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A UNIQUENESS THEOREM FOR LINEAR ELLIPTIC EQUATIONS WITH DOMINATING DERIVATIVE WITH RESPECT TO $\overline{Z}$

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Abstract. The interior uniqueness theorem for analytic functions was generalized by M.B. Balk to the case of polyanalytic functions of order n. He proved that, if the zeros of a polyanalytic function have an accumulation point of order n, then this function is identically zero. M.F. Zuev generalized this result to the case of metaanalytic functions. In this paper, we generalize the interior uniqueness theorem to solutions of linear homogeneous elliptic differential equations of order n with analytic coefficients whose senior derivative is the n-th power of the Cauchy–Riemann operator.

In [1], [2] the author proved the interior uniqueness theorem for solutions of linear elliptic equations with constant coefficients in the case of a principal type differential operator, i.e. in the case of an operator with only maximal order derivatives. Here we establish a similar theorem for linear elliptic equations with a dominant n-th order partial derivative with respect to $\overline{z}$ with variable coefficients in the case of the differential operator endowed with derivatives of lower order. Similar results on polyanalytic and metaanalytic functions can be found in [3]. Article [4] contains some boundary uniqueness theorems for solutions of linear elliptic equations with constant coefficients.

Consider $E \subset \mathbb{C}$, let $z_0 = x_0 + iy_0$ be some point (not necessarily from E);

$$y - y_0 = k(x - x_0) \tag{1}$$

be a line passing through the point z_0. A sequence of straight lines $\{y - y_0 = k_n(x - x_0)\}$ is called converging to line (1) if $\lim_{n\to\infty} k_n = k$.

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A set E thickens to the point z_0 along the straight line given by equation (1) if E contains such a sequence of points z_n converging to z_0 that the sequence of lines passing through the points z_0 and z_n converges to line (1).

A point z_0 is called an order m accumulation point for a set E if E thickens to the point z_0 at least along m different lines [3].

Lemma. *Let D be a domain in the complex plane $\mathbb{C}$. If a point $z_0 \in D$ is an accumulation point of the function $f(z) \in C^n(D)$ order n zeros then $f(z) = O((z - z_0)^n),\ z \to z_0$.*

Proof. Without loss of generality we assume that the domain D contains the origin and $z_0 = 0$. Let zeros of f thicken to the point $z = 0$ along n different lines $y = k_j x,\ j = 1, \dots, n$. Consider the Taylor formula in the neighborhood of $z = 0$

$$f(z) = \sum_{j=0}^{n} \frac{1}{j!} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^j f(0) + o(z^n). \tag{2}$$

The proof proceeds by induction. By the statement assumption $f(0) = 0$. Suppose that all the coefficients with powers of $x^j y^k$ vanish in equation (2) for $j + k < m < n$. Then

$$f(z) = \sum_{j=m}^{n} \frac{1}{j!} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)^j f(0) + o(z^n). \tag{3}$$

Since the function $f(z)$ zero set E thickens to the point $z = 0$ along the line $y = k_j x$ relation (3) yields

$$0 = \sum_{l=0}^{m} \binom{m}{l} k_j^l \frac{\partial^m f(0)}{\partial x^{m-l} \partial y^l},\ j = 1, \dots, n, \tag{4}$$

here $\binom{m}{l}$ is the number of l-combinations from a set of m elements.

Since rank of system (4) equals $m + 1$ all the derivatives

$$\frac{\partial^m f(0)}{\partial x^{m-l} \partial y^l},\ l = 0,\ 1, \dots, m < n, \tag{5}$$

vanish. This completes the proof.

Consider the elliptic equation

$$\frac{\partial^n f}{\partial \bar{z}^n} + \sum_{0 \le k+j \le n-1} A_{kj}(z) \frac{\partial^{k+j} f}{\partial z^k \partial \bar{z}^j} = 0, \tag{6}$$

here

$$z = x + \mathrm{i}y,\ \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y} \right),\ \frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \right),$$

and $A_{kj}(z)$ are functions analytic on the real variables $x,\ y \in D \subset \mathbb{C}$.

We now prove the following uniqueness theorem for equation (6):

Theorem. *If a point $z_0 \in D$ is an accumulation point of order n zeros for the regular equation (6) solution $f(z)$ in D then $f(z) \equiv 0$.*

Proof. Assume again that $z_0 = 0$ and that the set of the solution $f(z)$ zeros thickens to the point $z = 0$ along different lines $y = k_j x,\ j = 1,\ 2, \ldots, n.$ According to Lemma all the derivatives

$$\frac{\partial^m f(0)}{\partial z^{m-l} \partial \overline{z}^l},\ l = 0,\ 1, \ldots, m < n,$$

vanish. Suppose now that $m \geq n$. Equation (6) solution $f(z)$ is an analytic function in $x,\ y$ by Theorem of F. John ([5], p. 144) and therefore admits the following Taylor series expansion in some neighborhood of the origin:

$$f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(z \frac{\partial}{\partial z} + \overline{z} \frac{\partial}{\partial \overline{z}} \right)^j f(0). \tag{7}$$

We now show by induction that all the solution $f(z)$ partial derivatives vanish at the point $z = 0$. We assume that all of the solution $f(z)$ partial derivatives of order not higher than $m - 1$ vanish at the origin. Consider $n \leq l \leq m$ and differentiate equation (6) $m - l$ times with respect to z and $l - n$ times with respect to $\overline{z}$:

$$\frac{\partial^m f(z)}{\partial z^{m-l} \partial \overline{z}^l} = - \sum_{k=0}^{n-1} A_{k,n-k-1}(z) \frac{\partial^{m-1} f(z)}{\partial z^{k+m-l} \partial \overline{z}^{l-1-k}} + Lf(z),\ n \leq l \leq m, \tag{8}$$

here L is a linear differential operator of no greater than $m - 2$ order. By the induction hypothesis all the terms on the right-hand side of equation (8) vanish at the point $z = 0$. Therefore all the order m derivatives on the left-hand side of equation (8) also vanish at this point. Now since the solution $f(z)$ zeros thicken to the point $z = 0$ along the line $y = k_j x$ relation (7) yields

$$\sum_{l=0}^{n-1} \binom{m}{l} (1 + \mathrm{i}k_j)^{m-l} (1 - \mathrm{i}k_j)^l \frac{\partial^m f(0)}{\partial z^{m-l} \partial \overline{z}^l} = 0,\ j = 1, \ldots, n. \tag{9}$$

Since system (9) determinant is not zero we have

$$\frac{\partial^m f(0)}{\partial z^{m-l} \partial \overline{z}^l} = 0,\ l = 0,\ 1, \ldots, n - 1.$$

Thus all the function $f(z)$ derivatives vanish at the point $z = 0$. Since this function is analytic in the variables x and y we have $f(z) \equiv 0$.

This completes the proof of the theorem.

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