Parallel Algorithm of Solving the Electromagnetic Wave Diffraction Problem on the Spherical Screen

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Abstract— The electromagnetic wave diffraction problem on a thin conducting spherical screen is reduced to pair summatorial equation relative to unknown coefficients of expansion into a series of spherical waves. This equation can be transformed to a regular infinite set of linear algebraic equations by integral-summatorial identities method. For all stages of numerical algorithm of solving the problem the parallel calculating processes are possible. At first, if field traces of outside source at the sphere are decomposed onto magnetic and electric parts, then magnetic and electric parts of the unknown field can be found independently. Secondly, if coefficients of field conjugation conditions at the sphere do not depend on longitude coordinate, then calculations also can be fulfilled independently for every number of the series coefficients. Thirdly, if by reduction of infinite set the finite set of linear equations of large dimension is obtained, then it can be solved by one of parallel algorithms. But the most effect can be obtained just at the stage of calculating the auxiliary integrals over screen.

1. INTRODUCTION

The electromagnetic wave diffraction problem on the conducting thin screen is formulated as the boundary value problem for Maxwell equations set. If screen is a part of spherical surface then it is expediently to seek a solution of Maxwell equations in the spherical system of coordinates. In this case it is possible to represent general solution of these equations as an expansion into a series by spherical harmonics. It is necessary to find indeterminate coefficients of expansion in such way that the corresponding boundary conditions and conjugation conditions are fulfilled on a screen and on the remaining part of sphere. In the classical case of the diffraction problem on a complete sphere (see, for example, [1]) these coefficients are found in explicit form. In the case when screen has an arbitrary spherical form the set of equations for determining unknown values becomes rather difficult. Its numerical solving needs large calculating resources.

The purpose of the present work is to determine steps at which the calculating algorithm can be divided into independent subproblems and then realized on the multiprocessor calculating complex.

2. THE FORMULATION OF THE DIFFRACTION PROBLEM

Suppose an infinitely thin ideally conducting plate $\mathcal{M}$ (a screen) is placed in the space and is a part of sphere with radius R. Denote by $\mathcal{N}$ the remaining part of sphere $\mathcal{S}$. It is possible to assume in the general case that media properties are different inside sphere and outside of it. But to simplify reasoning, we assume that space is filled by homogeneous isotropic medium.

Suppose there is a source generating harmonic electromagnetic wave outside of screen. We should seek a field appearing by diffraction of this wave on a screen.

The mathematical formulation of the diffraction problem consists of the following. We should seek solutions of Maxwell equations set (complex amplitudes) inside sphere and outside of it

$$\operatorname{rot} H = i\omega\varepsilon_0\varepsilon\, E, \quad \operatorname{rot} E = -i\omega\mu_0\mu\, H$$

(it is assumed that dependence of field components on time has the form $e^{i\omega t}$). By this solutions should satisfy conditions at infinity, boundary conditions on $\mathcal{M}$ and conjugation conditions on $\mathcal{N}$. Boundary conditions and conjugation conditions are standard: tangent components of vector E vanish on the conducting surface and tangent components of vector E and H are continuous on the two media interface. Conditions at the infinity will be formulated later.

We choose the spherical coordinates system (r, θ, α), $\theta \in [0, \pi]$ is a latitude, $\alpha \in [0, 2\pi]$ is a longitude. In the plane of variables α and θ development of sphere is a rectangle.

3. MAXWELL EQUATIONS IN THE SPHERICAL COORDINATES SYSTEM

Maxwell equations set in the spherical coordinates system is a set of six equations with partial derivatives with respect to six complex-valued functions $E_r,\ E_\theta,\ E_\alpha,\ H_r,\ H_\theta,\ H_\alpha$ by three spatial variables. Each of the unknown functions can be decomposed (in any domain bounded by spheres with center in the origin of coordinates) into Fourier series by variable α of the form

$$A(r,\theta,\alpha)=\sum_{m=-\infty}^{+\infty} A_m(r,\theta)e^{im\alpha},$$

which converges in the classical (or in the generalized) sense. Set of equations for the Fourier coefficients with number m (to simplify formulas this number is not shown) has the form

$$\begin{aligned}
\frac{\partial(\sin\theta\,H_\alpha)}{\partial\theta}-im\,H_\theta &= i\omega\varepsilon_0\varepsilon\,r\sin\theta\cdot E_r, & \frac{\partial(\sin\theta\,E_\alpha)}{\partial\theta}-im\,E_\theta &= -i\omega\mu_0\mu\,r\sin\theta\cdot H_r,\\
im\,H_r-\sin\theta\,\frac{\partial(r\,H_\alpha)}{\partial r} &= i\omega\varepsilon_0\varepsilon\,r\sin\theta\cdot E_\theta, & im\,E_r-\sin\theta\,\frac{\partial(r\,E_\alpha)}{\partial r} &= -i\omega\mu_0\mu\,r\sin\theta\cdot H_\theta,\\
\frac{\partial(r\,H_\theta)}{\partial r}-\frac{\partial H_r}{\partial\theta} &= i\omega\varepsilon_0\varepsilon\,r\,E_\alpha, & \frac{\partial(r\,E_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta} &= -i\omega\mu_0\mu\,r\,H_\alpha.
\end{aligned}$$

If boundary conditions and conjugation conditions in the diffraction problem can be divided into independent conditions for the Fourier coefficients of field components also, then this problem splits into set of independent subproblems. At this stage *first possibility* of paralleling of algorithm appears.

Second possibility is connected with the fact that any solution of the set of Equation (2) can be represented as a sum of two partial solutions: of magnetic type for $E_r=0$ and of electric type for $H_r=0$.

We have for magnetic field by

$$\begin{aligned}
E_r &= 0, & E_\theta &= \frac{\omega\mu_0\mu m}{\sin\theta}\,u, & E_\alpha &= i\omega\mu_0\mu\,\frac{\partial u}{\partial\theta},\\
H_r &= \Big(\frac{\partial^2(r\,u)}{\partial r^2}+k^2r\,u\Big), & H_\theta &= \frac{1}{r}\,\frac{\partial^2(r\,u)}{\partial\theta\,\partial r}, & H_\alpha &= \frac{im}{r\sin\theta}\,\frac{\partial(r\,u)}{\partial r},
\end{aligned}$$

where u being a potential function. This function should satisfy the potential equation

$$r\,\frac{\partial^2(r\,u)}{\partial r^2}+\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\Big(\sin\theta\,\frac{\partial u}{\partial\theta}\Big)+\Big(k^2r^2-\frac{m^2}{\sin^2\theta}\Big)\,u=0.$$

Potential function v of electric field should satisfy just the same equation, by this field components

$$\begin{aligned}
E_r &= \Big(\frac{\partial^2(r\,v)}{\partial r^2}+k^2r\,v\Big), & E_\theta &= \frac{1}{r}\,\frac{\partial^2(r\,v)}{\partial\theta\,\partial r}, & E_\alpha &= \frac{im}{r\sin\theta}\,\frac{\partial(r\,v)}{\partial r},\\
H_r &= 0, & H_\theta &= -\frac{\omega\varepsilon_0\varepsilon m}{\sin\theta}\,v, & H_\alpha &= -i\omega\varepsilon_0\varepsilon\,\frac{\partial v}{\partial\theta}.
\end{aligned}$$

4. AXIS SYMMETRICAL FIELD

Suppose that components of electromagnetic field do not dependent on the coordinate α. For $m=0$ set of Equation (2) has the form

$$\begin{aligned}
\frac{\partial(\sin\theta\,H_\alpha)}{\partial\theta} &= i\omega\varepsilon_0\varepsilon\,r\sin\theta\cdot E_r, & \frac{\partial(\sin\theta\,E_\alpha)}{\partial\theta} &= -i\omega\mu_0\mu\,r\sin\theta\cdot H_r,\\
-\frac{\partial(r\,H_\alpha)}{\partial r} &= i\omega\varepsilon_0\varepsilon\,r\,E_\theta, & -\frac{\partial(r\,E_\alpha)}{\partial r} &= -i\omega\mu_0\mu\,r\,H_\theta,\\
\frac{\partial(r\,H_\theta)}{\partial r}-\frac{\partial H_r}{\partial\theta} &= i\omega\varepsilon_0\varepsilon\,r\,E_\alpha, & \frac{\partial(r\,E_\theta)}{\partial r}-\frac{\partial E_r}{\partial\theta} &= -i\omega\mu_0\mu\,r\,H_\alpha.
\end{aligned}$$

This set of equations splits into two independent subsystems and any its solution can be written down as a sum of two partial solutions as it was in the case $m\neq 0$.

When we consider the axis symmetrical problem, it is convenient to choose another potential functions. The potential function u of magnetic field should satisfy equation

$$r\,\frac{\partial^2(ru)}{\partial r^2}+\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\,\frac{\partial(\sin\theta\cdot u)}{\partial\theta}\right)+k^2r^2\,u=0,$$

the potential function v of electric field should satisfy just the same equation. The complete field components are expressed by two potential functions in the following way:

$$E_\alpha = u, \qquad H_r = \frac{-1}{i\omega\mu_0\mu}\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot u)}{\partial\theta}, \qquad H_\theta = \frac{1}{i\omega\mu_0\mu}\frac{1}{r}\frac{\partial(ru)}{\partial r},$$
$$H_\alpha = v, \qquad E_r = \frac{1}{i\omega\varepsilon_0\varepsilon}\frac{1}{r\sin\theta}\frac{\partial(\sin\theta\cdot v)}{\partial\theta}, \qquad E_\theta = \frac{-1}{i\omega\varepsilon_0\varepsilon}\frac{1}{r}\frac{\partial(rv)}{\partial r}.$$

It is easy to find partial solutions of potential equations by Fourier method:

$$u_n(r,\theta) = \frac{1}{\sqrt{kr}} Z_{n+1/2}(kr)\,\Theta_n(\theta), \qquad \Theta_n(\theta) = \mathrm{const}\, P_n^{(1)}(\cos\theta), \quad n = 1, 2, \dots$$

here $Z_{n+1/2}(\cdot)$ are cylindrical functions (solutions of Bessel equations) and $P_n^{(1)}(\cdot)$ are joint Legendre functions. We choose constant multipliers in such way that functions $\Theta_n(\theta)$ should be normalized with weight $\sin\theta$ on segment $[0,\pi]$.

Let us consider the diffraction problem for axis symmetrical wave at the ring spherical screen $\theta_1 < \theta < \theta_2$. We have for a wave being generated by exterior source, for $r > R$ and for $r < R$

$$u^0(r,\theta) = \sum_{n=1}^{+\infty}\left[\overrightarrow{u}_n^0 \frac{1}{\sqrt{kr}} H^{(2)}_{n+1/2}(kr) + \overleftarrow{u}_n^0 \frac{1}{\sqrt{kr}} H^{(1)}_{n+1/2}(kr)\right]\Theta_n(\theta),$$
$$v^0(r,\theta) = \sum_{n=1}^{+\infty}\left[\overrightarrow{v}_n^0 \frac{1}{\sqrt{kr}} H^{(2)}_{n+1/2}(kr) + \overleftarrow{v}_n^0 \frac{1}{\sqrt{kr}} H^{(1)}_{n+1/2}(kr)\right]\Theta_n(\theta).$$

It is assumed that this wave can contain both spherical harmonics coming from infinity and harmonics outgoing to infinity.

Outgoing from sphere waves determine potential functions

$$u^-(r,\theta) = \sum_{n=1}^{+\infty}\overrightarrow{u}_n^- \frac{1}{\sqrt{kr}} H^{(2)}_{n+1/2}(kr)\,\Theta_n(\theta), \qquad v^-(r,\theta) = \sum_{n=1}^{+\infty}\overrightarrow{v}_n^- \frac{1}{\sqrt{kr}} H^{(2)}_{n+1/2}(kr)\,\Theta_n(\theta).$$

The unknown potential field functions inside sphere have the form

$$u^+(r,\theta) = \sum_{n=1}^{+\infty} u_n^+ \frac{1}{\sqrt{kr}} J_{n+1/2}(kr)\,\Theta_n(\theta), \qquad v^+(r,\theta) = \sum_{n=1}^{+\infty} v_n^+ \frac{1}{\sqrt{kr}} J_{n+1/2}(kr)\,\Theta_n(\theta).$$

5. REGULAR SLAE OF THE DIFFRACTION PROBLEM

It is easy to see that boundary conditions and conjugation conditions on a sphere (for $r = R$) are independent for magnetic field and for electric field. Consider problem for magnetic field. Functions u and $\dfrac{\partial u}{\partial r}$ should be continuous on $\mathcal{N}$, i.e., conditions $u^- = u^+$ and $\dfrac{\partial u^-}{\partial r} = \dfrac{\partial u^+}{\partial r}$ should fulfill, and conditions $u^0 + u^- = 0$ and $u^0 + u^+ = 0$ should fulfill on a screen $\mathcal{M}$.

Note that field from exterior source is given in the whole space, i.e., both inside sphere and outside of it. In the case when spherical surface $r = R$ is a media interface (screen is placed on the surface of the dielectric ball), the diffraction problem should be solved in two stages. At first, it is necessary to find a solution of the reflection (and refraction) problem for given wave from the ball, and then it is necessary to pass to the problem on perturbation of founded in the whole space field by conducting thin screen.

By conditions for $r = R$ it follows that for all n

$$\overrightarrow{u}_n^-\, H^{(2)}_{n+1/2}(kR) = u_n^+\, J_{n+1/2}(kR) = c_n,$$

here c_n are new unknown coefficients. These coefficients should satisfy pair summatorial equation

$$\sum_{n=1}^{+\infty} c_n\,\Theta_n(\theta) = -\sum_{n=1}^{+\infty}\Big[\overrightarrow{u}_n^0\, H^{(2)}_{n+1/2}(kR) + \overleftarrow{u}_n^0\, H^{(1)}_{n+1/2}(kR)\Big]\Theta_n(\theta), \quad \theta\in\mathcal{M},$$
$$\sum_{n=1}^{+\infty}\gamma_n\, c_n\,\Theta_n(\theta) = 0, \quad \theta\in\mathcal{N}, \qquad \gamma_n = \frac{[H^{(2)}_{n+1/2}]'(kR)}{H^{(2)}_{n+1/2}(kR)} - \frac{[J_{n+1/2}]'(kR)}{J_{n+1/2}(kR)}.$$

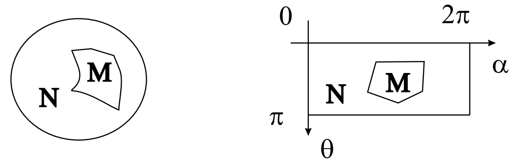

Figure 1: A screen on a sphere and a development of sphere.

Here $\mathcal{M} = [\theta_1, \theta_2]$ and $\mathcal{N}$ is a segment, supplementing $\mathcal{M}$ up to $[0, \pi]$.

This pair equation can be transformed into regular infinite set of linear algebraic equations by integral-summatorial identities [2, 3].

Third possibility of paralleling is the algorithm of solving SLAE, which appears by reduction method.

6. GENERAL CASE

In the case when $m \neq 0$, magnetic and electric potential functions are expressed by cylindrical and joint Legendre functions also,

$$u_n(r, \theta) = \frac{1}{\sqrt{kr}} Z_{n+1/2}(kr)\, P_n^{(m)}(\cos\theta), \quad n \geq |m|.$$

Radial multipliers are chosen in correspondence with required wave orientation. In the case of outgoing from sphere to infinity waves Hankel functions of the 2-nd kind are used, and for coming waves Hankel functions of the 1-nd kind are used. Both these functions should be present in the unknown field inside sphere. Moreover, they should have equal coefficients (as there are no sources of field inside sphere, so only standing waves can exist there).

In general case unknown coefficients of field decomposition have two indexes: m, n. But in the case when plane wave run into the spherical screen

$$E^0 = x_0\, A e^{-ikz}, \qquad H^0 = y_0\, \frac{A}{W} e^{-ikz},$$

it is necessary to consider only two values of parameter m: $m = -1$, $m = 1$.

In a difference from axis symmetrical case, if the exterior field is only magnetic or only electric wave, then the unknown field contains both magnetic and electric components. Some simplification is possible only in the case when spherical screen is a ring or a spherical segment. To such screens at variables plane α, θ (see Fig. 1) correspond strips $\theta_1 < \theta < \theta_2$ and $\alpha_1 < \alpha < \alpha_2$.

The numerical experiment has shown that the most economy effect of calculating resources is obtained by parallel calculating coefficients of SLAE of the form

$$I_{m,n;p,q} = \iint_{\mathcal{M}} \sin\theta\, S_{m,n}(\theta, \alpha)\, S_{p,q}(\theta, \alpha)\, d\theta\, d\alpha,$$

where $S_{m,n}(\theta, \alpha)$ are spherical functions. This is *fourth possibility* of paralleling of algorithm.

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