ISSN 0016-7932, Geomagnetism and Aeronomy, 2016, Vol. 56, No. 6, pp. 716–723. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © V.Yu. Belashov, E.S. Belashova, 2016, published in Geomagnetizm i Aeronomiya, 2016, Vol. 56, No. 6, pp. 755–762.

Nonlinear Dynamics of 3D Beams of Fast Magnetosonic Waves Propagating in the Ionospheric and Magnetospheric Plasma

V.Yu. Belashov[a, b, *] **and E. S. Belashova**[c]

[a]*Kazan Federal University, Kazan, 420008 Russia*
[b]*Kazan State Power Engineering University, Kazan, 420066 Russia*
[c]*Kazan National Research Technical University named after A.N. Tupolev, Kazan, 420111 Russia*
**e-mail: vybelashov@yahoo.com*

Received January 5, 2016; in final form, March 5, 2016

Abstract—On the basis of the model of the three-dimensional (3D) generalized Kadomtsev-Petviashvili equation for magnetic field $h = B_{\sim}/B$ the formation, stability, and dynamics of 3D soliton-like structures, such as the beams of fast magnetosonic (FMS) waves generated in ionospheric and magnetospheric plasma at a low-frequency branch of oscillations when $\beta = 4\pi nT/B^2 \ll 1$ and $\beta > 1$, are studied. The study takes into account the highest dispersion correction determined by values of the plasma parameters and the angle $\theta = (\mathbf{B},\mathbf{k})$, which plays a key role in the FMS beam propagation at those angles to the magnetic field that are close to $\pi/2$. The stability of multidimensional solutions is studied by an investigation of the Hamiltonian boundness under its deformations on the basis of solving of the corresponding variational problem. The evolution and dynamics of the 3D FMS wave beam are studied by the numerical integration of equations with the use of specially developed methods. The results can be interpreted in terms of the self-focusing phenomenon, as the formation of a stationary beam and the scattering and self-focusing of the solitary beam of FMS waves. These cases were studied with a detailed investigation of all evolutionary stages of the 3D FMS wave beams in the ionospheric and magnetospheric plasma.

DOI: 10.1134/S0016793216060049

1. INTRODUCTION. BASIC EQUATIONS

The objective of this work is to study the formation, structure, stability, and dynamics of multidimensional (two-dimensional (2D) and three-dimensional (3D)) soliton-like structures generated at a low-frequency branch of oscillations in the ionospheric and magnetospheric plasma when $\beta = 4\pi nT/B^2 \ll 1$ and $\beta > 1$. These processes are described by the following equation (Belashov, 2014):

$$\partial_t u + \hat{A}(t,u)u = f,\ \ f = \kappa\int_{-\infty}^{x}\Delta_\perp u\,dx, \qquad (1)$$
$$\Delta_\perp = \partial_y^2 + \partial_z^2,$$

which corresponds to the dispersion law in the limiting case of long waves (Karpman, 1973):

$$\omega_{1,2} = \frac{v_A k}{2\sqrt{1+k^2c^2/\omega_{pe}^2}}\left\{\left[(1+\cos\theta)^2 + \frac{k^2c^2}{\omega_{pi}^2}\times\frac{\cos^2\theta}{1+k^2c^2/\omega_{pe}^2}\right]^{1/2} \pm\left[(1-\cos\theta)^2 + \frac{k^2c^2}{\omega_{pi}^2}\times\frac{\cos^2\theta}{1+k^2c^2/\omega_{pe}^2}\right]^{1/2}\right\}$$

($\omega_{pe} = \left(4\pi n_e e^2/m\right)^{1/2}$ and $\omega_{pi} = \left(4\pi n_i e^2/M\right)^{1/2}$ are the electronic and ionic plasma frequencies, respectively, $v_A = B^2/4\pi n_i M$ is the Alfvén velocity, M is the ionic mass, θ is the angle between the direction of wave vector and magnetic field $\mathbf{B}$) and, depending on differential operator $\hat{A}$, describes waves propagating longitudinally and transversely relative to the external magnetic field. The case in which the lower sign (Alfven mode) is implemented in the dispersion relation was studied in detail in (Belashov, 2015); here we are interested in the case with a "plus" sign, when operator $\hat{A}$ has the form

$$\hat{A}(t,u) = \alpha u\partial_x - \partial_x^2(\nu - \gamma_1\partial_x - \gamma_2\partial_x^3), \qquad (2)$$

and the equation (1) represents the generalized generalized Kadomtsev–Petviashvili (KP) equation (Belashov–Karpman (BK) equation (Karpman, 1991; Belashov, 2005)), and in case when $\beta = 4\pi nT/B^2 \ll 1$ at $\omega < \omega_{Bi} = eB/Mc$ (where ω_{Bi} is an ion–cyclotron frequency); $|\mathbf{k}|r_D \ll 1$, describes the propagation of fast magnetosonic (FMS) waves in a magnetized plasma at $k_x^2 \gg \mathbf{k}_\perp^2$, $v_x \ll v_A$ near the cone of angles relative to the magnetic field $\mathbf{B}$ (which is assumed to be homogeneous) $\theta = \arctan(M/m)^{1/2}$ (Belashov, 1994). In this case, function u has the meaning of a dimensionless amplitude of magnetic field of the wave, $h = B_{\sim}/B$, while the coefficients of terms describing nonlinear, dissipative, and dispersive effects are determined by the plasma parameters and the angle $\theta = (\mathbf{B}, \mathbf{k})$.

Equations (1) and (2) can not be analytically integrated. Therefore, in order to study the stability of multidimensional solutions, we will use the approach developed in (Belashov, 1991) and investigate the Hamiltonian boundness for equations (1) and (2) upon its deformations, which conserve the system momentum, by solving the corresponding variational problem. We will also carry out an asymptotic analysis of multidimensional solutions in an analytical investigation of this system. The equations were integrated numerically with specially developed methods and codes described in detail in (Belashov, 2005) in order to study the evolution of 3D solutions, including the propagation of a 3D beam of FMS waves in the magnetized plasma. These problems will be considered below for the set of equations (1) and (2).

STABILITY AND ASYMPTOTICS OF 2D AND 3D SOLUTIONS

The problem of the stability of solitary wave solutions for KP and BK models remains highly relevant and is widely discussed in many literature sources related to the soliton theory (for example, Belashov, 2005; Liu, 1997; Pava, 2009; Esfahani, 2011). As for the BK dissipation-free equation, it has already been solved analytically (Belashov, 1991); herein, in the investigation of the stability of (1) and (2) solutions with $\nu = 0$, we will follow the method developed in the above-mentioned paper in the context of the problem under discussion. Let us make a transformation of coordinates and rewrite equations (1) and (2) with $\nu = 0$ in a Hamiltonian form:

$$\partial_t u = \partial_x(\delta\mathbf{H}/\delta u), \tag{3}$$

where

$$\mathrm{H} = \int\left[-\frac{\varepsilon}{2}(\partial_x u)^2 + \frac{\lambda}{2}\left(\partial_x^2 u\right)^2 + \frac{1}{2}(\nabla_\perp\partial_x v)^2 - u^3\right]d\mathbf{r}, \tag{4}$$

$\partial_x^2 v = u$, $\varepsilon = \gamma_1|\gamma_2|^{-1/2}$, and $\lambda = \operatorname{sgn}\gamma_2$. Stationary solutions of equation (3) are obtained from the variational problem $\delta(\mathrm{H} + vP_x) = 0$ ($P_x = \frac{1}{2}\int u^2 d\mathbf{r}$ is a projection of system momentum on the x-axis, v has the meaning of a Lagrange factor), illustrating the fact that all finite solutions of equation (3) are stationary points of Hamiltonian (4) under a fixed P_x. In accordance with Lyapunov's theorem, the stationary points of a dynamic system which realized the Hamiltonian maximum or minimum are absolutely stable; if the extremum is local, then locally stable solutions are possible. Unstable states correspond to the Hamiltonian monotonic dependence on its variables, i.e., the case in which the stationary point is a saddle one. Hence, it is necessary to prove the Hamiltonian boundness (from below) at a fixed P_x. Let us consider scale transformations in a real vector space R (Belashov, 2014) $u(x, \mathbf{r}_\perp) \to \zeta^{-1/2}\eta^{(1-d)/2}u(x/\zeta, \mathbf{r}_\perp/\eta)$ (where d is a problem dimension, and $\zeta, \eta \in \mathrm{R}$) that conserve the momentum projection P_x. Hamiltonian (as a function of ζ, η parameters) will take the form

$$H(\zeta,\eta) = a\zeta^{-2} + b\zeta^2\eta^{-2} - c\zeta^{-1/2}\eta^{(1-d)/2} + e\zeta^{-4}, \tag{5}$$

where $a = -(\varepsilon/2)\int(\partial_x u)^2 d\mathbf{r}$, $b = (1/2)\int(\nabla_\perp\partial_x v)^2 d\mathbf{r}$, $c = \int u^3 d\mathbf{r}$, and $e = (\lambda/2)\int(\partial_x^2 u)^2 d\mathbf{r}$. The necessary conditions of extremum existence are defined by the following set of equations: $\partial_\zeta \mathrm{H} = 0$, $\partial_\eta \mathrm{H} = 0$, while sufficient conditions for the Hamiltonian minimum are provided by a set of inequalities

$$\begin{vmatrix}\partial_\zeta^2 \mathrm{H}(\zeta_i,\eta_j) & \partial_{\zeta\eta}^2 \mathrm{H}(\zeta_i,\eta_j)\\ \partial_{\eta\zeta}^2 \mathrm{H}(\zeta_i,\eta_j) & \partial_\eta^2 \mathrm{H}(\zeta_i,\eta_j)\end{vmatrix} > 0,\quad \partial_\zeta^2 \mathrm{H}(\zeta_i,\eta_j) > 0.$$

The joint solution of these equations and inequalities yields the following results (Fig. 1). In the 3D case ($d = 3$ in the equations), 3D solutions are absolutely stable at $\lambda = 1$, $\varepsilon > 0$. For $\lambda = 1$, $\varepsilon \le 0$ the locally stable solutions can be observed, when the condition $ab^2e/c^4 < 9/512$ is satisfied for the integral Hamiltonian coefficients (5). Hence, we easily proved the possible existence of absolutely and locally stable solutions in the BK model and found the stability conditions for the 3D soliton solutions. It should be noted that the BK equation takes into account the dispersion correction of the next order, in contrast to the usual KP equation, and has stable 3D solutions, in contrast to the KP model (Kuznetsov, 1986). The used approach, when applied to the problem of FMS wave beam propagation in the magnetized plasma (see the following section), makes it possible to prove, for instance, that a 3D beam propagating at an angle of θ to the magnetic field is not focused; it becomes stationary and stable in the cone of angles $\theta < \arctan(M/m)^{1/2}$ when the following condition is valid (Belashov, 2014):

$$(m/M - \cot^2\theta)^2[\cot^4\theta(1 + \cot^2\theta)]^{-1} > 4/3.$$

As follows from the asymptotic analysis of possible solutions of equations (1) and (2), the asymptotics of solutions at $\gamma_2 > 0$ and $\gamma_1 = \pm 1$ in terms of $w = u(x, |\mathbf{r}_\perp|, t)/V$ are defined as follows:

(a) when the velocity of wave propagation along the x-axis is $V > 0$ and $\gamma_1 = -1$ and when $V < 0$, $\gamma_1 = -1$ (upper and lower signs, respectively):

$$w = A_1 \exp\left\{(2\gamma_2)^{-1/2}\left[C^2 + \sqrt{C^4 \pm 4\gamma_2}\right]^{1/2}\chi\right\}; \quad (6)$$

(b) when $V < 0$, $\gamma_1 = 1$:

$$w = A_2 \exp\left\{\left(2C^{-1}\gamma_2^{-1/2}\right)^{-1}\left(2C^{-2}\gamma_2^{1/2} - 1\right)^{1/2}\chi\right\} \times \cos\left\{\left(2C^{-1}\gamma_2^{-1/2}\right)^{-1}\left(2C^{-2}\gamma_2^{1/2} + 1\right)\chi + \Theta\right\}, \quad (7)$$

where A_1, A_2 and Θ are arbitrary constants, $C = |V|^{-1/4}$ and $\chi = x + |\mathbf{r}_\perp| + (\kappa - V)t$.

As follows from equations (6) and (7), equations (1) and (2) with $\nu = 0$, depending on the V and γ_1 signs, can have soliton solutions $u(x, |\mathbf{r}_\perp|, t)$ with both monotonous and oscillating asymptotics. It should also be noted that the solutions at $\gamma_1 = 0$ and any $\gamma_2 > 0$ have the form of $w = (A_1 + A_2\chi/C)\exp(\chi/\gamma_2^{1/4} C)$ and consequently also represent solitons with monotonous asymptotics (Karpman, 1991).

NONLINEAR EFFECTS IN PROPAGATION OF FMS WAVES IN MAGNETIZED PLASMA

It should be recalled that FMS waves can propagate in magnetized plasma at $\beta = 4\pi nT/B^2 \ll 1$ in the frequency region of $\omega < \omega_{Bi} = eB/Mc$, while in equations (1)–(5), from the physical point of view, function u is a dimensionless amplitude of magnetic field of the wave: $h = B_{\sim}/B$. The dispersion law at $|\mathbf{k}| r_D \ll 1$, $k_x^2 \gg \mathbf{k}_\perp^2$, and $v_x \ll v_A$ will have the following form:

$$\omega \approx v_A k_x \left(1 + \mathbf{k}_\perp^2/k_x^2 + \chi(\theta) D^2 k_x^2\right), \quad (8)$$

where $\mathbf{k}_\perp$ is a transversal (relative to the wave propagation direction) component of the wave vector, v_x is x-component of ionic velocity, D is a dispersion length, and θ is the angle between wave vector component $\mathbf{k}_x$ and the external magnetic field $\mathbf{B}$. It also should be recalled that the term "low dispersion" means that the primary nonlinear process is a three-wave interaction of low-amplitude waves; the low nonlinearity condition is based on the small angle between the interacting waves. At a relatively high ionic temperature, $\beta > m/M$, the dispersion length in (8) is defined by the following equation (Belashov, 1994):

$$\chi(\theta) D^2 = \frac{c^2}{2\omega_{pi}^2}\cot^2\theta - \frac{1}{2}\rho_i^2\left(3 - \frac{11}{4}\sin^2\theta\right), \quad (9)$$

where $\rho_i = v_{Ti}/\omega_{Bi}$ is an ionic Larmor radius. Under this process the plasma is quasineutral, because $\omega \ll \omega_{pi} = (4\pi n_i e^2/M)^{1/2}$. According to (9), the dispersion is positive (phase velocity grows with growth in $|\mathbf{k}|$),

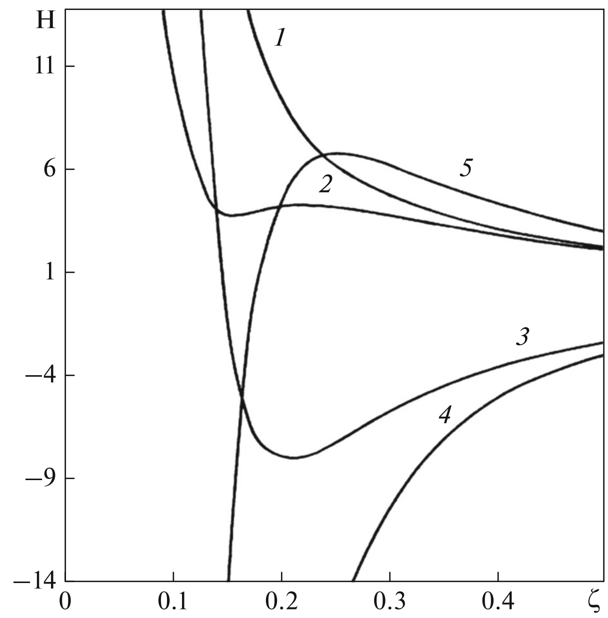

Fig. 1. Changing of $H(\zeta, \eta)$ at $d = 3$ for different values of integral coefficients along the lines of $\eta = (2b/c)\zeta^{5/2}$ at: (1) $a = 1.0$, $b = 1.0$, $c = 1.0$, $e = 0.025$; (2) $a = 1.0$, $b = 1.0$, $c = 1.0$, $e = 0.017$; (3) $a = -0.5$, $b = 1.0$, $c = 0.5$, $e = 0.02$; (4) $a = -0.5$, $b = 1.0$, $c = 0.5$, $e = -0.02$; (5) $a = 1.0$, $b = 1.0$, $c = 0.5$, $e = -0.02$.

except for angle regions near $\theta = 0$ and $\theta = \pi/2$. With propagation that is almost transversal relative to the magnetic field $\mathbf{B}$, when $|\pi/2 - \theta| \le (\beta/4)^{1/2}$, the dispersion is negative and is defined by effects related to the finiteness ρ_i. It is known that propagation of the low-amplitude FMS wave with a narrow angle distribution can be described by the KP equation ((1), (2) at $\nu = \gamma_2 = 0$) (Manin, 1983). For such angles, when dispersion is positive for low $|\mathbf{k}|$ (at a relatively high ionic temperature), the 3D FMS wave packet in the plasma with $\beta > m/M$ does not form stable stationary solutions and spreads for the angles $|\pi/2 - \theta| < (m/M)^{1/2}$ or collapses outside of this cone (Kuznetsov, 1986). (It should be noted that a similar phenomenon is occasionally termed as wave "self-compression" (Tsytovich, 1995)). The last case, when a relatively intensive FMS wave beam is limited in the $\mathbf{k}_\perp$-direction, can be characterized by the self-focusing phenomenon (Belashov, 2005). This problem was solved for the first time in (Manin, 1983) via the averaging of initial equations and subsequent numerical solution. Yet, the relation (9) will be not valid for the angles $\theta < (kc/\omega_{pi})^{1/2}$, which are characterized by intensive reconstruction of the oscillation dispersion mechanism. At $\beta < m/M$ the dispersion can be defined for any angle θ based from the hydrodynamic equations, and the FMS wave structure

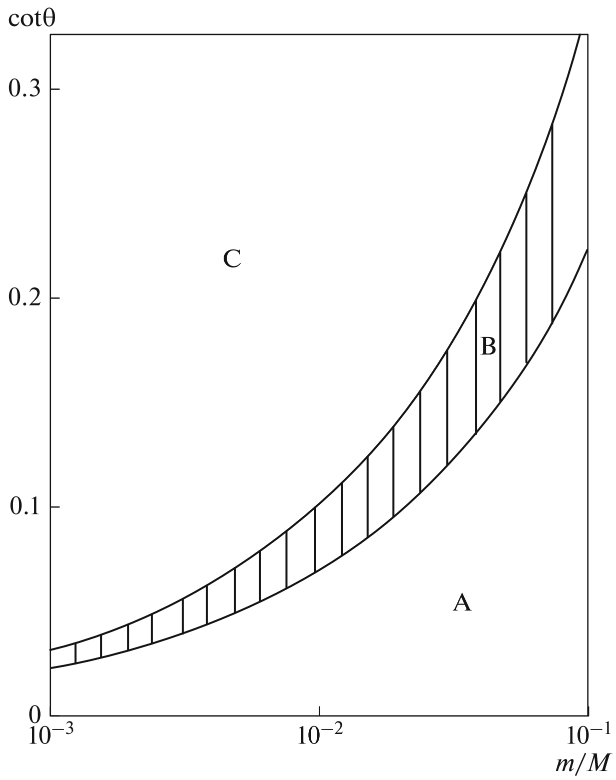

Fig. 2. Dispersion behavior for FMS waves with respect to the angle θ and ratio m/M.

will depend in this case on sign of the dispersion coefficient

$$\gamma_1 = -v_A\chi(\theta)D^2 = v_A\frac{c^2}{2\omega_{pi}^2}\left(\frac{m}{M} - \cot^2\theta\right),$$

which is defined by angle θ; in particular: the dispersion is negative for propagation that is almost transversal if $|\pi/2 - \theta| \le (m/M)^{1/2}$, and it is positive for all other angles. The KP equation can also be used in this case; and for a relatively intensive FMS wave beam, which is limited in the $\mathbf{k}_\perp$-direction, we can expect self-focusing of a beam propagating at such angles θ to the magnetic field, where the dispersion is positive (Zakharov, 2012).

In both cases, $\beta > m/M$ and $\beta < m/M$, it is necessary to take into account the fact that $\gamma_1 \to 0$ near the cone of angles, where dispersion changes the sign. Obviously, this does not mean that the dispersion disappears in this region. It just means that the description based on the KP equation model in its standard form is not correct in this case. Near the cone $|\pi/2 - \theta| \le (\beta/4)^{1/2}$, where $\gamma_1 \to 0$ at $\beta > m/M$, the results of (Manin, 1983) should be clarified and even reconsidered. For example, the relation (8) should be supplemented by a dispersion term of the next order, which will play a major role in this case (Belashov, 2005). An analogous situation is possible at $\beta < m/M$ near the cone of angles $\theta = \arctan(M/m)^{1/2}$. In both cases, the dispersion relation takes the form of $\omega \approx v_A k_x \times \left[1 + \mathbf{k}_\perp^2/2k_x^2 + c_0^{-1}(-\gamma_1 k_x^2 + \gamma_2 k_x^4)\right]$. The dispersion correction of the next order can be obtained by decomposition of the full dispersion equation into the Taylor's series by k (Belashov, 2005). In the case of $\beta < m/M$, which is considered in detail below, we obtain

$$\gamma_2 = v_A\frac{c^4}{8\omega_{pi}^4}\left[3\left(\frac{m}{M} - \cot^2\theta\right)^2 - 4\cot^4\theta\left(1 + \cot^2\theta\right)\right].$$

Hence, the dispersion character becomes much more complicated and is now defined by correlation of signs of coefficients γ_1 and γ_2 (Fig. 2). At $\gamma_1 > 0$, $\gamma_2 < 0$, we observe negative dispersion (region B in Fig. 2), while the dispersion is "mixed" at $\gamma_{1,2} > 0$ (region A) and $\gamma_{1,2} < 0$ (region C) (the dispersion sign is different for low and high k). In this case, low-amplitude FMS waves with a narrow angle distribution will be described by the BK equation [Karpman, 1991]; in the non-dissipative case, it will have the form (Belashov, 2014)

$$\partial_x\left(\partial_t h + \alpha h\partial_x h + \gamma_1\partial_x^3 h + \gamma_2\partial_x^5 h\right) = -(v_A/2)\Delta_\perp h, \quad (10)$$

where $\alpha = (3/2)v_A\sin\theta$. In this case, a nonlinear term $\alpha h\partial_x h$, which is the result of sound velocity renormalization, reflects a low probability of other nonlinear processes caused by vector nonlinearity. In contrast to the KP equations, the solutions of equation (10) are characterized by a more complicated structure and dynamics, which is related to the ratio of values and signs of the dispersion coefficients γ_1 and γ_2. Hence, it is found in the case of $\beta < m/M$, in contrast to the case of $\beta > m/M$ (which was considered in Manin, 1983), that the 3D beam of FMS waves, which propagates in the plasma at angle θ to the external magnetic field, is not self-focused and becomes stationary and stable in the cone of angles $\theta < \arctan(M/m)^{1/2}$ when the following conditions are satisfied:

$$\left(\frac{m}{M} - \cot^2\theta\right)^2\left[\cot^4\theta\left(1 + \cot^2\theta\right)\right]^{-1} > 4/3,$$

or, in other words, when $\gamma_{1,2} > 0$ in (10). This conclusion is confirmed by our analytical (see the previous section) and numerical (Karpman, 1991) results for 3D solitary wave structures propagating in low dispersive media, where the presence of the highest dispersion correction in BK (as opposed to KP) stops the wave collapse at the initial stage of development of self-focusing instability. This result is of key importance, because, prior to the works of (Karpman, 1991; Belashov, 1991), neither analytical nor numerical studies identified 3D stable wave structures such as 3D solitons. It is the accounting for higher-order dispersion effects that made it possible to found 3D stable soliton solutions in the BK equation model, in contrast to the results obtained for the standard KP equation model (Zakharov, 2012).

In order to study the dynamics of an FMS wave beam characterized by a narrow angular distribution, we solved the boundary problem (in contrast to (Karpman, 1991), in which the Cauchy problem was considered). We numerically integrated the corresponding equation, because the exact analytical solutions of the BK equation, even for the non-dissipative case, are not currently known.

Let us consider the problem of modeling of the FMS wave beam dynamics in the magnetized plasma. It is assumed that there is a 3D FMS beam propagating in the plasma at angle θ to the external magnetic field near the cone of angles $\theta = \arctan(M/m)^{1/2}$. Using the substitutions $x \to -st$, $y \to -s\kappa^{1/2}y$, $z \to -s\kappa^{1/2}z$, $t \to sx$, $h \to -(6/\alpha)h$, $s = |\gamma_2|^{1/4}$, and $\kappa = v_A/2$, we obtain from (10)

$$\partial_t\left(\partial_x h + 6h\partial_t h - \varepsilon\partial_t^3 h - \lambda\partial_t^5 h\right) = \Delta_\perp h, \qquad (11)$$

where $\varepsilon = \gamma_1|\gamma_2|^{-1/2}$, $\lambda = \operatorname{sgn}\gamma_2$. Equation (11) describes the FMS wave beam propagating along the x-axis from the boundary $x = 0$. If it is assumed that $\Delta_\perp = \partial_\rho^2 + (1/\rho)\partial_\rho$, and also $h_0 = h(t,0,\rho) = \cos(mt)\exp(-\rho^2)$, then the boundary condition is defined as the FMS wave beam localized in the (y,z) plane and time-periodic axially symmetric beam of FMS waves.

Equation (11) with boundary condition h_0 was integrated numerically. The series of performed numerical experiments related to the modeling of FMS beam propagation at its different intensities at the boundary of $x = 0$, h_0 and different angles θ (see above the cases A, B, and C) made it possible to obtain the following results. In region A (corresponding to $\lambda = 1$, $\varepsilon > 0$), as in (Belashov, 2005) and (Belashov, 2014), the spatial evolution of the FMS wave beam at the initial stage at any h_0 results in beam focusing, which is related to the predominant role of nonlinear processes in this time range. Meanwhile, as in the usual KP equation, we observe (Fig. 3, curves 1 and 2) beam compression in a transversal ρ-direction in the course of its propagation along the x-axis, such that its transversal characteristic size $l_\rho(x) \sim l_\rho(0)\,h(0)/h(x)$ decreases with simultaneous fast growth of the beam intensity in its axis with an increment of $\Gamma = (1/2W)\times dW/dt \sim 2$ (where $W = \langle h^2\rangle/4\pi$ is a wave energy in the volume unit), which is just slightly dependent on ε. In this case, the characteristic dimensions of the

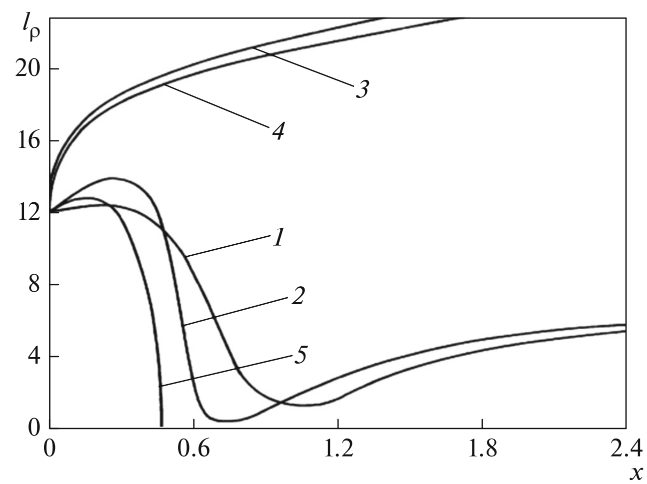

Fig. 3. Changing of a cross-section of the wave beam under its propagation in the x-direction: (*1*) $\lambda = 1$, $\varepsilon = 1.34$; (*2*) $\lambda = 1$, $\varepsilon = 2.24$; (*3*) $\lambda = -1$, $\varepsilon = 1.34$; (*4*) $\lambda = -1$, $\varepsilon = -1.34$; (*5*) $\lambda = 0$, $\varepsilon = -1.34$.

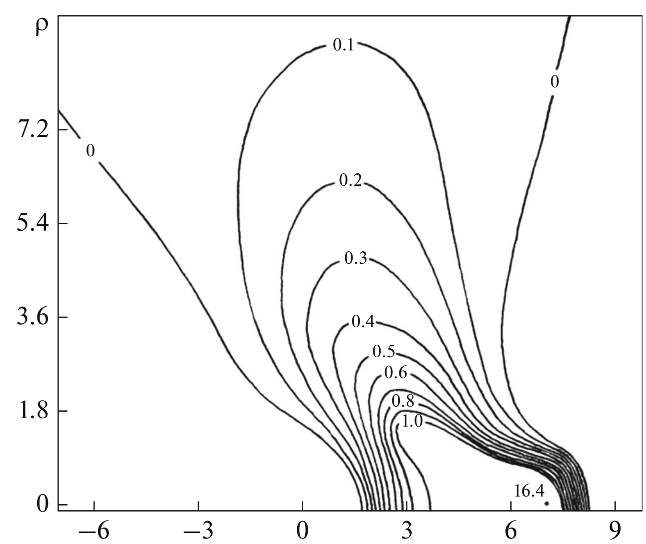

Fig. 4. Solution in the (x, ρ) plane at $\lambda = 1$, $\varepsilon = 0.89$ corresponding to the amplitude maximum stage.

beam, which represents a wave pulse, decrease, its "wings" start lagging behind its central part, and self-focusing instability develops (Fig. 3, curves 1 and 2; Fig. 4). This evolution type is also characterized by an increase in P and a decrease (at low $\varepsilon > 0$) in the Hamiltonian H in the system due to a nonlinear term, which grows at this evolution stage much faster than dispersion terms.

Under further growth of t, due to a decrease in a transverse size of wave pulse l_ρ (Fig. 3), the term, which is proportional to the fifth derivative in the equation (11), starts playing a predominant role (it is well-seen in the analysis of variations in the integral

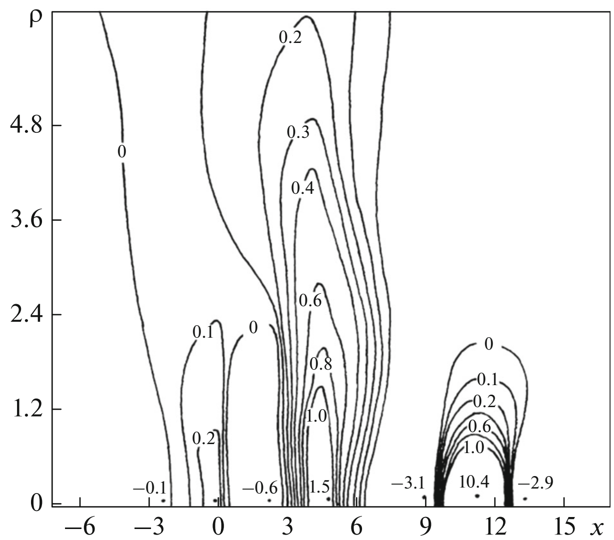

Fig. 5. Solution in the (x, ρ) plane at $\lambda = 1$, $\varepsilon = 0.89$.

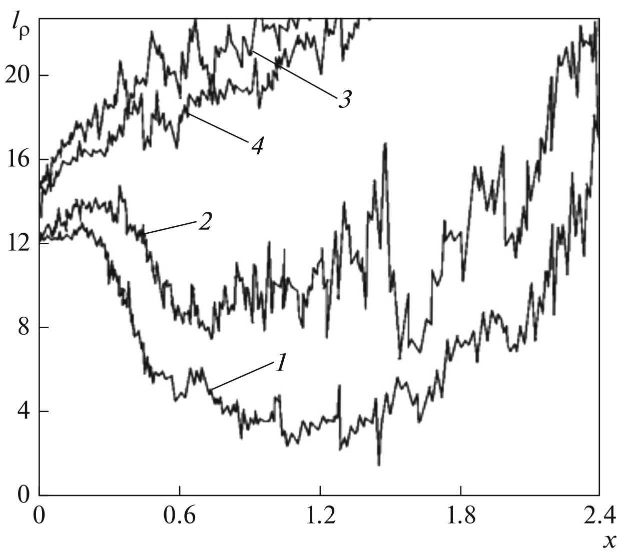

Fig. 6. Variations in a cross section of the beam under its propagation along the x-axis in the plasma with $\eta = \eta(t, x, \rho)$ at a standard deviation of $\sigma = 0.04$ for the same λ and ε values as in Fig. 3.

terms making up the Hamiltonian H). As a result, the "collapse" of the wave pulse wings behind its main part does not lead to a rapid increase in the field intensity and singularity formation in the major peak region, which is typical for the standard KP equation model with $\gamma_1/\kappa > 0$ (Belashov, 2005); as result, a ring region of the elevated field concentration is formed (Fig. 5). Further evolution of this structure will make it possible to form additional peaks in the x-axis behind the pulse (Fig. 5). In this case, the wave pulse stops being compressed and starts defocusing (Fig. 3, curves 1 and 2). This stage is completed by the formation of a stationary wave beam, i.e., by transition into the regime of $h_{\max}(x) = \text{const}$ and $l_\rho(x) = \text{const}$, which corresponds the results described in the second section. The role of the term, which is proportional to the fifth derivative in the equation (11), along with the role described above, consists of the appearance of small-scale oscillations forming a regular oscillatory structure of the tail (Fig. 5).

In regions B and C (Fig. 2), which correspond to $\lambda = -1$ and $|\varepsilon| \geq 0$, a sonic wave scatters with propagation along the x-axis at any beam intensity $h(0)$ at the boundary (Fig. 3, curves 3 and 4), just as in the electromagnetic wave self-action process in the media, where derivatives $\partial^2\omega/\partial k_x^2$ and $\partial^2\omega/\partial k_\perp^2$ have different signs (for example, this phenomenon is characteristic for ion–cyclotron waves, whistlers, etc.) (Litvak, 1983).

Figure 3 demonstrates that, at $\lambda = 0$, when (11) transits into the KP equation with a negative dispersion, there are no solutions as a self-focusing beam of FMS waves. Therefore, the self-focusing effect is not observed in the considered model when $\lambda = 0$. According to the test numerical experiments for the model (11) with $\lambda = 0$, self-focusing is possible only at $\varepsilon < 0$ (Fig. 3, curve 5), when the FMS wave beam described by this model does not correspond to any real situation (Belashov, 2005).

Hence, according to the study results based on the BK equation model (11), the self-focusing phenomenon of the FMS beam, which propagates in the plasma at the angles to the magnetic field near the cone of $\theta = \arctan(M/m)^{1/2}$, cannot be observed, in contrast to the standard KP equation model, even if the dispersion for low k is positive. In this case, however, together with the beam scattering, we can observe nonlinear stationary propagation. It should also be noted that equations (10) and (11) at $|\pi/2 - \theta| \gg (m/M)^{1/2}$ should also be supplemented with terms proportional to mixed derivatives, because $|\mathbf{k}_\perp| \geq k_x$ in this case and the dispersion equation acquires terms proportional to $k_x^i |\mathbf{k}_\perp^j|$, where $i, j = 1, 2$, etc.

It should be noted in the conclusion that it is also necessary under the ionospheric and magnetospheric plasma conditions to take into account the effect of stochastic fluctuations of the wave field $h(t, x, \mathbf{r}_\perp)$ on wave beam evolution, which should be taken into consideration in basic equations. Hence, equation (11) should be supplemented with a term such as $\eta(t, x, \rho)$ and rewritten as follows:

$$\partial_t\left(\partial_x h + 6h\partial_t h - \varepsilon\partial_t^3 h - \lambda\partial_t^5 h + \eta(t, x, \rho)\right) = \Delta_\perp h. \quad (12)$$

In (Belashov, 1995), equation (12) at $\lambda = 0$ for the case of low-frequency fluctuations, when $\eta = \eta(t)$, was integrated analytically. The results can be easily applied to (12) with $\eta = \eta(t)$. The interpretation of results obtained in (Belashov, 1995) in terms of this problem is indicative of the fact that even low stochastic fluctuations of the wave field will lead to the decay of the wave pulse upon its propagation, accompanied by the wave transformation into an oscillatory structure. Meanwhile, in the case of $\eta = \eta(t, x, \rho)$, the analytical study of the corresponding process becomes extremely complicated, and (Belashov 2014) carried out numerical integration of equation (12) with a stochastic term, which is a function of time and space coordinates. Figure 6 demonstrates the results of numerical modeling of the FMS wave beam evolution in a medium with stochastic fluctuations of the wave field in the form of Gaussian noise in the model (12) with $\eta = \eta(t, x, \rho)$. The obtained results are qualitatively similar to the case of $\eta = \eta(t)$: a decrease in amplitude of the FMS wave beam upon its propagation with the subsequent wave destruction (comparison with Fig. 3).

CONCLUSIONS

In the course of this investigation, we studied analytically and numerically the problem of the stability and dynamics of 3D soliton-like structures, such as a beam of FMS waves, which are formed in a low-frequency branch of oscillations in plasma, for the cases when $\beta \equiv 4\pi nT/B^2 \ll 1$ and $\beta > 1$. The study was based on the model of the 3D BK-equation for the magnetic field $h = B_{\sim}/B$, upon the assumption of homogeneity of the external magnetic field $\mathbf{B}$, and takes into account the highest dispersion correction determined by the plasma parameters and the angle $\theta = (\mathbf{B}, \mathbf{k})$. According to the results, in contrast to the KP equation model, when the FMS wave beam propagates at the angles to the external magnetic fields near the cone of $\theta = \arctan(M/m)^{1/2}$, the self-focusing phenomenon is not observed, even if dispersion for low k is positive. It is proved that on a level with the magnetic sound scattering the nonlinear stationary beam propagation can be observed; the analytical and numerical methods made it possible to prove the possibility of the formation of stable 3D solitary beams of FMS waves in the course of evolution. It is demonstrated that the presence of stochastic wave field fluctuations in the medium reduces the FMS wave beam amplitude upon propagation, followed by beam destruction.

Our work did not explicitly take into account the possible effects of the external magnetic field inhomogeneity that can take place in the Earth's ionosphere and magnetosphere. For instance, the field inhomogeneity can result in soliton acceleration (Popel et al., 1995) and other phenomena related to the imbalance between nonlinear and dispersive effects, for example, in soliton deformations and destruction, as happens during the propagation of nonlinear wave structures in the variable dispersion media (see, for example, (Belashov, 2005), (Belashova, 2006)). The latter can be caused in plasma as heterogeneity of its composition in space, which will lead to spatial dependence of the values such as m/M, n_i and, consequently, $\omega_{pi} = \left(4\pi n_i e^2/M\right)^{1/2}$, and also the field $\mathbf{B}$ heterogeneity, and then $v_A = B^2/4\pi n_i M$. In this case, the dispersion coefficients γ_1, γ_2 in the BK equations will also become functions of spatial coordinates. Hence, the heterogeneity effect can be taken into account in our model if we assume that $\mathbf{B} = f(\mathbf{r})$, while γ_1, γ_2, and v_A are functions of $\mathbf{r}$ in equations (10), (11). However, such generalization of the BK equation is beyond the scope of the research presented in this paper.

It should be noted in conclusion that the results can be directly applied to nonlinear wave dynamics in the ionospheric and magnetospheric plasma, because, in our opinion, they will contribute to a better understanding of the physics of nonlinear wave processes and may be useful in the interpretation of the results of laboratory and space experiments related to the excitation, evolution, and interaction of FMS wave solitons and of self-action effects, such as wave collapse and wave beam self-focusing.

ACKNOWLEDGMENTS

The work is performed according to the Russian Government Program of Competitive Growth of Kazan Federal University.

REFERENCES

Belashov, V.Yu., On the stability of two- and three-dimensional solitons in weakly dispersive media, *Dokl. Akad. Nauk SSSR*, 1991, vol. 320, no. 1, pp. 85–89.

Belashov, V.Yu., Nonlinear effects for FMS waves propagating in magnetized plasma, *Plasma Phys. Controlled Fusion*, 1994, vol. 36, pp. 1661–1669.

Belashov, V.Yu., Dynamics of KP equation solitons in media with low-frequency wave field stochastic fluctuations, *Phys. Lett. A.*, 1995, vol. 197A, pp. 282–289.

Belashov, V.Yu., Dynamics of multidimensional nonlinear wave structures of the of the solitons and vortex types in dispersive complex media. Theory, simulation, applications, in *Book of Abstracts of the 17th International Congress on Plasma Physics, ICPP-2014, Lisbon, Portugal*, 2014a, p. 35.

Belashov, V.Yu., Nonlinear effects of the dynamics of fast magnetosonic waves in a plasma: Self-focusing and stabilization of beams, in *Trudy XXIV Vserossiiskoi nauchoi konferentsii "Rasprostranenie radiovoln" RRV-24* (Proceedings of the XXIV All-Russian Scientific Conference "Radiowave Propagation" RWP-24), Irkutsk: ISZF SO RAN, 2014b, vol. 3, pp. 5–12.

Belashov, V.Yu. and Belashova, E.S., Nonlinear dynamics of the 3D Alfvén waves in plasma of ionosphere and magnetosphere, *J. Atmos. Sol.–Terr. Phys.*, 2015, vol. 136, pp. 150–154.

Belashov, V.Yu. and Vladimirov, S.V., *Solitary Waves in Dispersive Complex Media. Theory, Simulation, Applications*, Berlin: Springer, 2005.

Belashova, E.S. and Belashov, V.Yu., *Solitony kak matematicheskie i fizicheskie ob"ekty* (Solitons as Mathematical and Physical Objects), Kazan: KGEU, 2006.

Esfahani, A., Instability of solitary waves of the generalized high-order KP equation, *Nonlinearity*, 2011, vol. 24, pp. 833–846.

Karpman, V.I., *Nelineinye volny v dispergiruyushchikh sredakh* (Nonlinear Waves in Dispersive Media), Moscow: Nauka, 1973.

Karpman, V.I. and Belashov, V.Yu., Evolution of three-dimensional nonlinear pulses in weakly dispersive media, *Phys. Lett. A*, 1991, vol. 154, pp. 140–144.

Kuznetsov, E.A. and Musher, S.L., Influence of collapsing acoustic waves on the structure of noncollision shock-waves in magnetized plasma, *Zh. Eksp. Teor. Fiz.*, vol. 91, no. 5, pp. 1605–1619.

Litvak, A.G., On one type of self-action of waves in plasma, *Fiz. Plazmy*, 1983, vol. 9, no. 3, pp. 495–500.

Liu, Y. and Wang, X.-P., Nonlinear stability of solitary waves of a generalized Kadomtsev–Petviashvili equation, *Comm. Math. Phys.*, 1997, vol. 183, pp. 253–266.

Manin, D.Yu. and Petviashvili, V.I., Self-focusing of magnetoacoustic waves across the magnetis field, *Pis'ma Zh. Eksp. Teor. Fiz.*, 1983, vol. 38, no. 9, pp. 427–430.

Pava, J.A., *Nonlinear Dispersive Equations. Existence and Stability of Solitary and Periodic Travelling Wave Solutions*, Providence, Rhode Island: American Mathematical Society, 2009.

Popel, S.I., Vladimirov, S.V., and Tsytovich, V.N., Theory of modulational interactions in plasmas in the presence of an external magnetic field, *Phys. Rep.*, 1995, no. 6, pp. 327–405.

Tsytovich, V.N., *Lectures on Non-Linear Plasma Kinetics.* Berlin: Springer, 1995.

Zakharov, V.E. and Kuznetsov, E.A., Solitons and collapses: Two evolution scenarios of nonlinear wave systems, *Phys.-Usp.*, 2012, vol. 55, no. 6, pp. 537–556.

Translated by E. Maslennikova