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On Some "Collateral" Effects in the Alpha-convex Theory

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Abstract—Some effects in the α-convex theory of the univalent functions are discussed in the light of the uniqueness problem for the critical point of the conformal radius.

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0. The class of α-convex functions $\mathfrak{M}_\alpha$ is defined for $\alpha \in \mathbb{R}$ as the preimage

$$\mathfrak{M}_\alpha = J_\alpha^{-1}(C) \tag{1}$$

of the Caratheodory class C by means of the operator $J_\alpha = J_\alpha(f,\zeta) = p(\zeta) + \alpha\zeta p'(\zeta)/p(\zeta)$, $p(\zeta) = \zeta f'(\zeta)/f(\zeta)$, acting on the family $\mathcal{LS}$ of all holomorphic functions $f(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots$ in the unit disk $\mathbb{D}$ with $f(\zeta)f'(\zeta)/\zeta \neq 0$, $\zeta \in \mathbb{D}$. The seminal result of the α-theory [1], namely, the inclusion

$$\mathfrak{M}_\alpha \subset S^* \ \text{ (the class of all starlike } f \text{ in } \mathbb{D} \text{ with } f(0) = f'(0) - 1 = 0), \tag{2}$$

have generated the series of works generalizing (2) both in parametrical (see [2–4]) and in functional (e.g., [5, 6]) directions.

1. If we write (2) in the working form

$$J_\alpha(f,\zeta) = \frac{1+\varphi}{1-\varphi}(\zeta), \quad \zeta \in \mathbb{D}, \quad \Rightarrow \quad f \in S^*, \tag{2'}$$

then the illusion can arise that the Schwarz lemma function φ gains the status of the "controlling parameter" for f to be in the class $\mathfrak{M}_\alpha$. But really this is not the case: if $\alpha \in (-1, 0)$, then

$$J_\alpha^{-1}\left(\frac{1+\zeta^2}{1-\zeta^2}\right) \notin \mathfrak{M}_\alpha.$$

Indeed, if $\alpha \neq -1/2$, then

$$J_\alpha^{-1}\left(\frac{1+\zeta^2}{1-\zeta^2}\right) = \zeta + \frac{1}{2\alpha+1}\zeta^3 + \ldots \equiv f_\alpha,$$

whence $|a_3| = |2\alpha+1|^{-1} > 1$, i.e. $|\{f_\alpha, 0\}| > 6$, where $\{f,\zeta\} = (f''/f')'(\zeta) - (f''/f')^2(\zeta)/2$, and the well-known Kraus–Nehari theorem implies $f_\alpha \notin \mathfrak{M}_\alpha$ (by the use of (2)). In the case $\alpha = -1/2$, moreover, the formal expression of f_α contains $\ln\zeta$!

The "solution" of this "phenomenon" is find in more pedantry form of (1), $\mathfrak{M}_\alpha = J^{-1}(C) \cap \mathcal{LS}$: we must keep in mind the domain of definition $\mathcal{LS}$ of the operator J_α. Nevertheless, this "detective" poses the following

Open problem. *Decribe $J_\alpha(\mathfrak{M}_\alpha)$ in C for any $\alpha \in \mathbb{R}$. Find the set of α's such that $J_\alpha(\mathfrak{M}_\alpha) = C$.*

The expression of J_α in terms of φ (see (2′)) presents a some way for the "morphogenesis" of such a description. If we have $\varphi(\zeta) = c\zeta^2 + \ldots$ in (2′), then $a_2 = 0$ and $(2\alpha+1)a_3 = c$. When $\alpha = -1/2$, this

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implies $c = 0$. Symmetrizing the situation, we consider the example $f_n(\zeta) = \zeta(1-\zeta^n)^{-2/n}$, for which $J_{-1/n}(f_n) = (1+\zeta^{2n})/(1-\zeta^{2n})$, in order to pose the following

Minimization Problem. *Find the minimal zero multiplicity $k = k(n)$ of $\varphi(\zeta) = c\zeta^k + \ldots$ over $\mathfrak{M}_{-1/n}$ under the correspondence $f \mapsto \varphi$ defining by $J_{-1/n}(f) = (1+\varphi)/(1-\varphi)$. It is clear that $k(n) \le 2n$.*

Let us note that $f_n \in \mathfrak{M}_\alpha$ exactly for $\alpha \in [-2/n, 0]$.

2. Returning to $n = 2$ the latter gives us (the sharpness in) the following

Proposition 1. *If $f \in \mathfrak{M}_\alpha$, $\alpha \in [-1, 0]$, and $a_2(f) = 0$, then we have the sharp estimate*

$$a_3(f) \le 1. \tag{3}$$

This is an easy consequence of (2) and the Kraus–Nehari theorem.

Proposition 1 may be considered as an extension to $[-1, 0]$ of the results of [7] or [4] about the estimate on $|a_3 - \lambda a_2^2|$ over $\mathfrak{M}_\alpha$ for $\alpha \ge 0$, but it is not the direct consequence of these results. Moreover, in view of $(2\alpha+1)a_3 = c$ and $|c| \le 1$ we can bring the inequality (3) to the estimate $a_3(f) \le \min\{1, |2\alpha+1|^{-1}\}$ for any $\alpha \in \mathbb{R}$.

Hypothesis. *The estimate $a_3(f) \le |2\alpha+1|^{-1}$ is sharp for $\alpha < -1$.*

3. We are interested in the situation $n = 2$. Let us note in passing that for $n = 2$ and $\alpha \in (-1, 0)$ the "control" for the inclusion $J_\alpha^{-1}\left(\frac{1+\varphi}{1-\varphi}\right) \in \mathfrak{M}_\alpha$ goes over from the parameter $c = \varphi''(0)/2$ to the parameter a_3. An interest to the behavior of a_3 can be based on its role in the uniqueness problem for the extreme of the conformal radius

$$R(\zeta) = |f'(\zeta)|(1-|\zeta|^2). \tag{4}$$

The parameter a_3 detects the appearance of the additional critical points near $\zeta = 0$: when a_3 moves across the value $1/3$, an elliptic maximum $\zeta = 0$ turns into the several critical points (in one of the typical cases—into hyperbolic saddle at $\zeta = 0$ and two elliptic maxima). It is important to determine the domain of the elliptic α's as the basin for the uniqueness α's.

Proposition 2. *The whole collection $\mathfrak{M}_\alpha \cap \{f''(0) = 0\}$ has an elliptic R-critical point at $\zeta = 0$ if and only if $\alpha \in (-\infty, -2) \cup (1, +\infty)$.*

As a hypothesis we can suppose that "the uniqueness scale" (for the zero critical point of (4)) has the center at $\alpha = -1/2$ ("the worst case"), surrounded by the "bad" segment $[-1, 0]$; an outcome into $[-2, 1]$ improves the situation up to the uniqueness for $\alpha \ge 1$ or $\alpha \le -2$.

Hypothesis. *The zero critical point of (4) is unique (with natural exclusions) if and only if α as in the Proposition 2.*

Theorem. *The union $\bigcup_{\alpha \ge 1} \mathfrak{M}_\alpha$ is the class of (no more than) uniqueness with the strip as the only exclusion. The value $\alpha = 1$ is the obstacle to the uniqueness extension to $\alpha < 1$.*

The proof is needed to the second statement only and consists of the study of the function

$$f_\alpha(\zeta) = \left\{\frac{1}{\alpha}\int_0^\zeta \frac{t^{1/\alpha-1}}{(1-t^2)^{1/\alpha}}dt\right\}^\alpha, \quad 0 < \alpha \le 1.$$

4. The final remark concerns with the best dominant problem for the subordination

$$p(\zeta) + \alpha\zeta\frac{p'}{p}(\zeta) \prec \frac{1+\zeta}{1-\zeta}, \tag{5}$$

where $\alpha > 0$, i.e. the problem of existence of the best univalent q in $\mathbb{D}$ with (5) implying $p \prec q$. (This problem and the *Open Problem* from the Section 1 may be formulated as mutually inverse). We want to formalize the notion of the best $p(\zeta) = 1 + b\zeta^n + \ldots$, $b \ne 0$, satisfying (5). The standard theory of dominants [6, 3, 2] provides their existence, but the sharpness take place only for $n = 1$, and this is the consequence of the structure of the definition ($p(\mathbb{D}) \subseteq q(\mathbb{D})$). Modifying the latter we obtain the following

Proposition 3. *The best dominant for (5) over the p's with expansion* $p(\zeta) = 1 + b\zeta^n + \ldots$, $b \neq 0$, $n \geq 1$, *is the univalent solution q of the differential equation*

$$q(\zeta) + n\alpha\zeta\frac{q'}{q}(\zeta) = \frac{1+\zeta}{1-\zeta}.$$

The sharpness of the dominant q for (5) means that the function $q(\zeta^n)$ satisfies (5) (when $\varphi(\zeta) = \zeta^n$; we exclude rotations for the brevity).

This remark has appeared as the background for the selection of the above f_α as an indicator to the passage “uniqueness—non-uniqueness”.

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