One-parametric families of Reimann surfaces

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We describe one-parametric families of rational functions $R(z,t)$ depending smoothly on a real parameter $t \in [0;1]$. For a fixed t the function $R(z,t)$ maps conformally the Riemann sphere $\widehat{\mathbb{C}}$ onto a Riemann surface $S(t)$ over $\widehat{\mathbb{C}}$. Let the critical points $a_k = a_k(t)$ of $R(z,t)$ have multiplicities m_k and corresponding critical values $A_k(t) = R(a_k(t),t)$ depend smoothly on t. We will also assume that $R(z,t)$ has poles of order n_j at points $b_j = b_j(t)$. Provided that we know the functions $A_k(t)$, we find a system of ODE for the critical points $a_k(t)$ and the poles $b_j(t)$.

In the case when there is no poles at finite points of the complex plane, i.e. $R(z,t)$ are polynomials by z, such problem was solved in [1]. In [2] we considered families of rational functions with simple critical points $a_k(t)$. Here we investigate the case of arbitrary multiplicities. We should note that in [3, 4] we considered the case of one-parametric families of elliptic functions with a unique pole in every period parallelogram and simple critical points.

Thus, let

$$R(z,t) = C_0 \int_{z_0}^{z} \frac{\prod_{k=1}^{M}(\zeta - a_k(t))^{m_k-1}\,d\zeta}{\prod_{j=1}^{N}(\zeta - b_j(t))^{n_j+1}} + C_1 \qquad (1)$$

be a family of rational functions with critical points $a_k(t)$ of order m_k and poles $b_j(t)$ of order n_j. Without loss of generality we can assume that $C_0 = 1$, $z_0 = a_1$, $C_1 = A_1$, and

$$\sum_{k=1}^{M}(m_k-1)a_k(t) - \sum_{j=1}^{N}(n_j+1)b_j(t) = 0$$

for all $t \in [0,1]$.

Theorem. *The critical points $a_k = a_k(t)$ and the poles $b_j = b_j(t)$ of the functions (1) satisfy the folllowing system of ODE:*

$$\dot a_l = \frac{H_l^{(m_l-1)}(a_l)}{(m_l-1)!}\,\dot A_l + \sum_{k=1,k\neq l}^{M} \frac{G_{kl}^{(m_k-2)}(a_k)}{(m_k-2)!}\,\dot A_k, \quad \dot b_j = \sum_{k=1}^{M} \frac{I_{kj}^{(m_k-2)}(a_k)}{(m_k-2)!}\,\dot A_k,$$

where

$$H_l(x) = \frac{\prod_{j=1}^{N}(x-b_j)^{n_j+1}}{\prod_{k=1,k\neq l}^{M}(x-a_k)^{m_k-1}}, \quad G_{kl}(x) = \frac{H_k(x)}{x-a_l}, \quad I_{kj}(x) = \frac{H_k(x)}{x-b_j}.$$

Here the dot above a letter means differentiation by t.

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