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ANALYTIC SOLUTION OF AN $\mathbb{R}$-LINEAR CONJUGATION PROBLEM IN THE CASE OF HYPERBOLIC INTERFACE

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Abstract. We derive an explicit analytical solution for the conjugation problem in a heterogeneous infinite planar structure consisting of two dissimilar homogeneous media with two branches of a hyperbola as an interface. We investigate both cases involving real and complex coefficients of the corresponding boundary condition.

Keywords: heterogeneous media, $\mathbb{R}$-linear conjugation problem, analytic functions.

1 INTRODUCTION

We consider problems in multi-phase continuous isotropic planar media with solutions representable in terms of a vector field that is both solenoidal and irrotational. In each phase, distinguished by the subscript $p = 1, \dots, N$, we define a vector field $\mathbf{v}(x, y) = (v_x, v_y) = \mathbf{v}_p(x, y)$, $(x, y) \in S_p$, $p = \overline{1, N}$, of the horizontal and vertical components v_x, v_y such that

$$\nabla \cdot \mathbf{v}_p = 0, \quad \nabla \times \mathbf{v}_p = 0. \tag{1.1}$$

The refraction boundary conditions between each two dissimilar neighboring phases S_p and S_q stipulate that the normal components of $\mathbf{v}_p$, $\mathbf{v}_q$ be continuous across each boundary $\mathcal{L}_{pq} = \partial S_p \cap \partial S_q$ and that the tangential components of $\hat{\rho}_p \mathbf{v}_p$, $\hat{\rho}_q \mathbf{v}_q$ also be continuous:

$$[\mathbf{v}_p(x, y)]_n = [\mathbf{v}_q(x, y)]_n, \quad [\hat{\rho}_p \mathbf{v}_p(x, y)]_\tau = [\hat{\rho}_q \mathbf{v}_q(x, y)]_\tau, \quad (x, y) \in \mathcal{L}_{pq}; \tag{1.2}$$

the constant parameter $\hat{\rho}_p$ corresponds to a phase property of a medium S_p. One can, for instance, take $\mathbf{v}_p = \mathbf{I}_p$ to be the current. Then $\hat{\rho}_p = \rho_p(1 - \mathrm{i}\,\beta_p)$ is the generalized electrical resistivity of a medium S_p with ρ_p as the resistivity and β_p as the Hall parameter. In the theory of seepage through porous media ([11], p. 52), $\mathbf{v}_p$ is the Darcian velocity, and $\hat{\rho}_p = \rho_p = 1/k_p$ with k_p being the constant hydraulic conductivity of the phase S_p. Problem (1.1)–(1.2) is studied extensively in other mathematically equivalent areas of continuous media mechanics. Nevertheless, till now there are just a few examples of specific heterogeneous structures for which exact analytic solutions of this problem were derived. It has been mainly done for structures with conics as an interface (e.g., [3], [9], [10], [7]) and for some regular double-periodic structures ([8], [1], [5]; see more references in [6]).

In [9], a two-phase medium with one branch of a hyperbola as an interface was investigated. The present paper is an immediate continuation of [9] in the case of two hyperbolic inclusions.

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2 STATEMENT OF THE PROBLEM

We consider a two-phase structure with an interface consisting of both branches of hyperbola

$$(x/a)^2 - (y/b)^2 = 1, \tag{2.1}$$

where a, b are given positive parameters.

Let S_1 and S_2 be the exterior and interior of the hyperbola (2.1) (Fig. 1). We suppose that both components of the phase S_2 are characterized by the same physical parameters.

Below complex variables are used, that is, $z = x + \mathrm{i}\, y$ and $v_p(z) = v_{px} - \mathrm{i}\, v_{py}$. In each phase, S_p ($p = 1, 2$), due to (1.1), a holomorphic function $v_p(z)$, is defined.

We denote

$$A_{pq} = \frac{\rho_p + \rho_q}{2\rho_p} - \mathrm{i}\, \frac{\rho_p\beta_p - \rho_q\beta_q}{2\rho_p}, \quad B_{pq} = \frac{\rho_p - \rho_q}{2\rho_p} - \mathrm{i}\, \frac{\rho_p\beta_p - \rho_q\beta_q}{2\rho_p}; \tag{2.2}$$

in particular,

$$A_{pq} = \frac{\rho_p + \rho_q}{2\rho_p}, \quad B_{pq} = \frac{\rho_p - \rho_q}{2\rho_p}. \tag{2.3}$$

Then, as shown in [2], p. 53, problem (1.1)–(1.2) is equivalent to the following one:

$$v_1(t) = Av_2(t) - B[t'(s)]^{-2}\overline{v_2(t)}, \quad t \in \mathcal{L}, \tag{2.4}$$

where $A = A_{12}$, $B = B_{12}$, and $t = t(s) \in \mathcal{L} = \mathcal{L}_{12}$ is a function of the arc length s of the contour $\mathcal{L}$. The derivative $t'(s) = \exp(\mathrm{i}\alpha(s))$ coincides with the unit tangential vector to the contour $\mathcal{L}$ at the point $t \in \mathcal{L}$ ($\alpha(s)$ is the angle between $t'(s)$ and the real axis).

We find a general solution to problem (2.4) in the class of functions for which the following asymptotics holds in the vicinity of infinity:

$$|v(z)| = \mathrm{o}(|z|) \quad \text{for } |z| \gg 1. \tag{2.5}$$

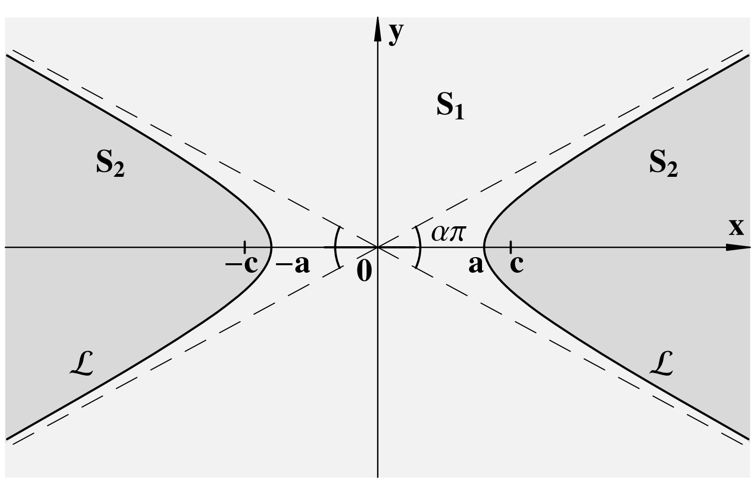

Figure 1. A medium with two hyperbolic inclusions.

First, we find a solution of the stated problem under the assumption that arbitrary real coefficients A and B of the boundary conditions (2.4) satisfy the only restriction $A \geqslant |B| \geqslant 0$. Then the corresponding solution of problem (2.4) can be easily written down for the coefficients $A = A_{12}$ and $B = B_{12}$ defined in (2.2) or (2.3).

3 SOLUTION OF THE PROBLEM ABOUT HYPERBOLIC INCLUSIONS FOR A REAL CASE

We note that an arbitrary solution of problem (2.4), under some additional supposition, can be represented as follows:

$$v(z) = v_R(z) + v_I(z), \tag{3.1}$$

where $v_R(z)$ and $v_I(z)$ are particular solutions of this problem satisfying the symmetry conditions

$$\begin{aligned} \overline{v_R(\overline{z})} &\equiv v_R(z), \qquad & \overline{v_R(-\overline{z})} &\equiv v_R(z), \\ \overline{v_I(\overline{z})} &\equiv -v_I(z), \qquad & \overline{v_I(-\overline{z})} &\equiv -v_I(z). \end{aligned} \tag{3.2}$$

Indeed, if $v(z)$ is an arbitrary solution of problem (2.4) with real coefficients A and B, then, due to the symmetry of the hyperbola $\mathcal{L} = \mathcal{L}_{12}$ about both coordinate axes, the functions $\overline{v(\overline{z})}$, $\overline{v(-\overline{z})}$, and $v(-z)$, as well as any their linear combination, are also solutions of this problem. We additionally suppose that any solution of our problem satisfies the symmetry condition $v(-z) \equiv v(z)$. The last condition, almost "evident" from a physical point of view, does not lead to any mathematical contradiction, as we will see later. Now our assertion follows from the identity

$$v(z) \equiv \frac{1}{2}(v(z) + \overline{v(\overline{z})}) + \frac{1}{2}(v(z) - \overline{v(\overline{z})}) = v_R(z) + v_I(z).$$

In accordance with identities (3.2), it suffices to define solutions $v_R(z)$, $v_I(z)$ in the first quarter, $\mathbb{C}_+^+ = \{z \colon \Re z > 0, \Im z > 0\}$, of the z-plane.

In [9], for the part $\mathcal{L}^+ \in \mathbb{C}_+^+$ of the right branch of hyperbola (2.1), the dependence of the function $t'(s)$ on $t \in \mathcal{L}^+$ was found in the form

$$\frac{\mathrm{d}t}{\mathrm{d}s} = (t^2 - c^2)^{1/4} \left(\sqrt{t^2 - c^2}\cos\pi\alpha - \mathrm{i}\, t \sin\pi\alpha\right)^{-1/2},$$

where $a + \mathrm{i}\, b = c\exp(\mathrm{i}\,\pi\alpha/2)$, $c > 0$, $0 < \alpha < 1$.

Let us consider the function

$$\zeta(z) = \frac{1}{c}(z + \sqrt{z^2 - c^2}) \tag{3.3}$$

and its inversion

$$z(\zeta) = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right). \tag{3.4}$$

The function (3.3) maps conformally the quarter plane $\mathbb{C}_+^+$ onto the domain $S^* = \{\zeta \colon \Re\zeta > 0, \Im\zeta > 0, |\zeta| > 1\}$ (Fig. 2), i.e., S^* is $\mathbb{C}_+^+$ with a dropped quarter of a unit disk. We denote by $\mathcal{L}_1^* = \{\zeta \colon \arg\zeta = \pi\alpha/2, |\zeta| > 1\}$ the image of $\mathcal{L}^+ = \mathcal{L} \cap \mathbb{C}_+^+$ and by S_1^*, S_2^* the corresponding portions of S^* that are the images of S_1^+, S_2^+, respectively.

Due to (3.3)–(3.4), one can get for $t = t(\tau) \in \mathcal{L}^+$, $\tau \in \mathcal{L}_1^*$:

$$\sqrt{t^2 - c^2} = \frac{c}{2}\left(\tau - 1/\tau\right), \quad t = \frac{c}{2}\left(\tau + 1/\tau\right).$$

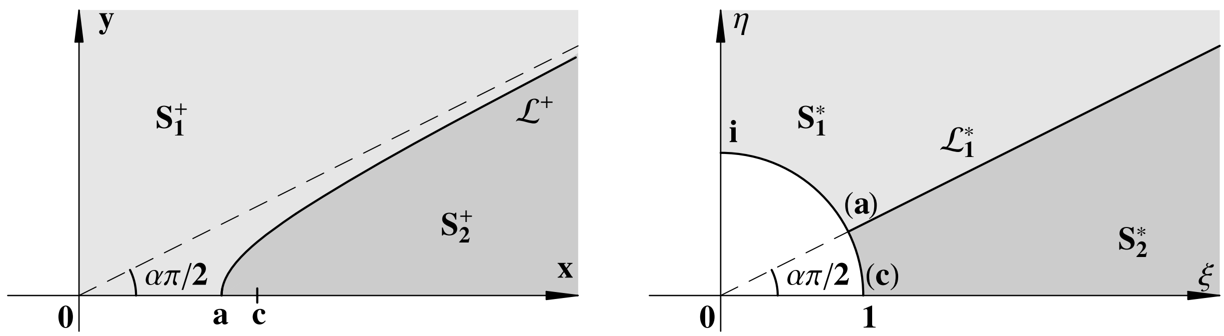

Figure 2. First quarter of z-plane and its image in the ζ-plane given by the function (3.3).

Thus, for the function $[t'(s)]^{-2}$ in the ζ-plane, the following representation holds:

$$\left(\frac{\mathrm{d}\,t}{\mathrm{d}\,s}\right)^{-2} = \frac{(\tau - 1/\tau)\cos\pi\alpha - \mathrm{i}\,(\tau + 1/\tau)\sin\pi\alpha}{\tau - 1/\tau} = \frac{\overline{\tau - 1/\tau}}{\tau - 1/\tau}, \quad \tau \in \mathcal{L}_1^*.$$

From the last relation, (2.3) and the corresponding identities (3.2), taking into account that the difference $\zeta - 1/\zeta$ is real on the real axis and pure imaginary on the unit circle, we get, with respect to the function

$$V(\zeta) = (\zeta - 1/\zeta)\, v_R\Big(\frac{c}{2}(\zeta + 1/\zeta)\Big), \tag{3.5}$$

the following boundary-value problem:

$$\begin{array}{ll} V_1(\tau) = AV_2(\tau) - B\overline{V_2(\tau)}, & \tau \in \mathcal{L}_1^*, \\ \operatorname{Im} V_2(\xi) = 0, & \xi > 1, \\ \operatorname{Re} V(\tau) = 0, & |\tau| = 1,\ 0 < \arg\tau < \pi/2, \\ \operatorname{Re} V_1(\mathrm{i}\,\eta) = 0, & \eta > 1. \end{array} \tag{3.6}$$

From (2.5) and definition (3.5) it follows that a solution of problem (3.6) must satisfy the additional conditions

$$|V(\zeta)| = \mathrm{o}(|\zeta|^2),\ |\zeta| \gg 1; \quad V(1) = 0. \tag{3.7}$$

Using the Riemann–Schwarz symmetry principle and the third boundary condition (3.6), we continue analytically the function $V(\zeta)$ from S^* into $\mathbb{C}_+^+$. For the continued piece-wise holomorphic function, we keep the same notation $V(\zeta) = \{V_2(\zeta),\ 0 < \arg\zeta < \pi\alpha/2;\ V_1(\zeta),\ \pi\alpha/2 < \arg\zeta < \pi/2\}$. Using boundary conditions (3.6), we arrive at the following problem with respect to this function:

$$\begin{array}{ll} V_1(\tau) = AV_2(\tau) - B\overline{V_2(\tau)}, & \tau \in \mathcal{L}^*, \\ \operatorname{Im} V_2(\xi) = 0, & \xi > 0, \\ \operatorname{Re} V_1(\mathrm{i}\,\eta) = 0, & \eta > 0, \end{array} \tag{3.8}$$

with auxiliary conditions (3.7). Two other conditions should be satisfied:

$$|V(\zeta)| = \mathrm{o}(|\zeta|^{-2}),\ |\zeta| \ll 1; \quad \overline{V(1/\overline{\zeta})} \equiv -V(\zeta). \tag{3.9}$$

Note that the first condition (3.9) follows from the second one and from the first condition in (3.7).

Let us consider the branch of the function $\omega(\zeta)=\zeta^\gamma$ with real power γ fixed in the ζ-plane with the cut along $\mathbb{R}_+=\{\xi\colon \xi>0\}$ which is positive at the upper side of the cut. For the chosen branch, the identities $\overline{\omega(1/\overline{\zeta})}\equiv 1/\omega(\zeta)$, $\overline{\omega(-\overline{\zeta})}\equiv \mathrm{e}^{-\mathrm{i}\,\pi\gamma}\omega(\zeta)$ are true in the upper half-plane. Hence, the function

$$V(\zeta)=\begin{cases} V_1(\zeta)=\mathrm{i}\,c_{11}(\mathrm{e}^{-\mathrm{i}\,\pi\gamma/2}\zeta^\gamma+\mathrm{e}^{\mathrm{i}\,\pi\gamma/2}\zeta^{-\gamma}), & \pi\alpha/2\leqslant \arg\zeta\leqslant\pi/2,\\ V_2(\zeta)=c_{12}(\zeta^\gamma-\zeta^{-\gamma}), & 0\leqslant\arg\zeta\leqslant\pi\alpha/2,\end{cases} \tag{3.10}$$

with arbitrary real c_{11}, c_{12} and $0<\gamma<2$ satisfies all conditions (3.7) through (3.9) with the exception of the first condition in (3.8). Since $\omega(\tau)=r^\gamma\mathrm{e}^{\mathrm{i}\,\pi\gamma\alpha/2}$ for $\tau\in\mathcal{L}^*$, by substitution of the function (3.10) into the first condition in (3.8) and equating the coefficients at $r^{\pm\gamma}$ for the parameters γ, c_{11}, c_{12}, we get the system

$$\begin{cases} c_{11}\sin[\pi\gamma(1-\alpha)/2]-c_{12}(A-B)\cos(\pi\gamma\alpha/2)=0,\\ c_{11}\cos[\pi\gamma(1-\alpha)/2]-c_{12}(A+B)\sin(\pi\gamma\alpha/2)=0.\end{cases} \tag{3.11}$$

The system (3.11) is linear and homogeneous with respect to c_{11} and c_{12}. It has a nontrivial solution if its determinant equals zero. This leads to the following equation with respect to γ:

$$(A-B)\cos[\pi\gamma(1-\alpha)/2]\cos\frac{\pi\gamma\alpha}{2}-(A+B)\sin[\pi\gamma(1-\alpha)/2]\sin\frac{\pi\gamma\alpha}{2}=0.$$

This equation transforms to the equivalent one of the form

$$\cos(\pi\gamma/2)-\varDelta\cos[\pi\gamma(1/2-\alpha)]=0, \tag{3.12}$$

where $\varDelta=B/A$. We are interested in solutions of Eq. (3.12) over the interval $(0,2)$.

Before analyzing Eq. (3.12), we remark that a construction of a function $v_I(z)$ satisfying the second pair of conditions (3.2) leads to the representation

$$V(\zeta)=\begin{cases} V_1(\zeta)=c_{21}(\mathrm{e}^{-\mathrm{i}\,\pi\gamma/2}\zeta^\gamma+\mathrm{e}^{\mathrm{i}\,\pi\gamma/2}\zeta^{-\gamma}), & \pi\alpha/2<\arg\zeta<\pi/2,\\ V_2(\zeta)=\mathrm{i}\,c_{22}(\zeta^\gamma-\zeta^{-\gamma}), & 0<\arg\zeta<\pi\alpha/2,\end{cases} \tag{3.13}$$

instead of (3.10); to the system

$$\begin{cases} c_{21}\cos[\pi\gamma(1-\alpha)/2]+c_{22}(A-B)\sin(\pi\gamma\alpha/2)=0,\\ c_{21}\sin[\pi\gamma(1-\alpha)/2]+c_{22}(A+B)\cos(\pi\gamma\alpha/2)=0,\end{cases} \tag{3.14}$$

instead of (3.11); and, correspondingly, to the equation

$$\cos(\pi\gamma/2)+\varDelta\cos[\pi\gamma(1/2-\alpha)]=0, \tag{3.15}$$

which differs from Eq. (3.12) by the sign of the second summand only.

Taking into account that $0<\alpha<1$ and $|\varDelta|<1$, it is clear (e.g., by plotting a corresponding graph) that each of transcendental equations (3.12) and (3.15) has only one root $\gamma\in(0,2)$ for all nonlimiting ($\varDelta\neq\pm1$) and nondegenerate ($\varDelta\neq0$) cases. Precisely, if $0<\varDelta<1$, then Eqs. (3.12) and (3.15) have roots $0<\gamma_1<1$ and $1<\gamma_2<2$, respectively. For $\varDelta<0$, an inverse situation takes place.

Remark. The permutation α to $1-\alpha$ does not change the considered equations, and the same solutions γ remain for the permuted angles.

If $\alpha=1/2$ ($\varDelta\neq\pm1$), then Eqs. (3.12) and (3.15) are solvable in an explicit form:

$$\gamma_1=\frac{2}{\pi}\arccos\varDelta,\quad \gamma_2=2-\gamma_1. \tag{3.16}$$

If $0 < \alpha < 1/2$ and $1 - 2\alpha = p/q > 0$ is a rational number, then the equations transform to algebraic ones with respect to $x = \cos(\pi\gamma/(2q))$:

$$\sum_{k=0}^{[q/2]} (-1)^k C_q^{2k} x^{q-2k} (1-x^2)^k = \pm\Delta \sum_{k=0}^{[p/2]} (-1)^k C_p^{2k} x^{p-2k} (1-x^2)^k,$$

where the square bracket denotes the integer part of a number. Hence, the roots of Eqs. (3.12) and (3.15) are defined explicitly by formulas analogous to (3.16). For two special angles, we have:

$$\begin{aligned} \gamma_1 &= \frac{4}{\pi} \arccos \frac{\Delta + \sqrt{8+\Delta^2}}{4}, \quad && \gamma_2 = \gamma_1(-\Delta), \quad (\alpha = 3/4, 1/4); \\ \gamma_1 &= \frac{6}{\pi} \arccos \frac{\sqrt{3+\Delta}}{2}, && \gamma_2 = \gamma_1(-\Delta), \quad (\alpha = 2/3, 1/3). \end{aligned}$$

For $\alpha = 7/8(1/8)$ and $\alpha = 5/8(3/8)$, the required solutions are defined via the formulas $\gamma_p = 8\pi^{-1} \arccos x_p$, where x_p are the corresponding roots of the equations

$$8x^4 - 8x^2 + 1 = \pm\Delta(4x^3 - 3x), \quad 8x^4 - 8x^2 + 1 = \pm\Delta x.$$

If $\alpha = 5/6(1/6)$, then $\gamma_p = 6\pi^{-1} \arccos x_p$, where x_p are the roots of the cubic equation

$$4x^3 - 3x = \pm\Delta(2x^2 - 1).$$

For all other rational values of α, different from those listed above, we get an algebraic equation of order higher than the fifth.

Now solutions of the systems (3.11)–(3.14) can be written down in the form

$$c_{12} = c_1 \Lambda(\gamma_1), \quad c_{11} = c_1, \qquad c_{22} = c_2 \Lambda(\gamma_2), \quad c_{21} = c_2, \tag{3.17}$$

where γ_1, γ_2 are the solutions of Eqs. (3.12) and (3.15), respectively,

$$\Lambda(\gamma) = \frac{\cos(\pi\gamma/2)}{B \sin(\pi\gamma\alpha)}, \tag{3.18}$$

and c_1, c_2 are arbitrary real constants.

Thus, the following theorem is proved.

Theorem 1. *If* $0 < |\Delta| = |B|/A < 1$, $a + \mathrm{i}\, b = c\mathrm{e}^{\mathrm{i}\,\pi\alpha/2}$, $\alpha \in (0,1)$, $\alpha \neq 1/2$, *then problem* (2.4) *for a pair of hyperbolic inclusions* (2.1) *has a general solution of the form*

$$\begin{aligned} v_1(z) &= \mathrm{i}\, c_1 \chi_1(z;\gamma_1) + c_2 \chi_1(z;\gamma_2), && z \in S_1^+, \\ v_2(z) &= c_1 \Lambda(\gamma_1) \chi_2(z;\gamma_1) + \mathrm{i}\, c_2 \Lambda(\gamma_2) \chi_2(z;\gamma_2), \quad && z \in S_2^+, \end{aligned} \tag{3.19}$$

where γ_1 *and* γ_2 *are the roots of Eqs.* (3.12) *and* (3.15)*, respectively, over the interval* $(0,2)$*. Single-valued branches of the functions*

$$\begin{aligned} \chi_1(z;\gamma) &= \frac{\mathrm{e}^{-\mathrm{i}\,\pi\gamma/2}(z + \sqrt{z^2-c^2})^\gamma + \mathrm{e}^{\mathrm{i}\,\pi\gamma/2}(z - \sqrt{z^2-c^2})^\gamma}{2\sqrt{z^2-c^2}}, \\ \chi_2(z;\gamma) &= \frac{(z + \sqrt{z^2-c^2})^\gamma - (z - \sqrt{z^2-c^2})^\gamma}{2\sqrt{z^2-c^2}}, \quad \chi_2(c;\gamma) = \gamma c^{\gamma-1}, \end{aligned} \tag{3.20}$$

are fixed in the corresponding domains S_1^+ *and* S_2^+ *and take imaginary and real values at the imaginary and real axes, respectively.* $\Lambda(\gamma)$ *is defined by relation* (3.18)*. Finally,* c_1 *and* c_2 *are arbitrary real parameters.*

If $\alpha = 1/2$*, then a general solution of problem* (2.4) *is defined by the formulas*

$$\begin{aligned} v_1(z) &= \mathrm{i}\, c_1\chi_1(z;\gamma) - c_2\chi_1(z;2-\gamma), & z \in S_1^+,\\ v_2(z) &= (A^2-B^2)^{-1/2}\left(c_1\chi_2(z;\gamma) + \mathrm{i}\, c_2\chi_2(z;2-\gamma)\right), & z\in S_2^+, \end{aligned} \tag{3.21}$$

where $\gamma = \gamma_1$ *is defined in* (3.16)*.*

Assertions of the formulated theorem follow from (3.13), (3.10), (3.5), (3.4), and (3.3). For $\alpha = 1/2$, on the basis of (3.16) through (3.18), it is not difficult to get that $\Lambda(\gamma) = 1/(A\sin(\arccos\Delta)) = 1/(A\sqrt{1-\Delta^2}) = (A^2-B^2)^{-1/2}$, $\Lambda(2-\gamma) = -\Lambda(\gamma)$, and thus representation (3.21) holds.

With the help of identities (3.2), one can get a solution of problem (2.4) defined in the whole z-plane:

$$v(z) = \begin{cases} v_R(z) + v_I(z), & z\in\mathbb{C}_+^+,\\ \overline{v_R(-\overline{z})} - \overline{v_I(-\overline{z})}, & z\in\mathbb{C}_-^+,\\ v_R(-z) + v_I(-z), & z\in\mathbb{C}_-^-,\\ \overline{v_R(\overline{z})} - \overline{v_I(\overline{z})}, & z\in\mathbb{C}_+^-, \end{cases}$$

where $\mathbb{C}_\pm^\pm = \{z\colon \pm\Re z > 0, \pm\Im z > 0\}$,

$$v_R(z) = c_1\begin{cases} \mathrm{i}\,\chi_1(z,\gamma_1), & z\in S_1^+,\\ \Lambda(\gamma_1)\chi_2(z,\gamma_1), & z\in S_2^+, \end{cases} \qquad v_I(z) = c_2\begin{cases} \chi_1(z,\gamma_2), & z\in S_1^+,\\ \mathrm{i}\,\Lambda(\gamma_2)\chi_2(z,\gamma_2), & z\in S_1^+. \end{cases}$$

Let now the function $v_1(z)$ take a certain value at $z = 0$:

$$v_1(0) = V_0 = v_{0x} - \mathrm{i}\, v_{0y}, \quad v_{0x}, v_{0y}\in\mathbb{R}, \tag{3.22}$$

Then the solution $v(z)$ of problem (2.4) is defined by Eqs. (3.19)–(3.21) in a unique way with

$$c_1 = v_{0x}c^{1-\gamma_1}, \quad c_2 = v_{0y}c^{1-\gamma_2}. \tag{3.23}$$

Remark 1. If the coefficients of the boundary condition (2.4) are defined by formulas (2.3), then nothing changes in the form of solutions given by Theorem 1. The only amendment is that $A^2 - B^2 = \rho_2/\rho_1$ in (3.21).

Remark 2. For the degenerate case $B = B_{12} \to 0$, we get consequently from (2.4), (3.12), (3.15), and (3.18) through (3.23) that $\rho_1\to\rho_2$, $A = A_{12}\to 1$, $\gamma_{1,2}\to 1$, $\Lambda(\gamma_{1,2})\to\pm 1$, $\chi_1(z;1) = -\mathrm{i}$, $\chi_2(z;1) = 1$, and, hence, $v_1(z) = v_2(z)\equiv V_0$, as it should be.

Remark 3. The limit situations $\Delta\to 1$ and $\Delta\to -1$ take place as $\rho_1\to\infty$ or $\rho_2\to 0$ and $\rho_2\to\infty$ or $\rho_1\to 0$, respectively. Then $\mathcal{L}$ becomes a streamline for the complex potential w_1 (w_2) if $\rho_2 = \infty$ ($\rho_1 = \infty$) and an equipotential line for both w_1, w_2 if $\rho_2 = 0$ or $\rho_1 = 0$. The corresponding complex potential could be easily found for such situations by using an appropriate conformal mapping. For example, if $\rho_1 = \infty$, then the medium S_1 is nonconductive. Hence, $v_1(z)\equiv 0$, and $\mathcal{L}$ is a streamline for flow in S_2. This means that $\psi(x,y)\equiv c = \text{const}$, $(x,y)\in\mathcal{L}$, for a corresponding complex potential $w_2(z) = \varphi(x,y) + \mathrm{i}\,\psi(x,y)$ ($w_2'(z) = v_2(z)$). It is clear that the required complex potential $w_2(z)$ could be obtained by a conformal mapping of the right component of S_2 onto the half-plane $\{w\colon \Im w > c\}$ under the condition $w_2(\infty) = \infty$. This mapping is given by the function

$$w_2(z) = \mathrm{i}\, c_1\left[\left(z+\sqrt{z^2-1}\right)^{1/\alpha} + \left(z-\sqrt{z^2-1}\right)^{1/\alpha}\right]/2 + \mathrm{i}\, c\,.$$

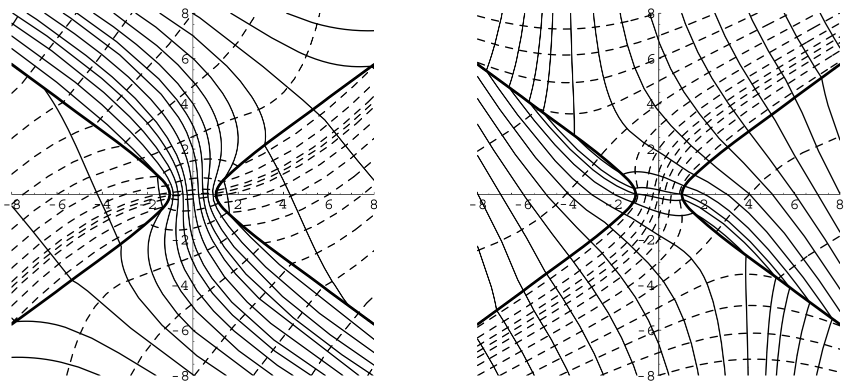

Figure 3. Left panel: $\rho_1 = 0.1,\ \rho_2 = 1,\ c_1 = 0.6,\ c_2 = 18$; right panel: $\rho_2 = 0.1,\ \rho_1 = 1,\ c_1 = 4,\ c_2 = 0.3$.

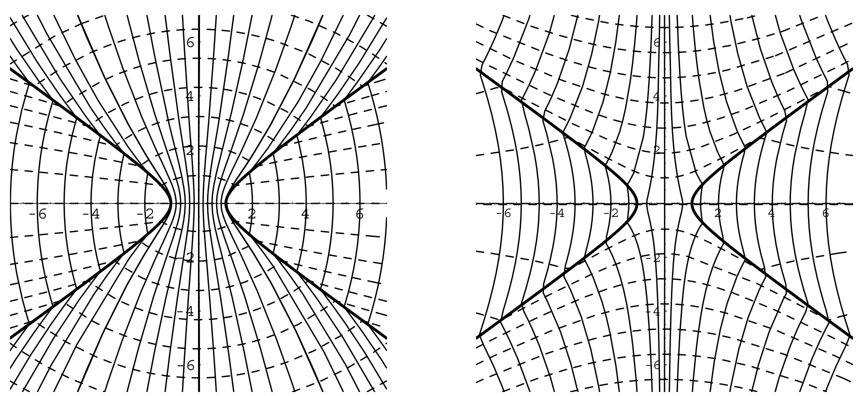

Figure 4. $c_1 = 0$ and $\rho_1 = 0.1$, $\rho_2 = 1$ at the left panel; $\rho_2 = 0.1$, $\rho_1 = 1$ at the right panel.

The derivative of this complex potential yields

$$v_2(z) = \mathrm{i}\, c_1 \frac{\left(z + \sqrt{z^2-1}\right)^{1/\alpha} - \left(z - \sqrt{z^2-1}\right)^{1/\alpha}}{2\sqrt{z^2-1}}.$$

Note that one can get the same complex velocity $v_2(z)$ as the corresponding limit case on the basis of Theorem 1. The last solution has the order $1/\alpha - 1$ at infinity. This order is less than 1 if and only if $1/2 < \alpha < 1$.

Figures 3–5 show the equipotentials (dashed) and streamlines corresponding to the velocity field $v(z)$ defined in (3.21).

4 SOLUTION OF THE PROBLEM ON HYPERBOLIC INCLUSIONS FOR THE CASE OF COMPLEX COEFFICIENTS (2.2)

In this case, problem (2.4) is solved by the following theorem about two-phase structures proved in [9].

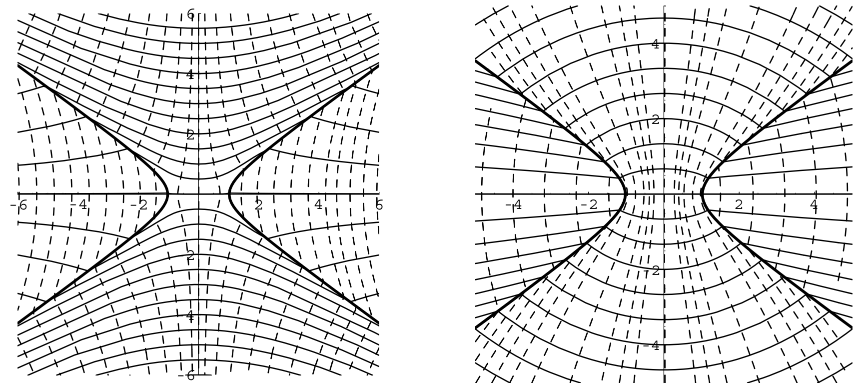

Figure 5. $c_2 = 0$ and $\rho_1 = 0.1$, $\rho_2 = 1$ at the left panel; $\rho_2 = 0.1$, $\rho_1 = 1$ at the right panel.

Theorem 2. *If A and B are arbitrary complex numbers satisfying the condition $|A| > |B|$ and if $v(z; |A|, |B|)$ is a solution of problem* (2.4) *with real coefficients $|A|$, $|B|$, then the function*

$$v(z; A, B) = \begin{cases} \sqrt{AB}v_1(z; |A|, |B|), & z \in S_1, \\ \sqrt{\overline{A}B}v_2(z; |A|, |B|), & z \in S_2, \end{cases}$$

solves problem (2.4) *with complex coefficients A, B.*

The last theorem allows one to easily write down a general solution of problem (2.4) in the case of complex coefficients (2.2) by means of Theorem 1. We leave this for the interested reader.

5 CONCLUSION

Using classical methods of complex analysis, an explicit analytical solution of the conjugation problem of flows at the hyperbolic border between two dissimilar heterogeneous media is derived.

It is interesting to point out that the general solution of a problem about one hyperbolic inclusion, found in [9] in the same class of function as in the present paper, was represented by a linear combination with arbitrary coefficients of three particular linear independent solutions. It was not quite clear, from a physical point of view, by what conditions one has to define these three coefficients. In this paper, we have derived a general solution involving only two parameters, which are uniquely defined by condition (3.22).

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