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METRIC DIMENSIONS, GENERALIZED INTEGRATIONS, CAUCHY TRANSFORM, AND RIEMANN BOUNDARY-VALUE PROBLEM ON NONRECTIFIABLE ARCS

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Abstract. We consider a nonrectifiable Jordan arc Γ on the complex plane $\mathbb{C}$ with endpoints a_1 and a_2. The Riemann boundary-value problem on this arc is the problem of finding a function $\Phi(z)$ holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ satisfying the equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\},$$

where $\Phi^\pm(t)$ are the limit values of $\Phi(z)$ at a point $t \in \Gamma \setminus \{a_1, a_2\}$ from the left and from the right, respectively. We introduce certain distributions with supports on nonrectifiable arc Γ that generalize the operation of weighted integration along this arc. Then we consider boundary behavior of the Cauchy transforms of these distributions, i.e., their convolutions with $(2\pi i z)^{-1}$. As a result, we obtain a description of solutions of the Riemann boundary-value problem in terms of a new version of the metric dimension of the arc Γ, the so-called approximation dimension. It characterizes the rate of best approximation of Γ by polygonal lines.

CONTENTS

Introduction

Let Γ be a Jordan arc on the complex plane $\mathbb{C}$ with endpoints a_1 and a_2. We consider the Riemann boundary-value problem on this arc, i.e., the problem of finding a function $\Phi(z)$ holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ satisfying the equalities

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}, \tag{1}$$

and $\Phi(\infty) = 0$ and certain restrictions on its behavior at the points a_1 and a_2. Here $\Phi^\pm(t)$ are the limit values of $\Phi(z)$ at a point $t \in \Gamma \setminus \{a_1, a_2\}$ from the left and right, respectively, and functions G and g are given. In the simplest case where $G \equiv 1$, the Riemann boundary-value problem turns into the so-called jump problem

$$\Phi^+(t) - \Phi^-(t) = g(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{2}$$

This boundary-value problem has a long history and a lot of applications, both traditional and new. The reader can find classic results on the problem in the famous monographs [5, 21]. These results are

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based on the assumption that the arc Γ is piecewise-smooth and rectifiable. A solution of the jump problem under this assumption is the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{g(t)dt}{t-z}, \tag{3}$$

and the Riemann boundary problem reduces to the jump problem by means of factorization. The same monographs [5, 21] contain information on the history of the Riemann boundary-value problem, related subjects, and traditional applications. Its new applications are connected with the theory of random matrices, nonclassical estimates for orthogonal polynomials, etc. (see, e.g., [2, 19]).

In the present work, we examine this problem for nonrectifiable arcs. The first results in this field are obtained by the author in the 1980s (see [10–12]). According to these results, problems (1) and (2) are solvable under the restriction

$$\nu > \frac{1}{2} \operatorname{dmb} \Gamma, \tag{4}$$

where ν is the Hölder exponent of the boundary data $g(t)$ and $G(t)$, and $\operatorname{dmb}\Gamma$ is the so-called box dimension of the set Γ (see the next section). Solutions cannot be represented by the Cauchy integral (3) because the integral over a nonrectifiable curve is not defined.

Recently, the author has introduced two new versions of the metric dimension of Γ, the so-called approximation dimension and refined metric dimension (see [15, 17]). It is proved that the replacement of the box dimension in (4) by one of these new characteristics improves this solvability condition. In addition, a certain generalization of the Cauchy integral is obtained which enables us to represent solutions of the Riemann boundary-value problem.

The results of articles [15, 17] are obtained for closed Jordan curves, and this assumption is essential. In the present paper, we prove their analogs for open Jordan arcs. In Sec. 1, we introduce the notions of approximation dimension and refined metric dimension for a nonrectifiable arc Γ. These dimensions characterizes the rate of its best approximations by polygonal lines. In Sec. 2, we consider certain distributions with supports on a nonrectifiable arc Γ, which generalize the operation of weighted integration along this arc. Then we study in Sec. 3 the boundary behavior of the Cauchy transforms of these distributions, i.e., their convolutions with $(2\pi i z)^{-1}$. As a result, we obtain in Secs. 4 and 5 a description of solutions of the Riemann boundary-value problem in terms of a new versions of metric dimension of the arc Γ.

1. Metric Dimensions

1.1. Hausdorff dimension and box dimension. We recall the notions of well-known metric dimensions, namely, the Hausdorff dimension and the box dimension.

Let Γ be a compact set on the complex plane $\mathbb{C}$. We fix $\varepsilon > 0$ and consider the family $T(\Gamma, \varepsilon)$ of all finite coverings $\tau = \{K_1, K_2, \dots, K_n\}$ of the set Γ by disks[1] K_j of radii $r_j \le \varepsilon$, $j = 1, 2, \dots, n$, where $n = \#(\tau)$ is the number of disks in the covering τ. We set

$$N(\Gamma, \varepsilon) := \inf \Big\{ \#(\tau) : \tau \in T(\Gamma, \varepsilon) \Big\},$$

$$H_p(\tau) := \sum_{j=1}^{\#(\tau)} r_j^p, \quad H_p(\Gamma, \varepsilon) := \inf \Big\{ H_p(\tau) : \tau \in T(\Gamma, \varepsilon) \Big\}.$$

The last function is monotone with respect to ε. Hence there exists

$$\lim_{\varepsilon \to 0} H_p(\Gamma, \varepsilon) := \mathfrak{h}_p(\Gamma)$$

[1] If Γ is a compact set in a metric space, then we consider its coverings by balls of this space.

(perhaps, it is infinite). The Hausdorff dimension of the set Γ is defined as

$$\operatorname{dmh}\Gamma := \inf\Big\{p : \mathfrak{h}_p(\Gamma) < \infty\Big\}$$

and its box dimension is defined as

$$\operatorname{dmb}\Gamma := \limsup_{\varepsilon\to 0}\frac{\log N(\Gamma,\varepsilon)}{-\log\varepsilon}$$

(see [4]). Obviously,

$$\operatorname{dmh}\Gamma \le \operatorname{dmb}\Gamma$$

for any set Γ. The dimension $\operatorname{dmb}\Gamma$ was introduced by A. N. Kolmogorov and V. M. Tikhomirov [18] as the upper metric dimension. The lower metric dimension

$$\operatorname{dmb}_*\Gamma := \liminf_{\varepsilon\to 0}\frac{\log N(\Gamma,\varepsilon)}{-\log\varepsilon}$$

was introduced earlier by L. S. Pontrjagin and L. G. Shnirelman [22]. They proved that

$$\inf\Big\{\operatorname{dmb}_* f(\Gamma) : f \text{ is an embedding of } \Gamma \text{ into a metric space}\Big\}$$

is equal to the topological dimension of Γ. In particular, this result implies the estimate

$$1 \le \operatorname{dmb}_*\Gamma \le \operatorname{dmb}\Gamma \le 2$$

for plane curves.

If Γ is a rectifiable curve, then

$$N(\Gamma,\varepsilon) \le [L\varepsilon^{-1}] + 1,$$

where $[\cdot]$ is the entire part and L is the length of Γ. Consequently,

$$\operatorname{dmh}\Gamma = \operatorname{dmb}\Gamma = \operatorname{dmh}_*\Gamma = 1.$$

The equality

$$\operatorname{dmh}\Gamma = \operatorname{dmb}\Gamma = \operatorname{dmh}_*\Gamma$$

is valid also for self-similar curves. For example, if Γ is the von Koch snowflake, then

$$\operatorname{dmh}\Gamma = \operatorname{dmb}\Gamma = \operatorname{dmh}_*\Gamma = \log_3 4.$$

But we can find plane sets Γ such that $\operatorname{dmh}\Gamma < \operatorname{dmb}\Gamma$.

Example 1. Let $\{a_k\}$ be a decreasing positive sequence such that $\sum\limits_{k=1}^{\infty} a_k = 1$ and the series $\sum\limits_{n=1}^{\infty} x_n$ diverges for $x_n = \sum\limits_{k=n}^{\infty} a_k$. We consider vertical segments

$$\sigma_n := \Big\{z = x_n + iy : 0 \le y \le x_n\Big\}$$

and their union

$$\sigma := \overline{\bigcup_{n\ge 1}\sigma_n}.$$

Let us divide the plane into squares with side $\varepsilon > 0$ and denote by $N^\diamond(A,\varepsilon)$ the number of squares intersecting a set $A \subset \mathbb{C}$. As is known,

$$N(A,\varepsilon) \asymp N^\diamond(A,\varepsilon)$$

for any compact set A, and we can replace N by $N^\diamond$ in the definition of the box dimension. We determine a number $n(\varepsilon)$ by the relation

$$a_{n(\varepsilon)+1} \le \varepsilon < a_{n(\varepsilon)}.$$

Then all segments with numbers $n \ge n(\varepsilon)$ are covered by N_1 squares filling the lower half of the square $[0, x_{n(\varepsilon)}] \times [0, x_{n(\varepsilon)}]$ under its diagonal. Hence

$$N_1 \asymp \varepsilon^{-2} x_{n(\varepsilon)}^2.$$

The remaining segments σ_k, $k = 1, 2, \dots, n(\varepsilon)-1$, are covered by N_2 squares, and any square intersects only one segment. Whence,

$$N_2 \asymp \varepsilon^{-1} \sum_{k=1}^{n(\varepsilon)-1} x_k$$

and

$$N^{\diamond}(\sigma, \varepsilon) \asymp \varepsilon^{-2} x_{n(\varepsilon)}^2 + \varepsilon^{-1} \sum_{k=1}^{n(\varepsilon)-1} x_k.$$

This relation enables us to evaluate $\operatorname{dmb} \sigma$ for a number of sequences $\{a_k\}$. In particular, for $x_n = 1/n^{\alpha}$, $0 < \alpha < 1$, we have

$$\operatorname{dmb} \sigma = \operatorname{dmb}_* \sigma = \frac{2}{1+\alpha}.$$

Now let us fix $p > 1$, $\varepsilon > 0$. The set

$$\sigma \setminus \{z : |z| < \varepsilon\}$$

is the union of a finite family of finite segments. Hence, this set has a covering τ' by disks of certain radius $\varepsilon' < \varepsilon$ such that $H_p(\tau') < \varepsilon$. Whence, the whole set σ has a covering

$$\tau := \tau' \cup \{z : |z| < \varepsilon\}$$

such that

$$H_p(\tau) < 2\varepsilon.$$

Thus, $\operatorname{dmh} \sigma = 1$, and we obtain an example of a set such that its Hausdorff dimension is strictly less than its box dimension.

Example 2. We fix $\beta \ge 1$ and consider the rectangles

$$R_n := \left\{ z = x + iy : x_n - \frac{a_n^{\beta}}{2} < x < x_n,\ 0 \le y < x_n \right\}, \quad n = 1, 2, \dots,$$

where x_n and a_n are the same values as in Example 1. Let $R := \bigcup_{n \ge 1} R_n$. Then the lower bound of the domain

$$D := \left\{ z = x + iy : 0 < x < 1,\ 0 < y < 1 \right\} \setminus R$$

is a zigzag arc Γ^* with endpoints 0 and 1. Obviously,

$$\operatorname{dmh} \Gamma^* = 1 < \frac{2}{1+\alpha} = \operatorname{dmb} \Gamma^* = \operatorname{dmb}_* \Gamma^*,$$

i.e., the Hausdorff dimension of the arc Γ^* is also less than its box dimension.

Example 3. Let us consider the spiral arc

$$S := \left\{ z = r \exp i\theta(r),\ 0 < r \le 1 \right\},$$

where $\theta(r)$ is a real decreasing continuous function defined for $0 < r \le 1$, $\lim_{r \to 0} \theta(r) = +\infty$. According to the monotonicity of the function $\theta(r)$, we conclude that the arc $S \setminus \{z : |z| < \varepsilon\}$ is rectifiable for any $\varepsilon > 0$. As above, this fact implies the equality $\operatorname{dmh} S = 1$. We consider the arcs

$$S_k := S \cap \left\{ z : (k+1)\varepsilon > |z| > k\varepsilon \right\}$$

and their coverings by disks of radius ε. We obtain the following bound:

$$N(S,\varepsilon) \asymp \varepsilon^{-2} r_{n(\varepsilon)}^2 + \varepsilon^{-1} \sum_{k=1}^{n(\varepsilon)-1} r_k,$$

where $r_k = k\varepsilon$ and the number $n(\varepsilon)$ is the greatest of the numbers n satisfying the inequality

$$\theta(r_n) - \theta(r_{n-1}) \geq 2\pi.$$

In particular, for $\theta(r) = r^{-p}$, $p \geq 1$, we obtain

$$\operatorname{dmb} S = \operatorname{dmb}_* S = \frac{2p}{p+1}.$$

1.2. Approximation dimensions and refined metric dimension. The approximation dimensions are introduced in [17] for closed curves. If Γ is a nonrectifiable closed curve bounding a finite domain D, then it is possible to exhaust D by polygons

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset D$$

such that the sequence of their boundaries Γ_n approximates Γ in a natural sense. We call this sequence an inner polygonal approximation. Then $\Delta_n := P_{n+1} \setminus P_n$ is either a doubly connected polygon or a finite family of simply connected polygons, $n = 1, 2, \ldots$. We use two metric characteristics of the rate of the polygonal approximation:

$$\lambda_n = \lambda(\Delta_n)$$

is sum of perimeters of all components of Δ_n and

$$w_n = w(\Delta_n)$$

is the diameter of the greatest disk $K \subset \Delta_n$. We denote by $E^+(\Gamma)$ the set of all values $p \geq 1$ such that

$$\sum_{n=1}^{\infty} \lambda_n w_n^{p-1} < \infty$$

for some inner polygonal approximation of the closed curve Γ and define the inner approximation dimension of this curve by the equality

$$\operatorname{dma}^+ \Gamma := \inf E^+(\Gamma).$$

The definition of the outer approximation dimension is similar.

Unlike the dimensions from the previous section that characterize any compact set, the approximation dimensions can characterize only closed curves. Below we offer a definition of approximation dimensions for plane arcs.

If an arc Γ curls in spiral at its endpoint (like the arc S from Example 3), then we cannot approximate it by polygonal lines with the same endpoints without intersections with Γ. Here we exclude such arcs from our consideration.

We say that a sequence of polygonal lines $G = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots\}$ is a polygonal approximation of an arc Γ from the left (from the right) if

(1) Γ_n begins at the point a_1, ends at the point a_2, and has no other common points with the arc Γ;
(2) the union $\Gamma \cup \Gamma_n$ bounds a domain D_n such that $D_{n+1} \subset D_n$ for any n;
(3) the direction of Γ is positive on ∂D_n for the approximation from the left and negative for the approximation from the right, $n = 1, 2, \ldots$;
(4) $\bigcap_{n=1}^{\infty} \overline{D_n} = \Gamma$.

If an arc Γ has a polygonal approximation, it is called a PA-arc. Obviously, the left and right approximations exist (or do not exist) simultaneously. PA-arcs have no spiral curls at their ends. The sum

$$M_d(G) = \sum_{n=1}^{\infty} \lambda(\Delta_n) w^{d-1}(\Delta_n) \tag{5}$$

is called the d-mass of the approximation G.

Definition 1. Let $E^+(\Gamma)$ (respectively, $E^-(\Gamma)$) be the set of all positive numbers d such that a PA-arc Γ has a polygonal approximation from the left (respectively, from the right) with a finite d-mass. Then the values

$$\operatorname{dma}^+ \Gamma := \inf E^+(\Gamma), \quad \operatorname{dma}^- \Gamma := \inf E^-(\Gamma)$$

are the left and right approximation dimensions of this arc.

The approximation dimensions of closed curves admit a definition in terms of polygonal decompositions of the domain D and its complement $\mathbb{C} \setminus \overline{D}$. This description is also valid for arcs: the polygons $\Delta_1, \Delta_2, \ldots$ compose a decomposition of the domain D_1. Therefore, we define the d-mass of the polygonal decomposition $\{\Delta_1, \Delta_2, \ldots\}$ by formula (5), and consider the sets $A^{\pm}(\Gamma)$ consisting of all positive numbers d such that there exists a domain D_1 satisfying the following conditions:

(1) the boundary ∂D_1 consists of Γ and a polygonal line (i.e., Γ is a PA-arc);
(2) the direction of Γ is the positive direction on ∂D_1 for the set $A^+(\Gamma)$ and negative for the set $A^-(\Gamma)$;
(3) the domain D_1 admits a polygonal decomposition with a finite d-sum.

Obviously, we can replace $E^{\pm}(\Gamma)$ by $A^{\pm}(\Gamma)$ in Definition 1. The dimensions obtained coincide with the approximation dimensions of an arc defined above.

The polygonal decomposition is a special case of a polygonal chain. A polygonal chain is a sequence of pairs $\{\Delta_n, s_n\}$, where Δ_n is a closed polygon and s_n is an integer number. A chain

$$C = \Big\{\{\Delta_1, s_1\},\ \{\Delta_2, s_2\},\ \ldots\Big\}$$

is said to be reducible if $s_1 = 1$, $s_n = \pm 1$ for $n = 2, 3, \ldots$, and the polygons Δ_1 and Δ_2 have a common side and satisfy the following restriction: for $s_2 = +1$ these polygons have no common interior points and for $s_2 = -1$ we have $\Delta_2 \subset \Delta_1$. We denote by $\Delta_1 + s_2\Delta_2$ the polygon congruent to $\Delta_1 \cup \Delta_2$ for $s_2 = +1$ and to $\overline{\Delta_1 \setminus \Delta_2}$ for $s_2 = -1$. The mapping $\mathfrak{G}$ is defined on the set of reducible chains by the equation

$$\mathfrak{G}(C) = \Big\{\{\Delta_1 + s_2\Delta_2, +1\},\ \{\Delta_3, s_3\},\ \ldots\Big\}.$$

If the chain $C_2 := \mathfrak{G}(C)$ is reducible, then we set $C_3 := \mathfrak{G}(C_2)$, and so on. If all chains $C \equiv C_1, C_2, C_3, \ldots$ are reducible and

$$\overline{\sum_{n=1}^{\infty} s_n \Delta_n} = \overline{D},$$

then we say that the chain C represents the domain D. The d-mass of the chain is defined by the same formula (5).

Definition 2. Let $\mathfrak{E}(\Gamma)$ be the set of all positive numbers d such that a PA-arc Γ is a part of the boundary of a domain D_1 representable by a polygonal chain with finite d-mass, and the full boundary of this domain is the union of Γ and a polygonal line. Then the number

$$\operatorname{dmr} \Gamma := \inf \mathfrak{E}(\Gamma)$$

is called the refined metric dimension of this arc.

The refined metric dimension is introduced for closed curves in [15]. Let us note that it is independent of the direction of Γ. Indeed, we have $\partial D_1 = \Gamma \cup \Gamma_1$, where Γ_1 is a polygonal line, and the direction of Γ coincides with the positive direction on ∂D_1. Then there exists a polygonal line Γ' with the same endpoints lying on the other side of the arc Γ. The polygonal lines Γ_1 and Γ' together bound a polygon Δ_0 and the arc Γ divides it into two domains. If the domain D_1 is representable by a chain

$$C = \Big\{\{\Delta_1, s_1\},\ \{\Delta_2, s_2\},\ \dots\Big\},$$

then its complement $\Delta_0 \setminus D_1$ is representable by the chain

$$C' = \Big\{(\Delta_0, +1),\ \{\Delta_1, -s_1\},\ \{\Delta_2, -s_2\},\ \dots\Big\},$$

and the d-masses of these chains differ by one term. Thus, the refined metric dimensions of Γ from the right and from the left coincide.

Theorem 1. *Any PA-arc Γ satisfies the inequalities*

$$1 \le \operatorname{dmr}\Gamma \le \operatorname{dma}^{\pm}\Gamma \le \operatorname{dmb}\Gamma \le 2. \tag{6}$$

For any number $d \in (1,2)$, there exist PA-arcs $\Gamma^*_{1,2,3}$ of box dimension d such that

$$\operatorname{dma}^{+}\Gamma^*_1 < d, \quad \operatorname{dma}^{-}\Gamma^*_2 < d, \quad \operatorname{dmr}\Gamma^*_3 < d.$$

Proof. Any polygonal decomposition is a polygonal chain with $s_n \equiv 1$. Hence

$$\operatorname{dmr}\Gamma \le \operatorname{dma}^{\pm}\Gamma.$$

The inequalities

$$1 \le \operatorname{dmr}\Gamma \le \operatorname{dmb}\Gamma \le 2$$

are proved in [15] and the proof is valid for arcs. The inequality

$$\operatorname{dma}^{\pm}\Gamma \le \operatorname{dmb}\Gamma$$

can be proved similarly. It remains to prove the existence of arcs $\Gamma^*_{1,2,3}$. Let us consider the arc Γ^* from Example 2 for $\alpha = 2d^{-1} - 1$. Then $\operatorname{dmb}\Gamma^* = d$. Let it be directed from the point 0 to the point 1. We set

$$Q = \Big\{z = x + iy : 0 < x < 1,\ -1 < y < 0\Big\}, \quad D_n := Q \cup \left(\bigcup_{k=1}^{n} R_k\right), \quad n = 1, 2, \dots.$$

We consider polygonal lines

$$\Gamma_{-1} = [0, -i] \cup [-i, 1 - i] \cup [1 - i, 1], \quad \Gamma_0 = [0, 1],$$

and Γ_n is the upper boundary of the domain D_n with endpoints 0 and 1 for $n \ge 1$. We consider these lines as directed from 0 to 1. Then the sequence $\{\Gamma_{-1}, \Gamma_0, \Gamma_1, \Gamma_2, \dots$ is a polygonal approximation of Γ^* from the right. Obviously, for $n \ge 1$, the difference Δ_n is the rectangle R_n and, consequently,

$$\lambda_n \asymp x_n, \quad w_n \asymp a_n^{\beta}$$

(we use the notation of Example 2). Thus, if $x_n = 1/n^{\alpha}$, $\alpha = 2d^{-1} - 1$, $d \in (1,2)$, then the p-mass of this approximation is finite for

$$p > 1 + \frac{1-\alpha}{\beta(1+\alpha)} = 1 + \beta^{-1}(d-1),$$

and we have

$$\operatorname{dma}^{-}\Gamma^* \le 1 + \frac{d-1}{\beta},$$

i.e., for $\beta > 1$ we set

$$\Gamma_2^* := \Gamma^*$$

and obtain

$$\operatorname{dma}^- \Gamma_2^* < d.$$

We take as the arc Γ_1^* the same arc Γ^*, but with the opposite direction, i.e., with starting point 1 and endpoint 0. Then the right polygonal approximations turns into left one, and

$$\operatorname{dma}^+ \Gamma_1^* \leq 1 + \frac{d-1}{\beta} < d$$

for $\beta > 1$.

Finally, inequality (6) enables us to choose an arc Γ^* with any direction in the capacity of the arc Γ_3^*. The theorem is proved. □

We do not find a polygonal approximation of the arc Γ_2^* from the left or of the arc Γ_1^* from the right with a finite p-mass for $p < d$. The author assumes that

$$\operatorname{dma}^+ \Gamma_2^* = \operatorname{dma}^- \Gamma_1^* = d.$$

We finish this section with another example.

Example 4. Let us consider the rectangles

$$R_n' := \left\{ z = x + iy : x_n - \frac{a_n^\beta}{2} < x < x_n,\ -x_n < y \leq 0 \right\}, \quad n = 1, 2, \ldots,$$

where x_n and a_n are the same values as in Example 2. These rectangles are symmetric to the rectangles R_n from Example 2 with respect to the real axis. We set

$$D := Q \cup \left(\bigcup_{k=1}^{\infty} R_{2k}' \right) \setminus \left(\bigcup_{k=1}^{\infty} R_{2k-1} \right),$$

where $Q = \Big\{ z = x + iy : 0 < x < 1,\ 0 < y < 1 \Big\}$. Then the lower bound Z of the domain D is a two-side zigzag arc with endpoints 0 and 1. The domain D is representable by the chain

$$\Big\{ (Q, +1),\ (R_1, -1),\ (R_2', +1),\ (R_3, -1),\ (R_4', +1),\ \ldots \Big\}.$$

A direct calculation of the mass of this chain shows that

$$\operatorname{dmr} Z \leq 1 + \frac{d-1}{\beta}.$$

We do not know a polygonal approximation of Z from any side such that its p-mass is finite for $p < d$. The author assumes that

$$\operatorname{dma}^+ Z = \operatorname{dma}^- Z = d,$$

i.e.,

$$\operatorname{dmr} Z < \operatorname{dma}^+ Z = \operatorname{dma}^- Z$$

for $\beta > 1$.

2. Integration over Nonrectifiable Arcs

A number of authors published papers on the construction and properties of an integral $\int_\Gamma f\,dz$ over a nonrectifiable curve Γ. Most of them (see [1, 6–8, 13, 14, 26]) developed in various ways the following idea, which goes back, apparently, to Whitney.[2]

Let Γ be a closed Jordan curve bounding a domain D. If it is rectifiable and $f(z)$ is continuously differentiable in $\overline{D}$, then by virtue of the Stokes formula

$$\int\limits_\Gamma f(z)\,dz = -\iint\limits_D \frac{\partial f}{\partial \overline{z}\,dz\,d\overline{z}}.$$

If Γ is not rectifiable, then the right-hand side of this equality can be used as the definition of the left-hand side. Here we modify this construction for Jordan arcs.

Let Γ be a PA-arc. Then we can find polygonal lines Γ' and Γ'' with the same starting and endpoints a_1 and a_2, lying from the left and from the right of the arc Γ, respectively. We obtain two domains D' and D'' such that

$$\partial D' = \Gamma \cup \Gamma', \quad \partial D'' = \Gamma \cup \Gamma'',$$

and a polygon P such that

$$\overline{P} = \overline{D' \cup D''}.$$

Let a function $F(z)$ be holomorphic in $P \setminus \Gamma$, have boundary values $F^{\pm}(t)$ from the left and from the right at any point $t \in \Gamma \setminus \{a_1, a_2\}$, and boundary values $F(t)$ on $\partial P \setminus \{a_1, a_2\}$. We assume that Γ has null square and $F(z)$ is integrable in neighborhoods of the points a_1 and a_2. As usually, we denote by C^∞ the set of all infinitely differentiable functions on the complex plane, and its subset $C_0^\infty(P)$ consists of functions with compact supports lying in a domain P. If the arc Γ is rectifiable, then for any $\omega \in C^\infty$ we have

$$\int\limits_\Gamma F^+(t)\omega(t)dt = \int\limits_{\Gamma'} F(t)\omega(t)dt - \iint\limits_{D'} F(z)\frac{\partial \omega}{\partial \overline{z}}\,dz\,d\overline{z},$$

$$\int\limits_\Gamma F^-(t)\omega(t)dt = \int\limits_{\Gamma''} F(t)\omega(t)dt + \iint\limits_{D''} F(z)\frac{\partial \omega}{\partial \overline{z}}\,dz\,d\overline{z},$$

$$\int\limits_\Gamma (F^+(t) - F^-(t))\omega(t)dt = \int\limits_{\partial P} F(t)\omega(t)dt - \iint\limits_P F(z)\frac{\partial \omega}{\partial \overline{z}}\,dz\,d\overline{z}.$$

We identify the function $F(\zeta)$ on the complex plane with the distribution

$$F : C_0^\infty \ni \omega \mapsto \iint F(\zeta)\omega(\zeta)d\zeta d\overline{\zeta}.$$

The previous consideration explains the following definition for integration over a nonrectifiable PA-arc.

Definition 3. Let P be a polygon such that $\Gamma \setminus \{a_1, a_2\} \subset P$. If a function F is holomorphic in $P \setminus \Gamma$, integrable in neighborhoods of the points a_1 and a_2, and have boundary values $F^{\pm}(t)$ from the left and from the right at any point $t \in \Gamma \setminus \{a_1, a_2\}$, then the distribution $\int_\Gamma [F]$ defined by the equality

$$\left\langle \int\limits_\Gamma [F], \omega \right\rangle := \langle \overline{\partial} F, \omega \rangle + \int\limits_{\partial P} F(t)\omega(t)dt \tag{7}$$

is the primary weighted integral over the arc Γ and $F^+(t) - F^-(t)$ is its weight.

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[2] Another approach to the construction of an integral over nonrectifiable curves is described in [16].

In what follows we write $\int\limits_\Gamma [F]\omega(t)dt$ instead of $\left\langle \int\limits_\Gamma [F], \omega \right\rangle$.

This distribution is independent of polygonal lines Γ' and Γ''. Indeed, let us consider other polygonal lines $\gamma' \subset D'$ and $\gamma'' \subset D''$ with the same endpoints. They bound a polygon $Q \subset P$, $\Gamma \subset \overline{Q}$. Then

$$\int\limits_{\partial P} F(t)\omega(t)dt - \iint\limits_P F(z)\frac{\partial\omega}{\partial\overline{z}}\,dz\,d\overline{z} = \int\limits_{\partial Q} F(t)\omega(t)dt - \iint\limits_Q F(z)\frac{\partial\omega}{\partial\overline{z}}\,dz\,d\overline{z}$$

by the Stokes formula.

Remark 1. The polygonal lines Γ' and Γ'' can be chosen so that their initial and final segments coincide. Removing these common segments from P, we obtain a new polygon P' containing the arc Γ inside. If F is holomorphic in P' and $\omega \in C_0^\infty(P')$, then

$$\int\limits_\Gamma [F]\omega(t)dt = -\iint\limits_P F(z)\frac{\partial\omega}{\partial\overline{z}}\,dz\,d\overline{z},$$

i.e.,

$$\int\limits_\Gamma [F] = \overline{\partial}F$$

on $C_0^\infty(P')$.

Let $\mathfrak{X}$ be a functional space such that $C^\infty(\mathbb{C})$ is dense in $\mathfrak{X}$ and $\mathfrak{X}C^\infty = \mathfrak{X}$, i.e., $f\omega \in \mathfrak{X}$ for any $\omega \in C^\infty, f \in \mathfrak{X}$. If a primary integration $\int[F]$ is continuous in $\mathfrak{X}$, then it is extendable up to a functional $\int[F]$ on $\mathfrak{X}$ and generates a family of distributions

$$\left\langle \int [F]f, \omega \right\rangle := \int [F]f(\zeta)\omega(\zeta)d\zeta, \quad f \in \mathfrak{X}. \tag{8}$$

We call them *integrations* and write $\int [F]f\omega d\zeta$ instead of $\left\langle \int [F]f, \omega \right\rangle$.

We define an appropriate space $\mathfrak{X}$ in terms of the Hölder condition

$$h_\nu(f, A) := \sup\left\{ \frac{|f(t') - f(t'')|}{|t' - t''|^\nu} : t', t'' \in A,\ t' \neq t'' \right\} < \infty, \tag{9}$$

where A is a compact set on the complex plane and $\nu \in (0, 1]$. Let $H_\nu(A)$ be the set of all functions f satisfying condition (9). It is a Banach space with norm

$$\|f\|_{C(A)} + h_\nu(f, A),$$

where

$$\|f\|_{C(A)} = \sup\Big\{ |f(\zeta)| : \zeta \in A \Big\}.$$

But C^∞ is not dense in this space. Introduce the notation

$$H^*(A, \nu) := \bigcup_{\mu > \nu} H_\mu(A).$$

We fix a sequence of exponents $\{\nu_j\}$ such that

$$1 > \nu_1 > \nu_2 > \dots > \nu_j > \nu_{j+1} > \dots$$

and

$$\lim_{j\to\infty} \nu_j = \nu.$$

Then the family of semi-norms $\{h_{\nu_j}(\cdot, A)\}$ together with the norm

$$\|f\|_{C(A)} := \sup\left\{|f(z)| : z \in A\right\}$$

turns $H^*(A,\nu)$ into a Fréchet space. Obviously, C^∞ is dense in $H^*(A,\nu)$ and

$$H^*(A,\nu)C^\infty = H^*(A,\nu).$$

Thus, we set

$$\mathfrak{X} = H^*(A,\nu).$$

Let us note that by the Whitney theorem (see, e.g., [23]) any function $f \in H_\nu(A)$ is extendable up to a function f^w satisfying the Hölder condition with the same exponent ν on the whole complex plane.

Let us study the continuity of primary integrations in $H^*(A,\nu)$.

Theorem 2. *Let $A = \overline{P}$, where a domain P contains a PA-arc Γ and* $\operatorname{dmr}\Gamma < 2$*. If a function F is holomorphic in $P\setminus\Gamma$, integrable with any degree $p > 1$ near points a_1 and a_2, and has boundary values from both sides on $\Gamma\setminus\{a_1, a_2\}$, then the primary integration is continuous in the space $H^*(A, \operatorname{dmr}\Gamma - 1)$.*

Proof. We fix values d and ν such that $\operatorname{dmr}\Gamma < d < 2$ and $1 > \nu > d - 1$. By the definition of the refined metric dimension, there exists a reducible chain

$$C = \Big\{\{\Delta_1, s_1\},\ \{\Delta_2, s_2\},\ \ldots\Big\}$$

with a finite d-mass, which represents a domain D, $\Gamma \subset \partial D$. Without loss of generality, we assume that P is a polygon, Γ divides it into two domains, D is one of these domains, and the direction of Γ coincides with the positive direction on ∂D. We set

$$\Gamma^* = \overline{\bigcup_{n\geq 1} \partial\Delta_n}.$$

Any function $\omega \in C^\infty$ satisfies the Hölder condition with any exponent $\mu \leq 1$. We restrict ω to Γ^*, apply the Whitney extension operator (see, e.g., [23]) to this restriction, and denote the obtained continuation by ω^*. By the well-known properties of the Whitney extension operator (see [23]), the function ω^* is defined on the whole complex plane, satisfies the Hölder condition with any exponent $\mu \leq 1$, and is equal to ω on the set Γ^*. In addition, it has partial derivatives of any order on $\mathbb{C} \setminus \Gamma^*$, and

$$|\nabla\omega^*(z)| \leq C h_\mu(\omega, A)\operatorname{dist}^{\mu-1}(z, \Gamma^*);$$

here and below we denote by C various constants. In particular,

$$|\nabla\omega^*(z)| \leq C h_1(\omega, A),$$

i.e., the first partial derivatives of ω^* are bounded. Obviously,

$$\overline{\partial}F = \overline{\partial}(F\chi^+) + \overline{\partial}(F\chi^-),$$

where $\chi^+(z)$ and $\chi^-(z)$ are characteristic functions of domains D and $P \setminus D$, respectively. We have

$$\begin{aligned}\left\langle \overline{\partial}F\chi^+, \omega\right\rangle = -\left\langle F\chi^+, \overline{\partial}\omega\right\rangle = -\iint\limits_D F(\zeta)\frac{\partial\omega}{\partial\overline{\zeta}}\, d\zeta\, d\overline{\zeta} \\ = -\sum_{n\geq 1} s_n \iint\limits_{\Delta_n} F(\zeta)\frac{\partial\omega}{\partial\overline{\zeta}}\, d\zeta\, d\overline{\zeta} = \sum_{n\geq 1} s_n \int\limits_{\partial\Delta_n} F(\zeta)\omega(\zeta) d\zeta.\end{aligned}$$

Thus,

$$\left\langle \overline{\partial} F \chi^+, \omega \right\rangle = -\sum_{n \geq 1} s_n \iint\limits_{\Delta_n} F(\zeta) \frac{\partial \omega^*}{\partial \overline{\zeta}} \, d\zeta \, d\overline{\zeta}.$$

Obviously, in the polygonal domain Δ_n the function ω^* is the Whitney continuation of the restriction of ω to $\partial \Delta_n$. Consequently, we can apply the following lemma from [15].

Lemma 1. *Let δ be a finite domain with rectifiable Jordan boundary γ, $f \in H_\nu(\gamma)$, and f^w be the Whitney continuation of f from γ. If $p < \dfrac{1}{1-\nu}$, then*

$$\iint\limits_{\delta} |\nabla f^w|^p \, dx \, dy \leq C h_\nu^p(f, \gamma) \lambda(\gamma) w^{1-p(1-\nu)}(\delta).$$

Here $\lambda(\delta)$ and $w(\delta)$ denote the perimeter of δ and the diameter of the maximal disk lying in δ. We fix a value p such that $d - 1 = 1 - p(1-\nu)$, i.e.,

$$p = \frac{2-d}{1-\nu}, \tag{10}$$

and obtain

$$\left| \left\langle \overline{\partial} F \chi^+, \omega \right\rangle \right| \leq C \|F\|_{L^q} S^{1/q} M_d^{1/p} h_\nu(\omega, A),$$

where S is the area of D, M_d is the d-mass of the representing chain, and $p^{-1} + q^{-1} = 1$. The value $\left\langle \overline{\partial} F \chi^-, \omega \right\rangle$ admits a similar bound. The inequality

$$\left| \int\limits_{\partial P} F(t) \omega(t) dt \right| \leq C \|\omega\|_{C(A)}$$

is obvious. Thus,

$$\left| \int\limits_{\Gamma} [F] \omega(t) dt \right| \leq C(h_\nu(\omega, A) + \|\omega\|_{C(A)}.$$

The proof is complete. $\square$

The primary integration $\int\limits_{\Gamma} [F]$ generates by formula (8) the family of integrations $\int\limits_{\Gamma} [F] f$, $f \in H^*(A, \operatorname{dmr} \Gamma - 1)$.

3. Cauchy Transform

Many authors in their recent publications (see, e.g., [20, 24, 25]) examined properties of the Cauchy transforms of various measures. If μ is a finite measure on the complex plane, then its Cauchy transform is the integral

$$\mathcal{C}\mu := \frac{1}{2\pi i} \int \frac{d\mu(\zeta)}{\zeta - z}.$$

In particular, if the support S of the measure μ is a rectifiable curve, $d\mu = f(\zeta) d\zeta$, where the function $f(\zeta)$ is integrable over the length of S, then we obtain the Cauchy integral

$$\mathcal{C}(f(\zeta) d\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

If φ is a distribution with compact support S on the complex plane, then its Cauchy transform is

$$\mathcal{C}\varphi := \left\langle \varphi, \frac{1}{2\pi i(\zeta - z)} \right\rangle,$$

where $z \notin S$, and φ is applied to the Cauchy kernel

$$E(\zeta - z) := \frac{1}{2\pi i(\zeta - z)}$$

as to a function of the variable ζ. One can consider it as the convolution $\varphi * E$, where $E = (2\pi i\zeta)^{-1}$.

Since E is the fundamental solution of the differential operator $\overline{\partial}$ (i.e., $\overline{\partial}E = \delta_0$, see [9]), $\overline{\partial}\mathcal{C}\varphi = \varphi$, and the function $\mathcal{C}\varphi(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus S$, it obviously vanishes at the point ∞.

Obviously, the support of the distribution $\int[F]f$ belongs to Γ. Therefore, we can apply it to the Cauchy kernel $\dfrac{1}{2\pi i(\zeta - z)}$ as a function of the variable ζ for $z \notin \Gamma$. Indeed, we apply this distribution to a function $\omega_z(\zeta) \in C^\infty$, which is equal to $\dfrac{1}{2\pi i(\zeta - z)}$ for $|\zeta - z| \geq \varepsilon$, where $0 < \varepsilon < \operatorname{dist}(z, \Gamma)$. We denote the Cauchy transform of the integration $\int_\Gamma[F]f$ by $\mathcal{C}[F]f$.

Theorem 3. *Let $A = \overline{P}$, where a domain P contains a Jordan arc Γ (perhaps, excluding its endpoints $a_{1,2}$). Let $F(z)$ be a holomorphic function in $P \setminus \Gamma$, integrable with any degree $p > 1$ near the points $a_{1,2}$, and having boundary values $F^\pm(t)$ at any point $t \in \Gamma \setminus \{a_1, a_2\}$. If*

$$f \in H^*(A, \operatorname{dmr}\Gamma/2),$$

then the Cauchy transform

$$\Phi(z) := \mathcal{C}[F]f(z)$$

has boundary values $\Phi^+(t)$ and $\Phi^-(t)$ from the left and right, respectively, at any point $t \in \Gamma \setminus \{a_1, a_2\}$, and

$$\Phi^+(t) - \Phi^-(t) = \big(F^+(t) - F^-(t)\big)f(t), \quad t \in \Gamma \setminus \{a_1, a_2\}. \tag{11}$$

Proof. Let Γ^* be the set from the proof of Theorem 2. We restrict the function $f \in H^*(A, \operatorname{dmr}\Gamma - 1)$ to this set, extend this restriction by means of the Whitney extension operator, and multiply the extension by a smooth compactly supported function which is equal to 1 on A. We denote the obtained function by f^*; it has a compact support and satisfies all properties of the Whitney extension. Then we conclude, by representation (7) and the continuity of the Whitney extension operator, that

$$\int\limits_\Gamma [F]f(\zeta)d\zeta = \int\limits_{\partial P} F(\zeta)f^*(\zeta)d\zeta - \iint\limits_P F(\zeta)\frac{\partial f^*}{\partial\overline{\zeta}}\,d\zeta\,d\overline{\zeta}. \tag{12}$$

This equality yields the representation

$$\mathcal{C}[F]f(z) = \frac{1}{2\pi i}\int\limits_{\partial P}\frac{F(t)f^*(t)dt}{t - z} + \chi_P(z)F(z)f^*(z) - \frac{1}{2\pi i}\iint\limits_P \frac{\partial f^*}{\partial\overline{\zeta}}\frac{F(\zeta)\,d\zeta\,d\overline{\zeta}}{\zeta - z}, \tag{13}$$

where $\chi_P(z)$ is the characteristic function of the domain P. The last terms of the representation are well-known integral operators (see, e.g., [9]). If

$$\frac{\partial f^*}{\partial\overline{\zeta}} \in L^p_{\text{loc}}$$

for $p > 2$, then this term is continuous on the whole complex plane and satisfy there the Hölder condition with exponent $1 - 2p^{-1}$. The exponent p is determined by Eq. (10); it exceeds 2 for $\nu > d/2$. The proof is complete. □

Remark 2. According to Remark 1, we obtain the representation

$$\mathcal{C}[F]f(z) = F(z)f^*(z) - \frac{1}{2\pi i}\iint\limits_{\mathbb{C}} \frac{\partial f^*}{\partial \overline{\zeta}} \frac{F(\zeta)\,d\zeta\,d\overline{\zeta}}{\zeta - z} \tag{14}$$

if F is holomorphic in $\mathbb{C} \setminus \Gamma$ and $F(\infty) = 0$.

The above-mentioned property of the last term of representation (13) means that on the whole complex plane it satisfies the Hölder condition with any exponent less than

$$\mu(\nu, d) := \frac{2\nu - d}{2 - d}, \tag{15}$$

where $d = \mathrm{dmr}\,\Gamma$ and $f \in H_\nu(A)$.

A point $t \in \Gamma \setminus \{a_1, a_2\}$ has a neighborhood $V(t)$ such that $V(t) \setminus \Gamma$ consists of two connected components $V^+(t)$ and $V^-(t)$. We say that a function $F(z)$ belongs to the class $\mathcal{H}_\mu(\Gamma)$ if

$$F(z) \in H_\mu(\overline{V^+(t)}), \quad F(z) \in H_\mu(\overline{V^-(t)}) \quad \forall t \in \Gamma \setminus \{a_1, a_2\}.$$

Obviously, the following assertion is valid.

Corollary 1. *Let*

$$f \in H_\nu(A), \quad \nu > \frac{\mathrm{dmr}\,\Gamma}{2}, \quad F(z) \in \mathcal{H}_\mu(\Gamma)$$

for $\mu = \mu(\nu, \mathrm{dmr}\,\Gamma)$. *Then under the assumptions of Theorem* 3 *the function*

$$\Phi(z) = \mathcal{C}[F]f(z)$$

belongs to the class $\mathcal{H}_\rho(\Gamma)$ *for any* $\rho < \mu(\nu, \mathrm{dmr}\,\Gamma)$.

4. Jump Problem

4.1. ***PA*-Arcs.** We consider the function

$$K_\Gamma(z) := \frac{1}{2\pi i} \log \frac{z - a_2}{z - a_1},$$

where the single-valued branch of the logarithm is determined by means of the cut along Γ and the condition $K_\Gamma(\infty) = 0$. Obviously,

$$K_\Gamma^+(t) - K_\Gamma^-(t) = 1, \quad t \in \Gamma \setminus \{a_1, a_2\},$$

i.e., K_Γ is a solution of the jump problem with unit jump, and the primary integration $\int\limits_\Gamma [K_\Gamma]$ has unit weight. If an arc Γ' has the same starting point and endpoint and has no other common points with Γ, then

$$K_\Gamma(z) - K_{\Gamma'}(z) = \pm\chi(z),$$

where $\chi(z)$ is the characteristic function of the domain bounded by $\Gamma \cup \Gamma'$. Therefore, if Γ is a PA-arc, then

$$K_\Gamma(z) = \frac{(-1)^j}{2\pi i} \log|z - a_j| + O(1), \quad j = 1, 2. \tag{16}$$

In addition,

$$K_\Gamma \in \mathcal{H}_1(\Gamma).$$

Thus, the Cauchy transform $\mathcal{C}[K_\Gamma]f(z)$ is a solution of the jump problem (2) for $f \in H_\nu(A)$, $\nu > \mathrm{dmr}\,\Gamma/2$, and belongs to $\mathcal{C}_\rho(\Gamma)$ for any $\rho < \mu(\nu, \mathrm{dmr}\,\Gamma)$. It is an immediate generalization of the Cauchy integral for nonrectifiable arcs and has logarithmic singularities at its endpoints.

Let us examine the uniqueness of a solution. In general, a solution of the jump problem on a nonrectifiable arc is not unique. If the Hausdorff dimension $\mathrm{dmh}\,\Gamma$ of the arc Γ exceeds 1, then there exist nontrivial functions holomorphic in $\overline{\mathbf{C}} \setminus \Gamma$ with null jump on Γ (see, e.g., [3]). But if $\mu > \operatorname{dmh}\Gamma - 1$, then any function $\Phi(z)$ satisfying the Hölder condition with exponent μ in a domain $D \supset \Gamma$ and holomorphic in $D \setminus \Gamma$ is holomorphic in D (this fact was proved by E. P. Dolzhenko; see [3]). We say that a function $\Phi(z)$ holomorphic in $\mathbf{C} \setminus \Gamma$ satisfies the *Hausdorff–Dolzhenko condition* (HD-condition) if

$$\Phi \in \mathcal{H}_\mu(\Gamma) \quad \text{for } \mu > \operatorname{dmh}\Gamma - 1.$$

If a solution of the jump problem (or the Riemann boundary-value problem) satisfies the HD-condition, then we call it an HD-solution. If an HD-solution of the jump problem in the class of functions with weak singularities at endpoints exists, then it is unique. According to Theorem 3, the Cauchy transform with $F = K_\Gamma$ gives an HD-solution of the jump problem if $f \in H_\nu(\Gamma)$ and

$$\operatorname{dmh}\Gamma - 1 < \frac{2\nu - \operatorname{dmr}\Gamma}{2 - \operatorname{dmr}\Gamma}. \tag{17}$$

Introduce the notation

$$\operatorname{dmu}\Gamma := \operatorname{dmr}\Gamma + (2 - \operatorname{dmr}\Gamma)(\operatorname{dmh}\Gamma - 1).$$

This value lies between 1 and 2. We consider it as a metric characteristic of dimensional type and call it the uniqueness dimension. Inequality (17) is equivalent to

$$\nu > \frac{\operatorname{dmu}\Gamma}{2}.$$

Thus, the following theorem hold.

Theorem 4. *If Γ is a PA-arc and $f \in H^*(\Gamma, \operatorname{dmr}\Gamma/2)$, then the jump problem* (2) *has a solution in the class of functions satisfying the condition*

$$\Phi(z) = O(|z - a_j|^{-\alpha}), \quad j = 1, 2, \quad \alpha = \alpha(\Phi) \in [0, 1). \tag{18}$$

This solution is representable by $\mathcal{C}[K_\Gamma]f^w(z)$, where f^w is the Whitney extension of f from Γ. In addition, if $f \in H^*(\Gamma, \operatorname{dmu}\Gamma/2)$, then $\mathcal{C}[K_\Gamma]f^w(z)$ is a unique HD-solution of the problem.

Let us also note the following solvability condition for the jump problem.

Theorem 5. *If $f_1 \in H^*(\Gamma, \operatorname{dmr}\Gamma/2)$ and f_2 is a function defined on Γ such that the jump problem* (2) *with $f = f_2$ has a solution integrable near $a_{1,2}$ with any degree $p > 1$, then this problem for $f = f_1 f_2$ has a solution with the same behavior near endpoints.*

Proof. Let F be the mentioned solution of the jump problem for $f = f_2$. Then by Theorem 3, the Cauchy transform $\mathcal{C}[F]f_1^w(z)$ is a solution of this problem for $f = f_1 f_2$. □

Theorem 4 implies certain lower bounds for the refined metric dimension. Let us consider the arc Γ^* from Example 2. We define the real, piecewise-linear function $f(x)$ on the segment $[0, 1]$ by the equalities

$$f(x_n) = 0, \quad f\left(x_n - \frac{a_n^\beta}{2}\right) = a^{\nu\beta}$$

and set

$$f(x + iy) := f(x), \quad x \in [0, 1].$$

This function satisfies the Hölder condition with exponent ν. Immediate estimates of the Cauchy integral over Γ^* with density f show that the jump problem has no solutions in the desired class for $\nu \leq d/2\beta$ (see a similar calculation in [15]). By Theorem 4 we conclude that

$$\operatorname{dmr}\Gamma^* \geq \frac{d}{\beta}.$$

The same bound is valid for the zigzag arc Z from Example 4.

4.2. General case. We consider here arcs with spiral curls at endpoints. The upper metric dimension is not defined for these arcs because they are not PA-arcs. But any arc is the union of a family of PA-arcs. We say that a point $t \in \Gamma$ is linearly attainable if there exists a line segment ι with endpoint t and without other common points with Γ. Let the set $\mathfrak{T}(\Gamma)$ consist of all linearly attainable points of the arc Γ.

Lemma 2. *The set $\mathfrak{T}(\Gamma)$ is dense in Γ.*

Proof. We consider two arbitrary points $a, b \in \Gamma$, $a \neq b$. Let $\gamma \subset \Gamma$ be a part of Γ connecting these points. Then the boundary of the closed convex hull of γ consists of extreme points of γ and segments that connect them. Obviously, it contains at least one point of γ, which differs from a and b. This point is linearly attainable. Hence, any subarc of Γ contains at least one point of the set $\mathfrak{T}(\Gamma)$. Thus, this set is dense in Γ. □

If $a_{1,2} \in \mathfrak{T}(\Gamma)$, then Γ is a PA-arc. If $a_j \notin \mathfrak{T}(\Gamma)$, then by Lemma 2 we can find a sequence of linearly attainable points of Γ converging to a_j, $j = 1, 2$. These points divide Γ into P-arcs. Thus, any Jordan arc is either a PA-arc or the union of a countable family of PA-arcs. The first case is considered in the preceding subsection. Let us prove the solvability of the jump problem in the second case.

Theorem 6. *Let*

$$\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k,$$

where Γ_k, $k = 1, 2, \dots,$ are PA-arcs, and

$$d = \sup\Big\{ \operatorname{dmr} \Gamma_k : k = 1, 2, \dots \Big\} < 2.$$

If $f \in H^*(\Gamma, d/2)$, then the jump problem (2) *has a solution.*

Proof. We obtain a solution as a result of the regularization of the series of Cauchy transforms $\mathcal{C}[K_{\Gamma_k}]f^w(z)$. Indeed, we can choose rational functions $R_k(z)$ with poles at the points $a_{1,2}$ such that the series

$$\Phi(z) := \sum_{k\geq 1} \Big(\mathcal{C}[K_{\Gamma_k}]f^w(z) - R_k(z) \Big) \tag{19}$$

converges to a holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function. This choice can be performed in just the same way as in the customary proof of the Mittag-Leffler theorem. The singularities at common endpoints of PA-arcs annihilate by virtue of representation (13). Thus, the function (19) is a solution of the jump problem. □

This solution is free of any restrictions on its growth at the points $a_{1,2}$. Now let us prove the solvability of the jump problem for arcs with curling ends in a certain class of functions with prescribed behavior at the endpoints.

If Γ is not a PA-arc, then its refined metric dimension and approximation dimensions are undefined. But we can modify our results in terms of the box dimension and kernel $K_\Gamma(z)$. Generally speaking, its growth at the endpoints can be arbitrarily fast. For example, for the spiral arc S from Example 3 we have $K_S(z) = O(\theta(|z|))$, $z \to 0$. Here we restrict ourselves to the case where $K_\Gamma(z)$ is integrable near the points $a_{1,2}$.

Theorem 7. *Let $A = \overline{P}$, where a domain P contains an arc Γ and $\operatorname{dmb} \Gamma < 2$. If a function F is holomorphic in $P \setminus \Gamma$, integrable with any degree $p > 1$ near the points a_1 and a_2, and has boundary values from both sides on $\Gamma \setminus \{a_1, a_2\}$, then the primary integration is continuous in the space $H^*(A, \operatorname{dmb} \Gamma - 1)$.*

This theorem is an analog of Theorem 2. Its proof can be obtained from the proof of Theorem 2 by replacement of the convergent chain by the Whitney square decomposition of $\mathbb{C} \setminus \Gamma$ (see [23]). This result was obtained in other terms in [7, 13].

Then we consider the family of integrations (8) and their Cauchy transforms for $F = K_\Gamma$. As a result, we obtain analogs of Theorem 3 and representation (13). Finally, we obtain the following analog of Theorem 4.

Theorem 8. *If K_Γ is integrable with any degree $p > 1$ near the points a_1 and $a_2 i$ and $f \in H^*(\Gamma, \operatorname{dmb}\Gamma/2)$, then the jump problem (2) has a solution in the class of functions satisfying the condition*

$$\Phi(z) = O(K_\Gamma(z)), \quad z \to a_{1,2}, \quad j = 1, 2. \tag{20}$$

This solution is representable by $\mathcal{C}[K_\Gamma]f^w(z)$, where f^w is the Whitney extension of f from Γ. In addition, if $f \in H^*(\Gamma, \operatorname{dmu}'\Gamma/2)$, where

$$\operatorname{dmu}'\Gamma := \operatorname{dmb}\Gamma + (2 - \operatorname{dmb}\Gamma)(\operatorname{dmh}\Gamma - 1),$$

then $\mathcal{C}[K_\Gamma]f^w(z)$ is a unique HD-solution of the problem.

The solvability of the jump problem for $f \in H_\nu(A)$, $\nu > \operatorname{dmb}\Gamma/2$, was proved in [12]. The representability of a solution by the Cauchy transform is, seemingly, a new result.

5. Riemann Boundary-Value Problem

Let us restrict ourself to PA-arcs. We assume that the coefficients G and g of problem (1) belong to the space $H^*(\Gamma, \operatorname{dmr}\Gamma/2)$ and $G(t) \neq 0$ for $t \in \Gamma$. Then $G(t) = \exp f(t)$, $f \in H^*(\Gamma, \operatorname{dmr}\Gamma/2)$, and the jump problem (2) with jump f has a solution

$$\Phi_0(z) := \mathcal{C}[K_\Gamma]f^w(z).$$

Representation (14) implies the estimates

$$\Phi_0(z) = \frac{(-1)^j f(a_j)}{2\pi i} \log|z - a_j| + O(1) \tag{21}$$

for $z \to a_j$, $j = 1, 2$. We set

$$v_j = \operatorname{Re}\frac{(-1)^j}{2\pi i} f(a_j), \quad \kappa_j =]v_j[+ 1,$$

where

$$]x[:= \max\{n \in \mathbb{N} : n < x\}, \quad j = 1, 2, \quad \kappa := \kappa_1 + \kappa_2,$$

and

$$X(z) := (z - a_1)^{-\kappa_1}(z - a_2)^{-\kappa_2} \exp\Phi_0(z).$$

Obviously, X satisfies estimates of the form (18) at the points $a_{1,2}$ and X^{-1} is bounded. In addition,

$$X^+(t) = G(t)X^-(t).$$

We can apply the customary factorization and reduce the Riemann boundary-value problem to the jump problem

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \quad t \in \Gamma \setminus a_1, a_2. \tag{22}$$

Let Γ' and Γ'' be polygonal lines with starting point a_1 and endpoints a_2, lying from the left and from the right of the arc Γ, respectively. We set

$$F(z) = \chi(z)X^{-1}(z),$$

where $\chi(z)$ is the characteristic function of the domain D' bounded by Γ and Γ'. Then the function $\Xi(z) = \mathcal{C}[F]g^*(z)$ is a bounded solution of problem (22). As a result, we obtain the following assertion.

Theorem 9. *If* Γ *is a* PA*-arc,* G *and* g *belong to* $H^*(\Gamma, \mathrm{dmr}\,\Gamma/2)$*, and* $G(t)$ *does not vanish on* Γ*, then for* $\kappa = 0$*, the function*

$$\Phi_*(z) = \Xi(z)X(z)$$

is a solution of the Riemann boundary-value problem (1) *in the class of functions satisfying condition* (18)*. If* $\kappa > 0$*, then the problem has a family of solutions*

$$\Phi(z) = \Phi_*(z) + X(z)P(z),$$

where $P(z)$ *is an arbitrary algebraic polynomial of degree less than* κ*. If* $\kappa < 0$*, then* $\Phi_0(z)$ *is a solution under* $-\kappa$ *solvability conditions.*

If G *and* g *belong to* $H^*(\Gamma, \mathrm{dmu}\,\Gamma/2)$*, then all these solutions satisfy the HD condition and the problem has no other HD solutions.*

In other words, if G and g belong to $H^*(\Gamma, \mathrm{dmu}\,\Gamma/2)$ and $G(t)$ does not vanish on Γ, then the HD-solvability pattern of problem (1) in the class of functions satisfying condition (18) repeats its solvability pattern in the classical case of a piecewise-smooth arc, and all its HD solutions and conditions of HD solvability are representable in terms of integrations and their Cauchy transforms.

This result is sharper than the results of [12] because the refined metric dimension $\mathrm{dmr}\,\Gamma$ is, generally speaking, less than the box dimension $\mathrm{dmb}\,\Gamma$.

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