol{z}}\times\nabla\psi}{B}, \\ \nabla^2\psi - f = -\rho. \end{gathered} \tag{16}$$

In dependence on the considering medium the functions and variables in Eqs. (16) will have various physical sense (Table 1), and the set (16) will get the form of one of described by the Eqs. (6)-(15) ones.

Table 1. Sense of the variables in dependence on type of medium

Function	Fluid, gas	Plasma
B	$B = 1$	module of a vector of a magnetic induction
ψ	flow function	potential of an electric field
ρ	z-component of vorticity	line density of a charge
f	$f = 0$	$f = 0$ – plasma with Coulomb interaction; $f = k^2\psi$ – plasma with shielded Coulomb interaction

Note that function f has various sense in dependence on considering model of medium. So, for an inviscid incompressible fluid and also for charged filaments (quasi-particles) with the Coulomb interaction $f = 0$, for filaments (particles) with the shielded Coulomb interaction $f = k^2\psi$. Further we consider only a case when $f = 0$, which corresponds to rotation of local vortical structures in a fluid or to evolution of the charged filaments (quasi-particles) in a homogeneous magnetic field. Generalization for $f = k^2\psi$ is rather trivial.

2. Modeling technique

For numerical simulation we used the contour dynamics (CD) method (Zabusky et al., 1979), to some extent modified (see (Belashov and Singatulin, 2003a) for detail). This has yielded us a possibility not only to observe evolution of single vortex, but also to study the interaction between vortices having different sizes, vorticities and symmetry orders (different modes), and also to simulate the 3D vortex structures. A general idea of CD method is that the interaction between the boundaries of the regions with constant ζ is considered, and due to this the dimension of the problem decreases on unit. Analytical solution of the Poisson equation (16) with $f = 0$ for flow function ψ has form (Belashov and Singatulin, 2003b)

$$\psi = -\frac{1}{2\pi}\iint \mathrm{d}\xi\,\mathrm{d}\eta\,[\ln r]\rho(\xi,\eta)\,, \tag{17}$$

where $\ln r$ is the Green's function of Eq. (2), and $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$. Then a value of velocity can be obtained by differentiation of integral (17), namely:

$$\boldsymbol{u}(x,y) = \rho_0 \oint_{\Gamma} [\ln r][\boldsymbol{e}_x\,\mathrm{d}\xi + \boldsymbol{e}_y\,\mathrm{d}\eta]. \tag{18}$$

Further, obtain the change of the contour coordinates with time by solving differential equation $\boldsymbol{u}(x,y) = \dot{x}\boldsymbol{e}_x + \dot{y}\boldsymbol{e}_y$. For the computer simulation of the vortex structures the contour's boundary is divided into N lattice points (moreover, the point quantity should be rather great), and the temporal evolution is computed for each point. Thus Eq. (18) is written in discrete form using the 3-layer difference scheme with approximation order $O(\tau^2)$:

$$\boldsymbol{x}_m^{p+1} - \boldsymbol{x}_m^{p-1} = 2\tau \sum_{n=1}^{N} (\Delta u)_n^p \left(\cos\theta_n^p \,\boldsymbol{e}_x + \sin\theta_n^p \,\boldsymbol{e}_y\right), \tag{19}$$

where $\mathrm{tg}\,\theta_n = \dfrac{y_m - y_n}{x_m - x_n}$, which stability is guaranteed by the condition $\tau \le h_n / \max\limits_{x,y}\{u, v\}$ where u and v are the x and y components of the velocity of the contour point, respectively. We omit here the details of the CD method and nuances of its modification for modeling of the FAVR evolution. You can find them in (Belashov and Singatulin, 2003a). Equation (19) allows us to found a value of velocity of each point of contour in dependence on influence to it of the points of both the same contour and the contour interacted with it. So, one can observe the time evolution of the vortex structure setting its initial form.

3. Numerical simulation and discussion

Let us consider the results of numerical simulation in terms of the vortex motion of the inviscid incompressible fluid, as more visual and directly applicable to physics of the atmosphere and hydrosphere.

For the first time the CD method has been used for simulation of evolution of 2D 2-vortex systems of FAVRs in (Zabusky et al., 1979), after that there was a whole series of similar studies of different authors in which, however, the problems of evolution of more general N-vortex systems and possible modes of vortical interaction depending on their initial configuration were not considered. For the first time such studies have been undertaken in (Belashov and Singatulin, 2001).

In general, to study the evolution of vortex structures with different symmetry orders it is necessary to insert a small amplitude perturbation $r = R_0[1 + \varepsilon\cos(m\alpha - \omega_m t)]$ (where R_0 is a conditional radius, ε is an eccentricity, m is symmetry order (mode), α is an angle and $\omega_m = \rho_0 (m-1)/2$ to the circle region with constant vorticity. But, taking into account that the results of evolution for one and two vortices with different m were described in detail in [7, 9], let us stay on results on interaction of FAVRs and consider the most simple cases of circle vortices when m=1 and, therefore, $\omega_m = 0$. As it was found in (Belashov and Singatulin, 2003d) for such FAVRs the result of the evolution depends on sign of vorticity ("polarity" of vortex) ζ [ζ = ρ in Eqs. (16)] and the distance δ between boundaries of vortices. We fulfilled a number of the series of numerical simulations for study of 2-vortex interaction, the interaction in the N-vortex systems, including interaction between the hydrodynamical vortex structures and the dust particles in a plasma, and also interaction of two 3D plane-rotating vortex structures within the framework of many-layer model of medium, in dependence on some parameters: initial distance between vortices, value and sign of their vorticities, and spatial configuration of the vortex system. Consider the examples of the basic results.

3.1 Two-vortex interaction

For two circle vortices having opposite polarities we observed that at an initial stage they approach and further move in the same direction, rotating in opposite directions (Fig. 1). Thus, the vortices practically don't interact independently on value of δ.

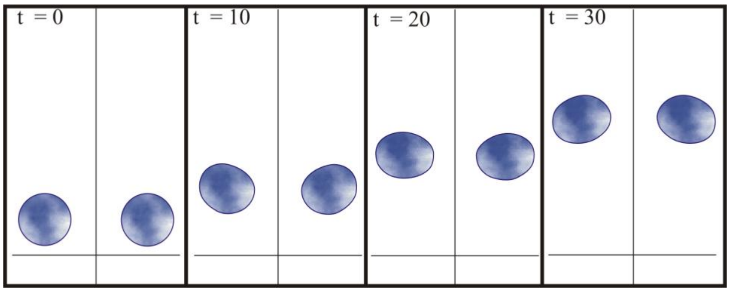

Fig. 1. Evolution of two circle vortices with opposite polarities ($\zeta_1 = -1$ and $\zeta_2 = 1$)

For the circle vortices having the same polarities the result of evolution depends essentially on δ. So, our results show that at interaction of a pair of circle vortices some cases can take place:

1. For rather big δ they, on a level with rotation about their own axes, rotate around of common center and one can observe a deformation of the vortices – they are drawn out, taking the form close to elliptical, but in due course return to an original state [Fig. 2(a)], thus their interaction is weak and it is reduced to a cyclic change of their shape (so-called "quasi-recurrence" phenomenon (Zabusky et al., 1979) is observed).

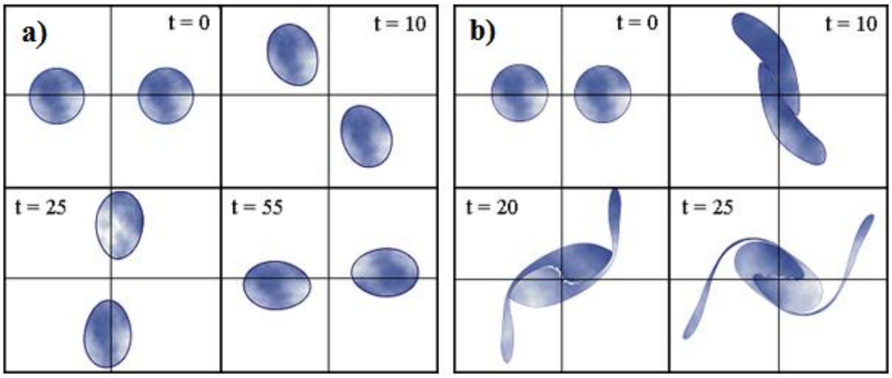

Fig. 2. Interaction of two vortices with $\zeta_1 = \zeta_2 = -1$ at initial distance between each other: a) δ = 2*d*; b) δ = *d*/2

2. With decreasing of a distance the vortices start ever more to be deformed during interaction, that results in formation of the cusps (Belashov and Singatulin, 2001). At further evolution it causes appearance of the filaments of vorticity (Belashov and Singatulin, 2001) (see. Fig. 3) and, as a result, the vortices disintegrate.

3. For rather small δ the vortices, on a level with rotation about their own axes and around of their common center, interact forming a common vortex region which consists of the vorticities of more small scales [Fig. 2(b)]. Thus, in this case the regime of active interaction with the "phase intermixing" takes place, and dif-

ferent configurations are possible too from small coupling of the vortices down to full junction of two vortices (Kozlov and Makarov, 1985).

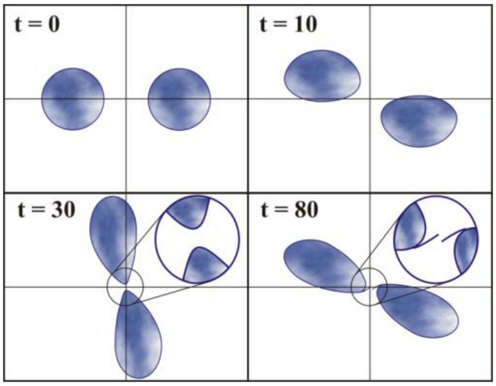

Fig. 3. Formation of filaments of vorticity

In our numerical experiments we have found that critical initial distance for two interacting vortices dividing these two types of interaction $\delta_{cr} = 3d/4$, where d is the vortex diameter.

Note, that qualitative character of interaction of the vortices with different symmetry orders is, in general, the same, but in this case the vortex structures with more high symmetry order m liable to more high deformation (the vortex filaments appear) and have the greater tendency to destruction (Belashov and Singatulin, 2003c).

To make more strong analysis we shall suppose, that the qualitative change (some kind of a "jump") in a character of interaction of two vortex regions happens with transition to a "phase intermixing" state. The problem is to find some generalized critical parameter describing the interaction of the vortices in the terms of such jump, which value would allow us to predict the qualitative character of the result of vortex interaction.

As such parameter we offer to use the following function of the basic characteristics of interacting vortex structures corresponding their state at $t = 0$:

$$\xi = \frac{S}{l^2}\frac{\zeta_1}{\zeta_2}\left(1-e_0\right)^{-1}\left(1+\sin^2\theta\right), \tag{20}$$

where S is the area of each interacting FAVR[1], l is the distance between their centers, ζ_1 and ζ_2 are the values of the vorticities (and $\zeta_1 \geq \zeta_2$), $e_0 = (e_1 + e_2)/2$ is the eccentricity averaged on two vortices, and $\theta = \theta_1 + \theta_2$ is the sum of angles of inclination of large axes of the vortex ellipses concerning a line, connecting their centers (see Fig. 4).

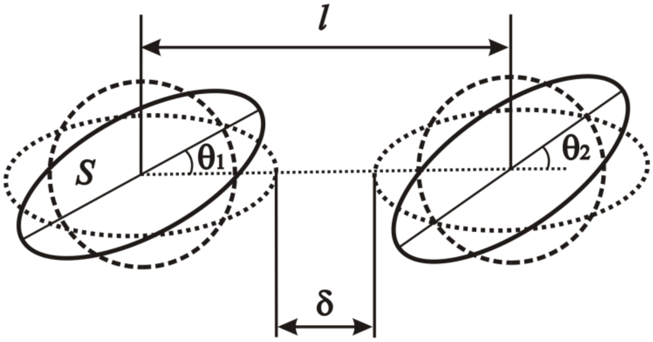

Fig. 4. Illustration to definition of the initial state parameters of vortex system

Let us introduce the following denotations for critical parameters corresponding an initial state of a vortex system and determining the transition to the "phase intermixing" state with change of the sizes and positional relationship of vortices, ratio of their vorticities, eccentricity and angle θ, respectively:

$$\alpha = S/l^2, \quad \beta = \zeta_1/\zeta_2, \quad \gamma = \left(1-e_0\right)^{-1}, \quad \theta_0 = 1+\sin^2\theta$$

and write function ξ (20) in the following form

$$\xi = \alpha\beta\gamma\theta_0 . \tag{21}$$

To justify the expediency of offered criterion (21) we fulfilled some series of numerical experiments in which the critical values of parameters α, β, γ and θ_0 for

[1] Suppose, for a determinacy, that the areas of interacting vortices $S_1 = S_2 = S$.

vortex regions of the circle and elliptical form, as the models most often meeting in numerous applications, were calculated.

With the purpose of finding of the critical value of the parameter α, the system consisting from two circle vortices with equal values of vorticities and radiuses was considered: at fixed distance between the centers of two vortices we increased their radiuses (and, accordingly, areas) until there was an interaction. Thus the parameters corresponding to the critical state of vortex pair were fixed. The quantities which uniquely determinate initial configuration of the system of two circle vortices are shown in Fig. 4.

In our numerical simulations for the cases corresponding the initial states between the centers of vortices $l = 1, 2, \ldots, 5$ we have found that the beginning of interaction in all cases responds the approximately same value of parameter α. The values of parameters, at which there is a qualitative change in the character of interaction – the transition from steadily rotated pair to the "phase intermixing" state, are shown in Table 2. So, the results of numerical simulations enable us to conclude that the critical value of parameter α, at which there is qualitative change in the interaction of the vortices, equals $\alpha_{cr} = 0.267$. For $\alpha < \alpha_{cr}$ the merging of the vortex regions does not happen during interaction, but as soon as parameter α reaches its critical value, there is a qualitative jump in behavior of the vortex system, and the vortices start to be intermixed.

Table 2

S	l	δ/l	α
0.267864	1	0.416	0.267864
1.067791	2	0.417	0.266947
2.399785	3	0.417	0.266642
4.271168	4	0.417	0.266947
6.669121	5	0.417	0.266764

The next series of the numerical simulations purposed a calculation of the critical value of the parameter β. Our results showed that the vortices with the greater value of β are exposed to the greater deformation, their filamentation (i.e. formation of the filaments of a vorticity) happens faster, thus the change in character of the interaction happens at the ratio of vorticities $\zeta_1/\zeta_2 > 1.11$ (remind, that $\zeta_1 \geq \zeta_2$), therefore, $\beta_{cr} = 1.11$.

To answer a problem on the critical value of the parameter γ a series of simulations for vortices of elliptical form was conducted (Fig. 4). We fixed the area *S*, at which the circle vortices still save a stable state, and at $S = \text{const}$ changed the eccentricities e_1 and e_2. Further, we found the critical value e_0, at which the vortex system loses its stability transferring to the "phase intermixing" state. Thus, we considered the cases corresponding the initial states between the centers of vortices *l* =1, 2, …, 5. The values of critical parameters, at which there is a qualitative change in the behavior of the system of two elliptical vortices, are presented in Table 3.

Table 3

l	δ/l	α	e_0
1	0.180	0.266033	0.863847
2	0.180	0.266033	0.866426
3	0.183	0.266033	0.863834
4	0.180	0.266490	0.863341
5	0.180	0.266764	0.863037

Numerical simulations have shown that there is the same for all cases a critical value of the averaged eccentricity, at which the "phase change" happens – $e_0 \approx 0.864$, that corresponds γ = 7.143. Thus, as one can see from Table 3 the ratio δ/*l* is also a constant in a critical region, however it cannot be used as the critical parameter for the description of interaction, because, at first, it takes different values for elliptical and circle vortices (see Table 2), secondly, it is less information as determines only a distance between boundaries of the vortices, nothing speaking about their form. Therefore, for definition of the function ξ we use parameter γ, expressed through the averaged eccentricity.

Further investigations have been connected with a finding of the critical angle of inclination (see Fig. 4) of the elliptical FAVRs for the initial state of a system, at which the evolution results in qualitative change in character of their interaction. The simulations fulfilled show that increase of the angle of declination of the vortex regions θ at t = 0 more than on 4° leads to the transition to the unstable state. Thus, we mean as angle of inclination the summing angle $\theta = \theta_1 + \theta_2$, and, for example, the case when $\theta_1 = \theta_2 = 2°$ is analogous to the case $\theta_1 = 4°,\ \theta_2 = 0$. As it follows from processing of the results of this series of the simulations, the critical value of corresponding parameter is $\theta_0 = 1.005$.

Summing all presented above results we can define a critical value of the generalized parameter ξ (21) as a multiplication of four parameters $\alpha_{cr}, \beta_{cr}, \gamma_{cr}$ and $\theta_{0\,cr}$:

$$\xi_{cr} = \alpha_{cr}\beta_{cr}\gamma_{cr}\theta_{0cr} = 2.129.$$

Numerical simulations for $|\,\xi| \geq \xi_{cr}$ with simultaneous variation of critical parameters α, β, γ and θ_0 corresponding to change of the sizes and positional relationship of vortices, the ratio of their vorticities, eccentricity and the summing angle of inclination of their major axes θ, respectively, have confirmed a capability and expediency of usage of the parameter ξ for prediction of character of interaction of the 2D vortex structures.

Note that obtained results, despite their general significance in theory of vortex dynamics, can help to predict the temporal behavior of 2-vortex system in real physical media such as atmosphere, hydrosphere and plasma.

3.2 Interaction in N-vortex systems

To study the interaction in more complex N-vortex systems we considered the problems with N=3 and N=4 in two variants: 1) for vortices linearly disposed at initial time, and 2) for vortices disposed at initial time in the corners of appropriate equilateral figures, and we used the critical parameter δ in the analysis of obtained results. Fig. 5(a) shows an example of simulation of the interaction for initially linear disposition of four vortices. One can see that for rather big and equal initial distance between vortices the evolution leads to formation of two vorticity regions as a result of more strong interaction of each of the "outer" vortices with closest "inner" vortex. Thus, the interaction of forming pairs is similar to that of two vortex case. In case $\delta_i = d/2$ we observed the formation of a complex vortex structure which consists of many vorticities of more small scales [Fig. 5(b)]. Further evolution of such structure leads to formation of complex turbulent field. Note that in last case we can also see that the interaction between outer vortices is stronger.

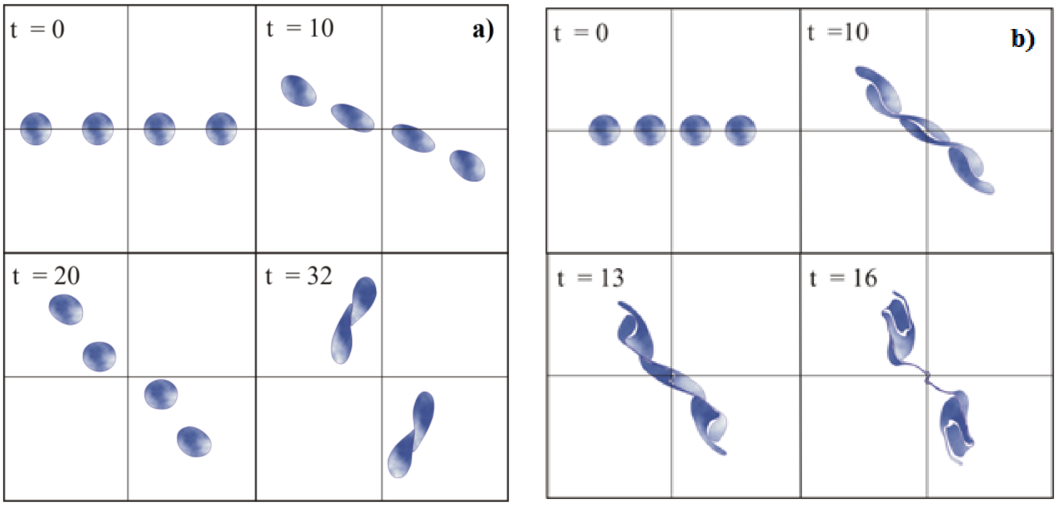

Fig. 5. Interaction of four linearly disposed vortices with $\zeta_1 = \zeta_2 = -1$:
a) $\delta_i = d$; b) $\delta_i = d/2$

This can be explained by the fact of more strong "attraction" of outer vortices to the "center of mass" of the vortex system because the outer vortex is attracted to the center by three other vortices, and the inner vortex is attracted to the center by two vortices and, to opposite side – by one outer vortex. To test this statement, in the next series of numerical experiments we have arranged outer and inner vortices on different initial distances. As a result, we observed the formation of common vortex structure from two inner vortices (see Fig. 6). The results obtained for the 4-vortex system and the simulations for the 3-vortex system showed that in both cases, owing to effect noted above, the critical initial value δ_{cr} dividing quasi-stationary and active types of interaction is less than that for 2-vortex case.

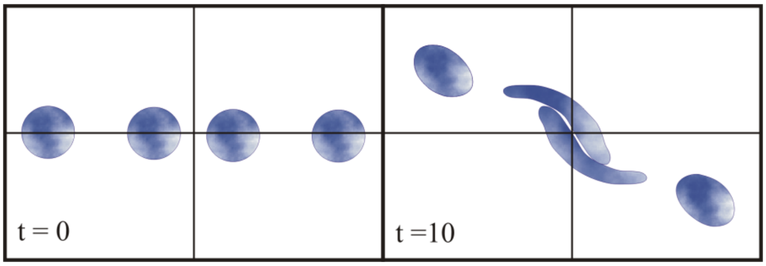

Fig. 6. Interaction of four linearly disposed vortices with
$\zeta_1 = \zeta_2 = -1$ for $\delta_{out} = d$ and $\delta_{inn} = d/2$

In the next series of numerical experiments we studied the interaction between the vortices disposed at initial time in the corners of appropriate equilateral figures. The following results were obtained. In case of evolution of three vortices with different signs of ζ being at initial time in the corners of triangle, we observed that a pair of them, having opposite polarities, behaves as well as pair of vortices with opposite polarities in 2-vortex case, and third vortex does not participate in interaction almost, practically independently on value of δ_i $(i = 1,2,3)$. The similar character of interaction is observed for four vortices with different signs of ζ being at $t = 0$ in the corners of square (see Fig. 7, numbering of the vortices – clockwise, since the upper left corner).

The character of interaction in the 3- and 4-vortex systems consisting of vortices having the same polarities depends essentially on the distances between them like that in the 2-vortex case. The examples of such interaction for $\delta = d/2 < \delta_{cr}$ and $\delta = d > \delta_{cr}$ are shown in Fig. 8. One can see that in the first case the four vortices are rotated forming one big vortex structure which consists of many vorticities of more small scales. In the second case we observed a "quasi-recurrence" phenomenon. Similar pictures take place in the 3-vortex systems when at $t = 0$ the vortices are in the corners of triangle on the distances $\delta < \delta_{cr}$ or $\delta > \delta_{cr}$ one from another.

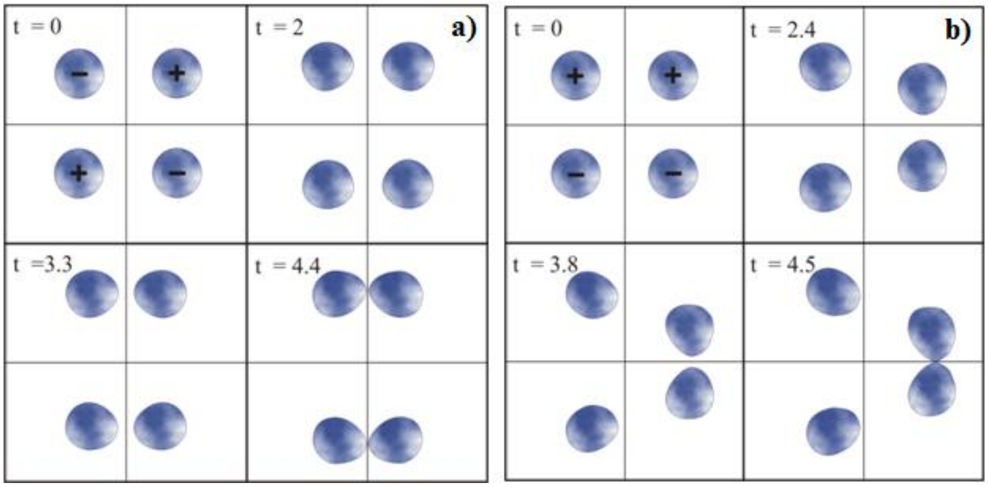

Fig. 7. Interaction of four vortices for $\delta_i = d$ with: a) $\zeta_1 = \zeta_3 > 0, \zeta_2 = \zeta_4 < 0$; b) $\zeta_1 = \zeta_2 > 0, \zeta_3 = \zeta_4 < 0$

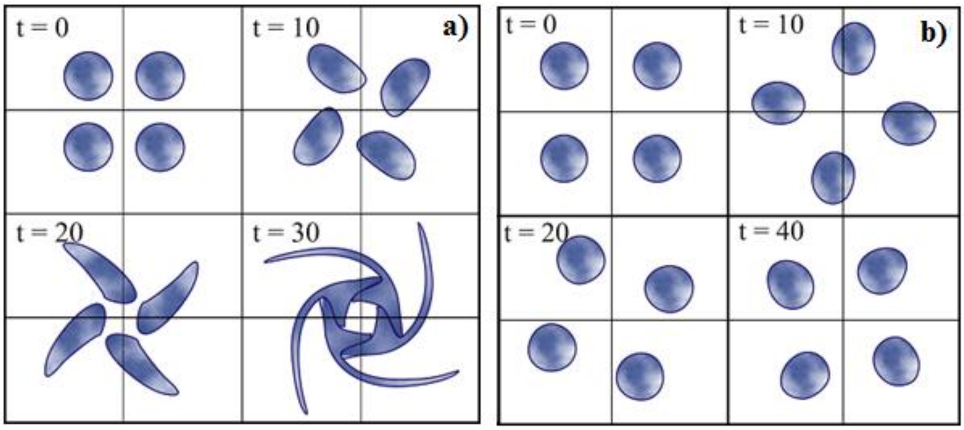

Fig. 8. Interaction of four vortices with the same polarities for: a) $\delta_i = d/2$; b) $\delta_i = d$

3.3 Three-dimensional vortices interaction

Our modification of the CD method enables also to simulate the interaction dynamics of the three-dimensional plane-rotating vortex structures in the "two-dimensional approximation" within the framework of multilayered model of medium. Fig. 9 shows an example of results of numerical simulation of interaction of two three-dimensional vortices with the exponential decreasing of their vorticity in (x, y)-planes of rotation with z-coordinate. One can see that, in the beginning, the vortices' central regions start to interact and only then other their areas are involved in the interaction. Such behavior is explained by stronger interaction of central regions, which locate at the relatively short distance each other and their vorticities have relatively big values, so that the ratio ζ/δ is big in comparison with that for top and bottom of vortices. More strong analysis, however, requires more detail study of the regimes of this interaction, that has been discussed in detail above in point 3.1 of the paper.

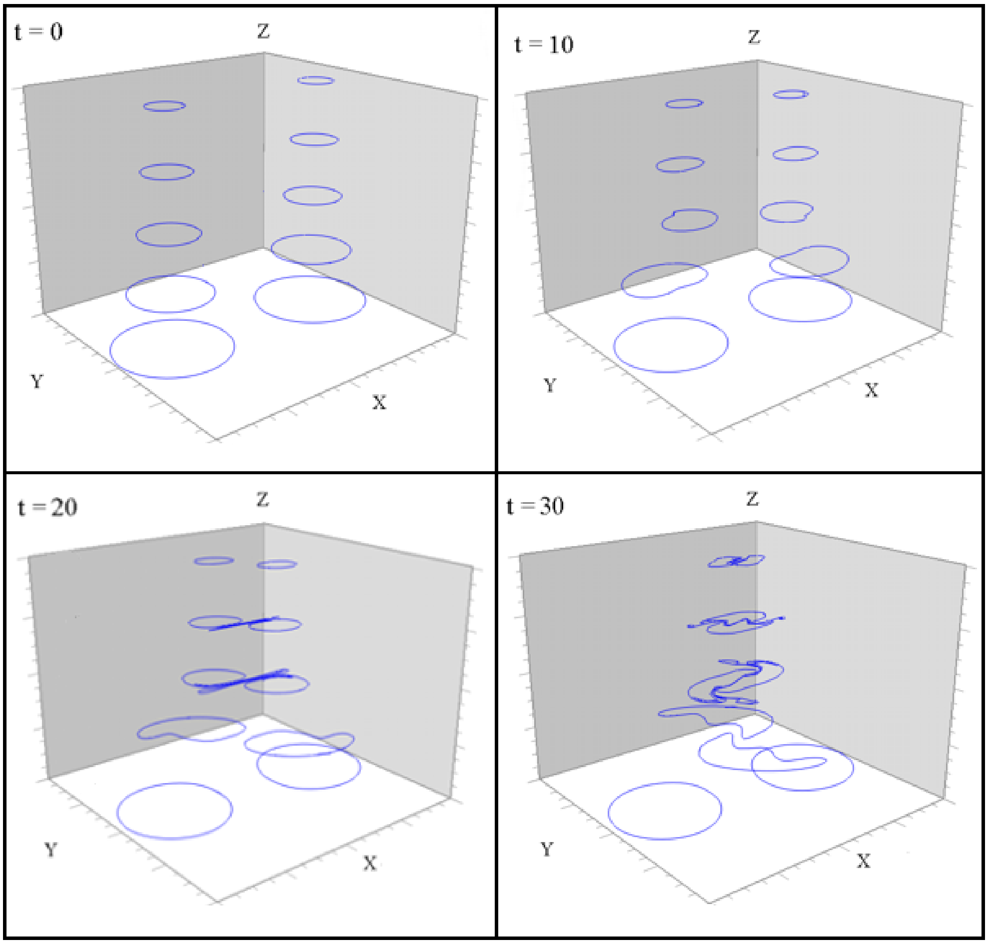

Fig. 9. Interaction of three-dimensional plane-rotating vortex structures in the many-layer model

4. Some examples of applications

Consider now some examples of applications of our results to the problems of study of vortex motions in the atmosphere, hydrosphere and in a plasma of ionosphere.

4.1 Vortical motions in the atmosphere and hydrosphere

Using our technique we studied numerically the evolution and interaction of synoptic vortices and vortical structures in a fluid such as atmosphere and hydrosphere. Figures 10 and 11 show the examples of our results on modeling of the evolution of the cyclonic type synoptic vortex in the atmosphere and of the 4-vortical interaction in the channel Naruto (Japan), respectively, in comparison with photos of real systems. Here we used our modified CD method for vortex structures with due account of scale parities of parameters of the model and real vortices which were simulated (see Table 4 below for detail).

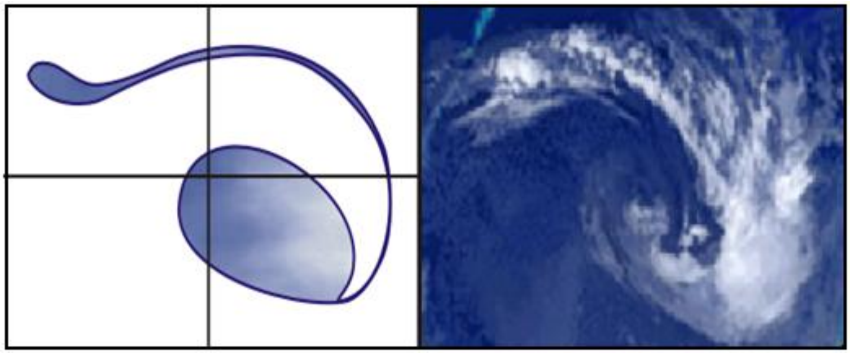

Fig. 10. Modeling of evolution of the cyclonic type synoptic vortex
(numerical result and satellite photo)

In these figures one can see that our numerical results are qualitatively coincided with the real systems which are simulated.

Using the quasi-2D approach with many-layer approximation of the 3D vortical structure by the FAVR system we studied also the time evolution of a tornado, and our model vortex (FAVR system) has been associated with real tornado from video-record (see Fig. 12). One can see that our simulation reflects the basic features of evolution of a tornado such as its form, spatial structure and dynamics of evolution. In particular, we investigated an influence of the perturbation imposed on the tornado axis on its dynamics. We established as a result, that small cross-section indignation leads to inappreciable fluctuations of an axis and, as a whole, does not influence on structure and stability of a vortex. Let us note also that vertical motions in tornado, which are sufficient in such 3D natural vortices, are taken into account implicitly by the modified CD method as each point of the contour of each layer interacts with each point of the contours of other layers. So, using our approach we can forecast tornado evolution and simulate interaction of such type of vortices.

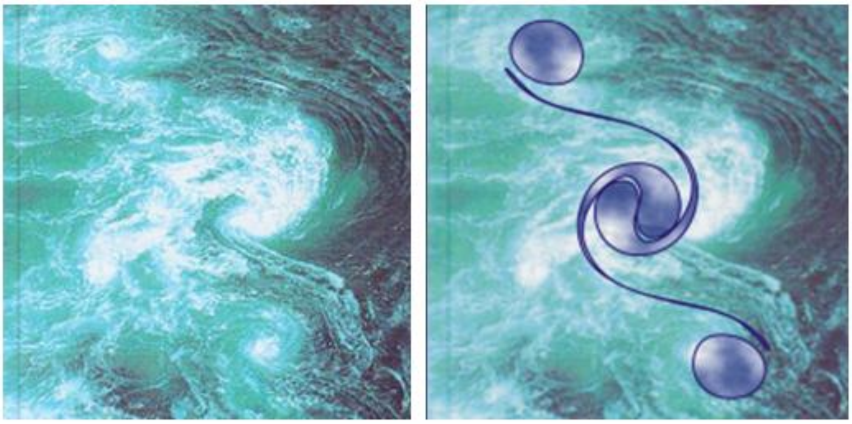

Fig. 11. Modeling of the 4-vortical interaction in channel Naruto, Japan (numerical result and air photography)

As we mentioned above, to make modeling it is necessary to know the scale parities of parameters of the model and the real system which is simulated. One can see some of them in Table 4.

4.2 Vortex structures in a plasma

Using 2D model of plasma of Taylor-McNamara (Tailor and McNamara, 1971) we studied the dynamics of charged filaments which represent streams of charged particles in a uniform magnetic field **B**. Fig. 13 shows the examples of our results for a few cases of the particles' streams with their cross-section perturbations. As is known such perturbations lead to deformations of a magnetic field in a zone polar cusp, which influence on dynamics of streams of the charged particles.

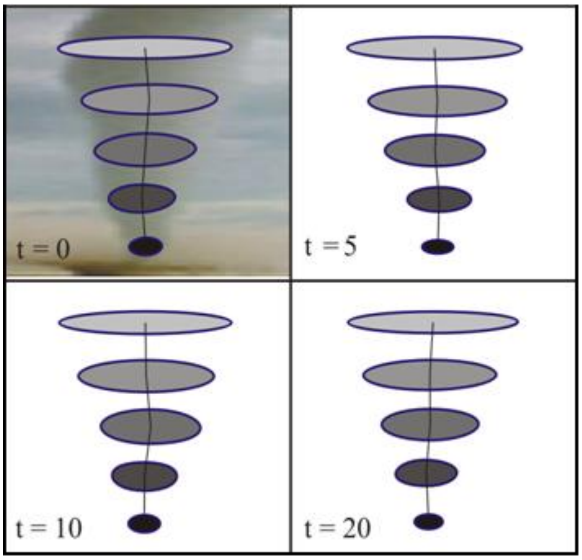

Fig. 12. Evolution of the 3D tornado vortex

Table 4. Scale parities of parameters of modeling and some real vortex systems

Parameter	Model values	Tornado	Tropical cyclones	Ocean vortices
R	1	10^2 *m*	10^5 *m*	2.5×10^4 *m*
V	1	100 *m/c*	10 *m/c*	2.5 *m/c*
ζ	1	1 c^{-1}	10^{-4} c^{-1}	10^{-4} c^{-1}
T	2π	2π *c*	$2\pi\times10^4$ *c*	$2\pi\times10^4$ *c*

We have found that the structures of vortical type are forming especially quickly and more intensively, than more amplitude of perturbations and quantity of the filaments participating in interaction, and also than more close to each other filaments are located. One can see also that the cross-section perturbations of velocity of a stream lead to its transition in a unstable state with formation of folds and complex vortical structures.

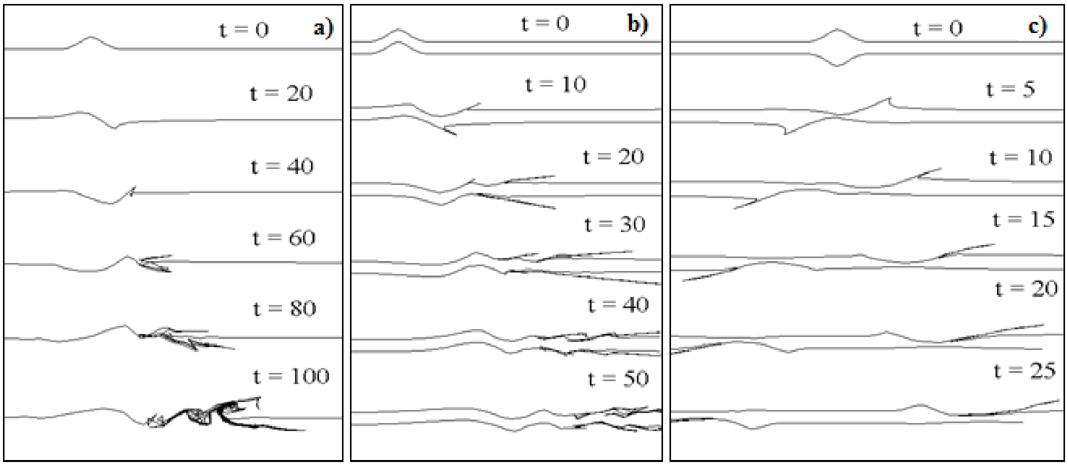

Fig. 13. Vortex structures formation at cross-section perturbations of the charged filaments: (a) one perturbed line; (b) and (c) two lines with perturbations of the same and opposite polarities

Next example is the interaction in the vortex-dust particles system. The theoretical analysis and the experimental results (Vaulina et al., 2001) show that in a plasma with gradient of dust charge the vorticity of dust particles can exist. (In particular, it was found that vertical vortices rotate with frequency 0.2-1.5 s^{-1}. Experiments were made in argon with the particles of melanin (the size of particles is about 3 microns). During electrical discharge the formation of two vortices with opposite signs of vorticity was observed.) This gives a possibility to study the interaction between the "hydrodynamic" vortex structures and dust particles by use of the CD-method considering the dust particles as vortices of very small scales (Belashov and Singatulin, 2003b). We studied numerically the interaction of the particles having nonzero value of a vorticity with the vortical area of greater size. The results of our numerical simulations showed that the character of interaction in this case depends on value of particles' vorticity. If this value is very small that the interaction does not observed. When the vorticity of dust particles becomes like vorticity of the "hydrodynamic" vortex, the interaction becomes significant. The examples of simulation for both linear dust layers and dust cloud are presented in Fig. 14, where one can see that the dust particles are involved by a vortex in large-scale rotation. This result is especially important for numerous possible applications in physics of an atmosphere and plasma where presence of dust particles practically always takes place.

Next example of the application of our approach is investigation of the evolution of plasma clouds in the ionosphere. Such clouds are formed in the ionosphere under influence of solar ionization of artificial injected Ba in rocket experiments at heights of the *F*-region of ionosphere (Mishin et al., 1989). An example of our modeling results is presented in Fig. 15.

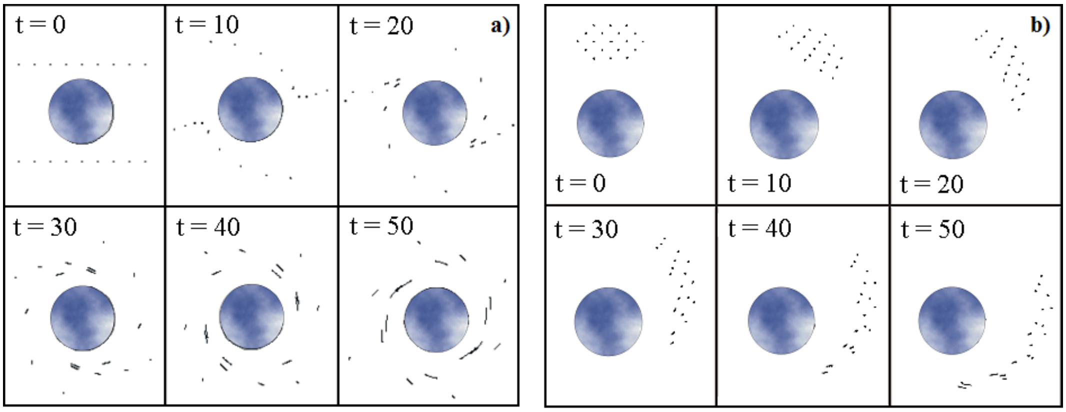

Fig. 14. Interaction of dust particles with rather big values of vorticity with "hydrodynamic" vortex: a) linear dust layers; b) dust cloud

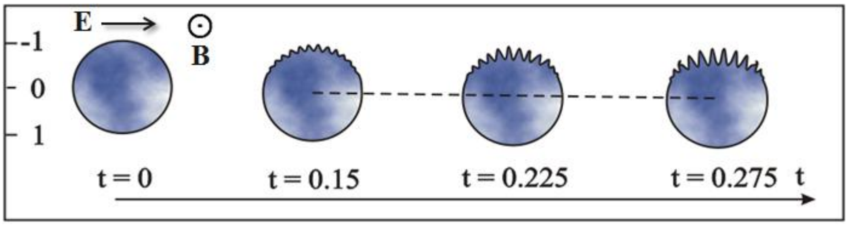

Fig. 15. Evolution of artificial electron-ionic ionospheric inhomogeneity, cross-section section

One can see, that such plasma structures, that lead to formation the aligned along magnetic field **B** electron-ionic irregularities (mainly in collision plasma with small $\beta = 4\pi nT / B^2$), diffusing across field **B** at evolution, get irregular “striped” structure. This effect is rather new because it was not found earlier [see, for example, (Overman et al., 1981)]. Our result coincides with the experimental data obtained in rocket experiments (Mishin et al., 1989). Such irregularities lead to development of nonlinearity in a *F*-layer and can lead to dispersion and fading of HF and VHF radio waves.

4.3 Other possible applications

Our approach can be useful in studies of other applications which are connected with dynamics of vortex and spiral structures in space and laboratory plasmas. One can note, for example, such of them as modeling of formation and evolution of vortical structures in astrophysics (such as spiral structure of Galaxies and solar flare activity associated with the dynamics of magnetic loops and magnetic tubes in the solar corona). Next examples are related to hydro- and aerodynamics (formation of vorticities and vortical chains at flowing of solid bodies by streams of gas and a fluid), and to the problem of magnetic confinement of plasma and controlled fusion, and also to some plasma technologies.

5. Conclusion

So, we have presented here the results of analysis and numerical simulation of evolution and interaction of the *N*-vortex structures of various configurations and different vorticities in the continuum including atmosphere, hydrosphere and plasma on the basis of the model described by Eqs. (16) in terms of the vortex motion of the inviscid incompressible fluid. We have found that in dependence on initial conditions the regimes of weak interaction with quasi-stationary evolution and active interaction with the "phase intermixing", when the evolution can lead to formation of complex forms of vorticity regions, are realized in the *N*-vortex systems. For the pair of the vortices at 2-vortex interaction we managed to find the function ξ having the sense of critical parameter which uniquely determines a qualitative character of their interaction. It was shown that for given initial conditions its value divides modes of active interaction and quasi-stationary evolution. Thus, comparing the value of ξ with its critical value ξ_{cr} we can predict the result of interaction of the vortices, namely: if $\xi < \xi_{cr}$ then "phase intermixing" of vortices is not observed with evolution, in the opposite case, when $\xi \geq \xi_{cr}$, the merging of vortices with further formation of the vorticities of more small scales is happen. For the vortices of the circle and elliptical (or close to elliptical) form, the value of generalized critical parameter $\xi_{cr} = 2.129$ corresponds to the "phase change" point. This result concerns only the systems which consist of two vortices. The generalization for a case of arbitrary number of vortex regions [in particular, for the 2D and quasi-3D *N*-vortex cases considered here and in (Belashov, 2016b)] requires padding investigations.

The results of simulation of evolution and interaction of the 2D and 3D vortex structures, including such phenomena as dynamics of the atmospheric synoptic vortices of cyclonic type and tornado [on the basis of the multilayered model of medium (Belashov, 2016a)], hydrodynamic 4-vortex interaction and also interaction in the systems of a type of "hydrodynamical vortex - dust particles" (when the dust particles are involved in rotation by hydrodynamic vortices), and dynamics of plasma clouds in the ionosphere of the Earth were presented. Other possible applications of the results obtained can be associated with study of dynamics of the Alfven vortices in plasma of the ionosphere and magnetosphere of the Earth (Pokhotelov et al., 1996), stability of vortex structures of different types and origins, including the quasi-geostrophic vortices in an ocean [Kozlov and Makarov, 1985], dynamics of the acoustic-gravity waves in the Earth atmosphere (Izvekova et al., 2015), and motions in dust devils on surfaces of Earth and Mars (Izvekova and Popel, 2016). The approach proposed in the paper enables also to study the motions in the hydrodynamic model of rotating fluid that corresponds to the screening interaction (Charney, 1948), and it can be useful for description of zonal flows in vortices in the ionospheric plasma (Benkadda et al., 2011).

We have shown that the generalized set (16) with $f=0$ can describe also the dynamics of quasi-particles with Coulomb interaction model [see Eqs. (10) and (11)], and the results obtained and presented in the paper can be easily extended to the 2D simple systems where the plasma is represented by charged filaments, aligned with a uniform magnetic field **B**, that move with the guiding-centre velocity $\mathbf{E}\times\mathbf{B}/B^2$. We have demonstrated the application of undertaken approach developed in (Belashov, 2015; Belashov, 2016b) to the problems of such plasma systems as streams of charged particles in a uniform magnetic field **B**. Note, that this approach can be useful and also for other 2D continuum models when $f \neq 0$ in the Poisson equation (16). They can describe the vortices or filaments with the non-Coulomb interaction. In the last case it is assumed that ions move with the guiding-centre velocity but electrons have a Boltzmann distribution, thus the additional term $f = k^2\psi$ describes the Debye screening – see models (10), (12), (13) and the Hasegawa-Mima model (Hasegawa and Mima, 1978) which includes additionally the ion equation of motion (14).

In conclusion, in the paper we have proposed the approach for investigations of the evolution and dynamics of the vortices of different type and origin in a continuum, have considered some problems on the basis of the modified CD method developed, and have shown that the results obtained have obvious applications in studies of the problems associated with the vortex movements in the atmosphere and hydrosphere, and in a plasma of the ionosphere and magnetosphere of the Earth.

Acknowledgment

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