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A Numerical Method for Finding Dispersion Curves and Guided Waves of Optical Waveguides

R. Z. Dautov, E. M. Karchevskii, and G. P. Kornilov
Kazan State University, Kremlevskaya ul. 18, Kazan, 420008 Russia
e-mail: Evgenii.Karchevskii@ksu.ru
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Abstract—The original problem in an unbounded domain is reduced to a linear parametric eigenvalue problem in a circle, which is convenient for numerical solution. The examination of the solvability of this problem is based on the spectral theory of compact self-adjoint operators. The existence of guided waves is proved, and properties of the dispersion curves are investigated. An algorithm for the numerical solution of the problem based on the discretization of the equations using the finite element method is proposed. Numerical results are discussed.

Keywords: waveguide, eigenfunctions, dispersion curves, finite element method

INTRODUCTION

Optical waveguides are dielectric cylindrical structures that can conduct electromagnetic energy in the visible and infrared parts of the spectrum. The waveguides used in optical communication are flexible fibers made of transparent dielectrics. The cross section of a waveguide usually consists of three regions: the central region (core) is surrounded by a cladding which, in turn, is surrounded by a protective coating. The refractive index of the core can be constant or can vary over the cross section; the refractive index of the cladding is usually constant. The coating is optically isolated from the core; for this reason, it is usually neglected in mathematical models, and it is assumed that the cladding is unbounded from the outside.

We use the classical model (see [1]), in which the waveguide is assumed to be unbounded and linearly isotropic; i.e., the refractive index n of the waveguide is invariable along the axis Ox_3 and is a piecewise continuous function of the transverse coordinates: $n = n(x)$, where $x = (x_1, x_2) \in \mathbb{R}^2$. The core, i.e., the domain Ω_i in the plane (x_1, x_2) is bounded, contains the origin, but is not necessarily simply connected. Within the cladding $\Omega_e = \mathbb{R}^2 \setminus \bar{\Omega}_i$, we have $n = n_\infty = \text{const} > 0$, and

$$\min_{x \in \mathbb{R}^2} n(x) \ge n_\infty, \quad n_+ = \max_{x \in \mathbb{R}^2} n(x) > n_\infty. \tag{1}$$

We assume that the magnetic permeability is everywhere equal to the magnetic permeability of free space μ_0.

The guided waves of a waveguide are defined as the particular solutions of Maxwell's equations

$$\mu_0 \partial \mathbb{H}/\partial t + \operatorname{curl} \mathbb{E} = 0, \quad \varepsilon \partial \mathbb{E}/\partial t - \operatorname{curl} \mathbb{H} = 0, \tag{2}$$

that have the form

$$\begin{bmatrix} \mathbb{E} \\ \mathbb{H} \end{bmatrix}(x, x_3, t) = \operatorname{Re}\left(\begin{bmatrix} E \\ H \end{bmatrix}(x)\exp[i(\omega t - \beta x_3)]\right). \tag{3}$$

Here, $\mathbb{E}$ and $\mathbb{H}$ are the vector intensities of the electric and magnetic fields, respectively; $E = (E_1, E_2, E_3)^T$ and $H = (H_1, H_2, H_3)^T$ are the amplitudes of these vectors; and

$$\int_{\mathbb{R}^2} (\varepsilon |E|^2 + \mu_0 |H|^2) dx < +\infty. \tag{4}$$

Here, $\varepsilon = \varepsilon_0 n^2$ is the dielectric constant, ε_0 is the dielectric constant of free space, $\omega > 0$ is the frequency of the electromagnetic oscillations, and $\beta > 0$ is the propagation constant.

Solutions (3) are plane harmonic waves propagating without distortions with the phase velocity ω/β in the direction of the axis Ox_3; the length of such a wave is $2\pi/\beta$. The physical meaning of condition (4) is that the energy of the wave is confined in a certain bounded domain of the waveguide cross section (surface waves are to be found).

Define the operations curl_β and div_β obtained from the operations curl and div by replacing the differentiation with respect to x_3 with the multiplication by $-i\beta$:

$$\mathrm{curl}_\beta H = \begin{bmatrix} \partial H_3/\partial x_2 + i\beta H_2 \\ -i\beta H_1 - \partial H_3/\partial x_1 \\ \partial H_2/\partial x_1 - \partial H_1/\partial x_2 \end{bmatrix}, \quad \mathrm{div}_\beta H = \frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial x_2} - i\beta H_3.$$

Substitute vectors (3) into Eqs. (2) to obtain the system of equations

$$\mathrm{curl}_\beta E = -i\omega\mu_0 H, \quad \mathrm{curl}_\beta H = i\omega\varepsilon_0 n^2 E. \tag{5}$$

Hence, using the identity $\mathrm{div}_\beta \mathrm{curl}_\beta(\cdot) = 0$, we obtain

$$\mathrm{curl}_\beta\left(\frac{1}{n^2}\mathrm{curl}_\beta H\right) = k^2 H, \quad \mathrm{div}_\beta H = 0, \tag{6}$$

$$\mathrm{curl}_\beta(\mathrm{curl}_\beta E) = k^2 n^2 E, \quad \mathrm{div}_\beta(n^2 E) = 0, \tag{7}$$

where $k = \sqrt{\mu_0\varepsilon_0}\,\omega$ is the wave number.

To design a numerical method, we must choose one of the systems of equations (6) or (7), which hold almost everywhere on the plane $\mathbb{R}^2$. For the finite element method, Eq. (6) is more convenient because the vector field H, in contrast to E, is continuous in $\mathbb{R}^2$ for all feasible values of the refractive index n. When H is determined, E can be found from the second equation in (5).

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$. Consider the problem

$$\text{find } (\beta, k) \in \mathbb{R}_+^2 \text{ and } H \in [L_2(\mathbb{R}^2)]^3, \text{ where } H \neq 0, \text{ satisfying (6).} \tag{$\mathcal{P}_1$}$$

An exact solution of problem ($\mathcal{P}_1$) is known for the waveguide with a circular cross section with a constant refractive index [1] (this solution is represented in the form of characteristic equations). The points (β, k)—the solutions of problem ($\mathcal{P}_1$) for $k \in (0, 6]$—are shown in Fig. 1a. An analysis of the characteristic equations shows that the points (β, k) belong to smooth monotonically increasing curves, which are called dispersion curves. In turn, the dispersion curves $\beta = \beta(k)$ lie on the cone

$$\Lambda = \{(\beta, k) : \beta/n_+ < k < \beta/n_\infty, \beta > 0\}$$

and have a linear asymptote as $k \longrightarrow \infty$. For each fixed $k > 0$, there is a finite number $m(k)$ of solutions $(k, \beta_i(k), H_i(k; x))$ of problem ($\mathcal{P}_1$), $m(k) \geq 2$. The function $m = m(k)$ $(k > 0)$ is nondecreasing, and $m(k) \longrightarrow \infty$ as $k \longrightarrow \infty$. The points k at which the function $m(k)$ is discontinuous are called cut-off points. At these points, the number of solutions of problem ($\mathcal{P}_1$) changes, and these points are of independent interest.

In the problem illustrated in Fig. 1, the waveguide radius is 1, the refractive index is $\sqrt{2}$, and $n_\infty = 1$. The dotted line indicates the boundaries of the regions Λ and K.

The properties described above are also true for waveguides with an arbitrary cross section and an arbitrary refractive index satisfying condition (1). A mathematical analysis based on the theory of unbounded self-adjoint operators can be found in [2]. In that paper, problem ($\mathcal{P}_1$) is considered as a problem of the form $A(\beta)H = k^2H$ with respect to the spectral parameter k^2, and the dependence $k = k(\beta)$ is studied. In [3], a similar technique is used to extend the results obtained in [2] to the case of waveguides with a variable magnetic permeability.

The results obtained in [2, 3] give a complete understanding of the qualitative properties of the spectrum of surface guided waves; however, in order to calculate the spectral characteristics of waveguides, numerical methods are needed (see survey [4] and [5, 6]). The statements of the problems used in [2, 3] are not quite convenient for obtaining numerical solutions. This is due to two specific features of those statements.

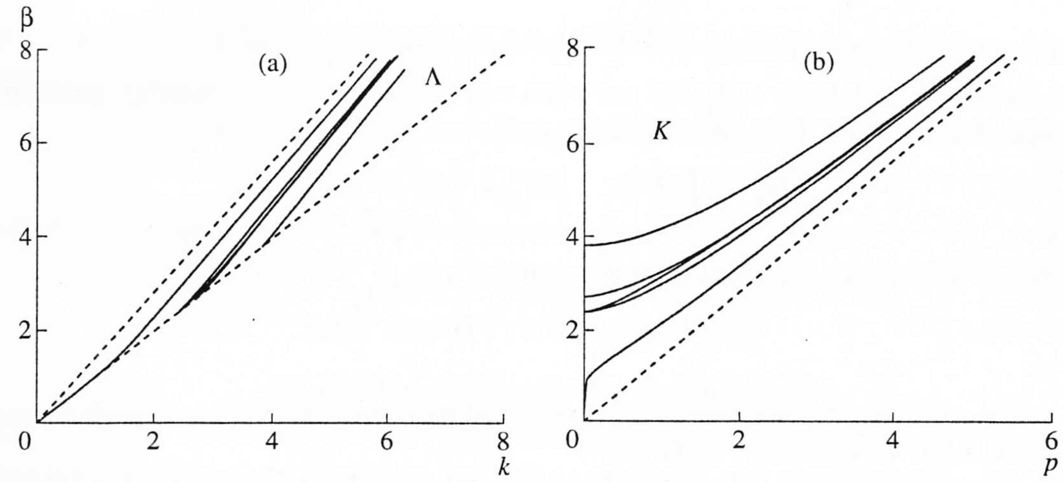

Fig. 1.

1. The problems are formulated for the entire plane $\mathbb{R}^2$. For a numerical solution, special measures must be taken to restrict the integration domain and to formulate additional boundary conditions.

2. Spectral problems (except for a point spectrum) have a continuous fragment of the spectrum. Although the location of this fragment is known exactly, a numerical solution requires that false approximate solutions be detected and discarded (see [4, 7]).

Statements of problems suggested in [5, 8] are free of those drawbacks. In those papers, nonlocal boundary value conditions (see [9–11]) are used to reduce the problems that were originally formulated for the entire plane $\mathbb{R}^2$ to equivalent problems in a circle. In [5, 8], the spectral problems are formulated in a circle $B_R \supset \Omega_i$; these problems have no continuous spectrum. Moreover, their spectrum is identical to the point part of the spectrum of problem $(\mathcal{P}_1)$. These statements are convenient for the finite element method. The cost of this advantage is that the spectral parameter appears in the equation in a nonlinear fashion; more precisely, the problems have the form $A(\beta, \lambda)H = \lambda H$, where A is a compact self-adjoint operator. The solution of such problems requires the use of special iterative methods.

In this paper, we propose a new formulation of the problem. In Section 1, the problem is first formulated in $\mathbb{R}^2$ in terms of the parameters (β, k), and, then, it is reduced to an equivalent linear eigenvalue problem[1] $\mathbf{A}(p)\mathbf{H} = \beta^2\mathbf{B}(p)\mathbf{H}$ in a bounded domain B_R. Here, the parameter is the transverse wave number $p = \sqrt{\beta^2 - k^2 n_\infty^2}$, $\mathbf{A}(p)$ and $\mathbf{B}(p)$ are compact and bounded self-adjoint operators, respectively; and $\mathbf{H} = (H_1, H_2)$ represents the first two components of the intensity vector $\mathbf{H}$. Setting $p = 0$ in the equation, we obtain an equation for finding the cut-off points (the cut-off equation). In Section 2, we examine the solvability of the problem and investigate some properties of the dispersion curves. The results obtained in this section can be immediately extended to the corresponding finite-dimensional problems stated in Section 3. In the final section, we discuss some numerical results.

1. EQUIVALENT STATEMENTS OF THE PROBLEM

1.1. Linear Spectral Problem on the Plane

First, we recall the statement of the problem given in [2]. Let $\bar{H}$ be the complex conjugate of H. For the complex-valued vector fields $H = (H_1, \dots, H_l)^\mathrm{T}$ and $H' = (H'_1, \dots, H'_l)^\mathrm{T}$ $(l \geq 1)$, we define

$$H \cdot H' = H_1 H'_1 + \dots + H_l H'_l, \quad |H|^2 = H \cdot \bar{H}, \quad \nabla H = (\nabla H_1, \dots, \nabla H_l),$$

$$\nabla H \cdot \nabla H' = \nabla H_1 \cdot \nabla H'_1 + \dots + \nabla H_l \cdot \nabla H'_l, \quad |\nabla H|^2 = \nabla H \cdot \nabla \bar{H}.$$

[1] An eigenvalue problem is called linear (respectively, nonlinear) if the spectral parameter appears in it linearly (respectively, nonlinearly).

Let $L_2(D)$ and $W_2^1(D)$ be the Lebesgue and Sobolev spaces of complex-valued scalar functions defined on the domain $D \subseteq \mathbb{R}^2$; and $H^l(D) = [L_2(D)]^l$ and $V^l(D) = [W_2^1(D)]^l$ be the corresponding spaces of l-dimensional vector functions. The following functionals define norms in these spaces:

$$|H|_D^2 = \int\limits_D |H|^2 dx, \quad \|H\|_D^2 = \int\limits_D (|\nabla H|^2 + |H|^2) dx.$$

The scalar product in $V^l(D)$ is defined in the conventional way:

$$(H, H')_D = \int\limits_D (\nabla H \cdot \nabla \bar{H}' + H \cdot \bar{H}') dx.$$

It is known that the spaces defined above are Hilbert, and the embedding $V^l(D) \subset H^l(D)$ is compact if D is bounded and has a sufficiently smooth boundary.

It was proved in [2] that the following problem is a weak formulation of problem $(\mathcal{P}_1)$: Find $(\beta, k, H) \in \Lambda \times V^3(\mathbb{R}^2) \backslash \{0\}$ such that, for any $H' \in V^3(\mathbb{R}^2)$, it holds that

$$\int\limits_{\mathbb{R}^2} \left(\frac{1}{n^2} \mathrm{rot}_\beta H \overline{\mathrm{rot}_\beta H'} + \frac{1}{n_\infty^2} \mathrm{div}_\beta H \overline{\mathrm{div}_\beta H'} \right) dx = k^2 \int\limits_{\mathbb{R}^2} H \cdot \bar{H}' dx. \qquad (\mathcal{P}_2)$$

Denote by $c(\beta; H, H')$ the Hermitian form on the left-hand side of $(\mathcal{P}_2)$.

Here, we call the reader's attention to an important advantage of the new formulation: the condition $\mathrm{div}_\beta H = 0$ (the second equation in (6)) is taken into account in the definition of the form $c(\beta; \cdot, \cdot)$ rather than in the definition of the space of solutions. As a consequence, this form is positive definite for $\beta > 0$ (see [2]):

$$c(\beta; H, H) = \frac{1}{n_+^2} \int\limits_{\mathbb{R}^2} (|\nabla H|^2 + \beta^2 |H|^2) dx \quad \forall H \in V^2(\mathbb{R}^2). \qquad (8)$$

This immediately implies the inequality $\beta^2/n_+^2 < k^2$. The coefficient $1/n_\infty^2$ in the definition of the form is chosen so as to ensure that, when $H' = 0$ in $\bar{\Omega}_i$, the identity $(\mathcal{P}_2)$ takes the form

$$c(\beta; H, H') = \frac{1}{n_\infty^2} \int\limits_{\Omega_e} (\nabla H \cdot \nabla \bar{H}' + \beta^2 H \cdot \bar{H}') dx = k^2 \int\limits_{\Omega_e} H \cdot \bar{H}' dx. \qquad (9)$$

Thus, H satisfies the homogeneous Helmholtz equation

$$-\Delta H + p^2 H = 0, \quad H \in V^3(\Omega_e), \quad p^2 = \beta^2 - k^2 n_\infty^2$$

in Ω_e. Hence, it follows that $p^2 > 0$ (see [12]); therefore, $k < \beta/n_\infty$ and

$$|H| = \exp(-p|x|) O(1/\sqrt{|x|}), \quad |x| \longrightarrow \infty.$$

When β is fixed, problem $(\mathcal{P}_2)$ is a linear eigenvalue problem with respect to k^2. The corresponding spectral problem has a noncompact resolvent (which is a consequence of the fact that the embedding $V^3(\mathbb{R}^2) \subset H^3(\mathbb{R}^2)$) is not compact), and its spectrum has the continuous part $\{k^2 \geq \beta^2/n_\infty^2\}$ in addition to the nonempty discrete part (see [2]).

Let us explain the nature of the dependence of problem $(\mathcal{P}_2)$ on the parameter β. For the vector fields $\mathbf{H} = (H_1, H_2)^\mathrm{T}$ and $H = (\mathbf{H}, H_3)^\mathrm{T}$ on $D \subseteq \mathbb{R}^2$, define $\nu = n_\infty^{-2} - n^{-2}$,

$$\mathrm{curl}\mathbf{H} = \partial H_2/\partial x_1 - \partial H_1/\partial x_2, \quad \mathrm{div}\mathbf{H} = \partial H_1/\partial x_1 + \partial H_2/\partial x_2,$$

$$a_D(\beta; H, H') = \int\limits_D \left(\frac{1}{n^2} \mathrm{curl}\mathbf{H} \mathrm{curl}\bar{\mathbf{H}}' + \frac{1}{n_\infty^2} \mathrm{div}\mathbf{H} \mathrm{div}\bar{\mathbf{H}}' + \frac{1}{n^2} \nabla H_3 \cdot \nabla \bar{H}_3' + \frac{\beta^2}{n_\infty^2} H \cdot \bar{H}' \right) dx,$$

$$b_D(H, H') = \int_D \nu \mathbf{H} \cdot \overline{\mathbf{H}}' dx,$$

$$c_D(H, H') = \int_D \nu(\nabla H_3 \cdot \overline{\mathbf{H}}' - \mathbf{H} \cdot \nabla \overline{H_3'}) dx.$$

Note that $\nu \geq 0$ in $\mathbb{R}^2$ and $\nu = 0$ in Ω_e. We also have the representation (see [2])

$$c(\beta; H, H') = a_{\mathbb{R}^2}(\beta; H, H') + i\beta c_{\mathbb{R}^2}(H, H') - \beta^2 b_{\mathbb{R}^2}(H, H'). \tag{10}$$

1.2. Nonlinear Eigenvalue Problem in a Circle

Following [8], we reduce problem $(\mathcal{P}_2)$ to a problem in a circle. Let $\Omega = B_R$ be an open circle of radius R with the boundary Γ such that $\Omega_i \subset \Omega$, $\Omega_\infty = \mathbb{R}^2 \backslash \overline{\Omega}$, and $V_0^3(\Omega_\infty) = \{H \in V^3(\Omega_\infty) : H|_\Gamma = 0\}$.

Definition 1. The vector function $H_\infty \in V^3(\Omega_\infty)$ is called a metaharmonic extension $H \in V^3(\Omega)$ to the domain Ω_∞ if $H_\infty|_\Gamma = H|_\Gamma$ and

$$a_{\Omega_\infty}(p; H_\infty, H') = 0 \quad \forall H' \in V_0^3(\Omega_\infty).$$

It was proved in [8] that a metaharmonic extension exists, is unique, and satisfies the identity (this also follows from (9), (10))

$$\int_{\Omega_\infty} (\nabla H_\infty \cdot \nabla \overline{H}' + p^2 H_\infty \cdot \overline{H}') dx = 0 \quad \forall H' \in V_0^3(\Omega_\infty). \tag{11}$$

For a given $H \in V^3(\Omega)$, we denote by $H_p \in V^3(\mathbb{R}^2)$ a function such that $H_p|_\Omega = H$ and $H_p|_{\Omega_\infty} = H_\infty$ (a gluing of H and H_∞). Let

$$K = \{(\beta, p) : \beta > 0, 0 < p < \sqrt{1 - (n_\infty/n_+)^2}\beta\}.$$

Consider the following problem: find $(\beta, p, H) \in \mathbb{R}_+^2 \times V^3(\Omega) \backslash \{0\}$ such that, for any $H' \in V^3(\Omega)$, it holds that

$$a_\Omega(p; H, H') + i\beta c_\Omega(H, H') - \beta^2 b_\Omega(H, H') + a_{\Omega_\infty}(p; H_p, H_p') = 0. \tag{$\mathcal{P}_3$}$$

The following theorem establishes the equivalence of problems $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$.

Theorem 1. *Suppose (β, k, H) is a solution of problem $(\mathcal{P}_2)$. Then, $(\beta, p, H|_\Omega)$ $(p = \sqrt{\beta^2 - k^2 n_\infty^2})$ is a solution of $(\mathcal{P}_3)$, where $(\beta, p) \in K$. Conversely, let (β, p, H) be a solution of problem $(\mathcal{P}_3)$. Then, $(\beta, p) \in K$, and (β, k, H_p), where $k = \sqrt{\beta^2 - p^2}/n_\infty$, is a solution of problem $(\mathcal{P}_2)$.*

Proof. Let (β, k, H) be a solution of $(\mathcal{P}_2)$. Taking into account (10) and the coincidence of the forms $b_{\mathbb{R}^2} = b_\Omega$ and $c_{\mathbb{R}^2} = c_\Omega$, we rewrite identity $(\mathcal{P}_2)$ as

$$(\mathcal{P}_2') : a_\Omega(p; H, H') + i\beta c_\Omega(H, H') - \beta^2 b_\Omega(H, H') + \alpha_{\Omega_\infty}(p; H, H') = 0.$$

Setting here $H' = 0$ in $\overline{\Omega}$, we obtain $H|_{\Omega_\infty} = H_\infty$ and $H = H_p$. Setting $H' = H_p'$ in $(\mathcal{P}_2')$, we obtain $(\mathcal{P}_3)$. Since $(\beta, k) \in \Lambda$, then $(\beta, p) \in K$.

Conversely, let (β, p, H) be a solution of $(\mathcal{P}_3)$. Then,

$$c(\beta; H_p, H_p') = a_{\mathbb{R}^2}(\beta - p; H_p, H_p') = \frac{\beta^2 - p^2}{n_\infty^2} \int_{\mathbb{R}^2} H_p \cdot \overline{H_p'} dx. \tag{12}$$

Setting $H'_p = H_p$ here and estimating the left-hand side from below using (8), we obtain

$$\beta^2/n_+^2 < (\beta^2 - p^2)/n_\infty^2 \Rightarrow 0 < p < \sqrt{1-(n_\infty/n_+)^2}\beta \Rightarrow (\beta, p) \in K.$$

Taking into account the definition of H'_p, we conclude from (10) that (β, k, H_p) is a solution of $(\mathcal{P}_2)$ for $k^2 = (\beta^2 - p^2)/n_\infty^2$. This completes the proof of the theorem.

In [8], the following representation of the form $a_{\Omega_\infty}(p; H_p, H'_p)$ was obtained (the derivation was based on the solution of problem (11) obtained using the method of separation of variables):

$$a_{\Omega_\infty}(p; H_p, H'_p) = \sum_{l=-\infty}^{\infty}\left(D_{|l|}(p)a_l(H)\overline{a_l(H')} + \frac{2\pi i l}{n_\infty^2}a_l(\mathbf{H}) \times \overline{a_l(\mathbf{H}')}\right). \tag{13}$$

Here, $\mathbf{H} \times \mathbf{H}' = H_1H'_2 - H_2H'_1$, (r, φ) are the polar coordinates of the point $x \in \Gamma$,

$$D_l(p) = \frac{2\pi}{n_\infty^2}[l + \mathbb{K}_l(pR)], \quad \mathbb{K}_l(z) = z\frac{K_{l-1}(z)}{K_l(z)},$$

$$a_l(H) = \frac{1}{2\pi}\int_0^{2\pi} H(R, \varphi)e^{-il\varphi}d\varphi,$$

and $K_l(z)$ is the modified Bessel function of order l (see [13]). Note the following properties of the functions $z \longrightarrow \mathbb{K}_l(z)$ $(l \geq 0, z \geq 0)$ (see [8]). These functions are positive, monotonically increasing, and $\mathbb{K}_l \in C^\infty(\mathbb{R}_+)$; $\mathbb{K}_0(z) = O(-1/\ln(z))$ as $z \longrightarrow 0$; $\mathbb{K}_l(z) \leq z$, and $\mathbb{K}'_l(z) \leq 2l$ when $l \geq 1$. These properties imply that Hermitian form (13) is bounded in $V^3(\Omega) \times V^3(\Omega)$ (see [8, and Lemma 1 below in this paper]).

In [8], problem $(\mathcal{P}_3)$ was considered as a nonlinear eigenvalue problem with respect to the parameter p. Using the theory of compact self-adjoint operators, it proved possible to investigate the solvability of this problem and establish some properties of the dispersion curves. However, it was not observed in [8] that, in spite of the degeneracy of the form b_Ω and definiteness of the form ic_Ω (these forms are Hermitian), problem $(\mathcal{P}_3)$ with respect to the parameter β is simpler. In the next section, we consider this problem.

1.3. Linear Eigenvalue Problem in a Circle

Let us represent Eq. $(\mathcal{P}_3)$ in the block form in accordance with the decomposition $H = (\mathbf{H}, H_3)^\mathrm{T}$, $\mathbf{H} = (H_1, H_2)^\mathrm{T}$. Below, we assume that $V = V^1(\Omega)$, $V^2 = V^2(\Omega)$, $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$, $\|\cdot\| = \|\cdot\|_\Omega$, and $|\cdot| = |\cdot|_\Omega$. For $p \geq 0$, define the operators $\mathbf{A}(p)$, $\mathbf{B}_0 : V^2 \longrightarrow V^2$, $\mathbf{C}_0 : V^2 \longrightarrow V$, and $L(p) : V \longrightarrow V$ using the corresponding forms (H, $H' \in V$ and $\mathbf{H}, \mathbf{H}' \in V^2$ are arbitrary functions):

$$(\mathbf{A}(p)\mathbf{H}, \mathbf{H}') = \int_\Omega\left(\frac{1}{n^2}\mathrm{curl}\mathbf{H}\mathrm{curl}\overline{\mathbf{H}}' + \frac{1}{n_\infty^2}\mathrm{div}\mathbf{H}\mathrm{div}\overline{\mathbf{H}}' + \frac{p^2}{n_\infty^2}\mathbf{H}\cdot\overline{\mathbf{H}}'\right)dx$$

$$+ \sum_{l=-\infty}^{\infty}\left(D_{|l|}(p)a_l(\mathbf{H})\cdot\overline{a_l(\mathbf{H}')} + \frac{2\pi i l}{n_\infty^2}a_l(\mathbf{H}) \times \overline{a_l(\mathbf{H}')}\right),$$

$$(\mathbf{B}_0\mathbf{H}, \mathbf{H}') = \int_\Omega \nu(x)\mathbf{H}\cdot\overline{\mathbf{H}}'dx, \quad (\mathbf{C}_0\mathbf{H}, H') = \int_\Omega \nu(x)\mathbf{H}\cdot\nabla\overline{H}'dx,$$

$$(L(p)H, H') = \int_\Omega\left(\frac{1}{n^2}\nabla H\cdot\nabla\overline{H}' + \frac{p^2}{n_\infty^2}H\overline{H}'\right)dx + \sum_{l=-\infty}^{\infty} D_{|l|}(p)a_l(H)\overline{a_l(H')}.$$

The properties of these forms and of the corresponding operators will be established in the next section.

Equation $(\mathcal{P}_3)$ is equivalent to the system of equations

$$\begin{pmatrix} \mathbf{A}(p) & -i\beta C_0^* \\ i\beta \mathbf{C}_0 & L(p) \end{pmatrix} \begin{pmatrix} \mathbf{H} \\ H_3 \end{pmatrix} = \beta^2 \begin{pmatrix} \mathbf{B}_0 & 0 \\ \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{H} \\ H_3 \end{pmatrix}, \tag{14}$$

where $C_0^* : V \longrightarrow V^2$ is the adjoint operator of $\mathbf{C}_0$.

Eliminating H_3 from this system (the operator $L(p)$ is positive definite for $p > 0$ because $D_{|l|}(p) \geq 0$ for all l), we obtain the following linear eigenvalue problem with respect to the spectral parameter β^2: find $(\beta, p, \mathbf{H}) \in K \times V^2 \backslash \{0\}$ satisfying the equations

$$\mathbf{A}(p)\mathbf{H} = \beta^2 \mathbf{B}(p)\mathbf{H}, \quad \mathbf{B}(p) = \mathbf{B}_0 + C_0^* L(p)^{-1} \mathbf{C}_0. \tag{$\mathcal{P}_4$}$$

The third component of the eigenfunction is determined by the formula

$$H_3 = -i\beta L^{-1}(p)\mathbf{C}_0\mathbf{H}. \tag{15}$$

It is obvious that problem $(\mathcal{P}_3)$ and linear problem $(\mathcal{P}_4)$ with determining H_3 by formula (15) are equivalent.

Remark 1. The operators $\mathbf{A}(p)$ and $\mathbf{B}(p)$ in problem $(\mathcal{P}_4)$ are defined on the space of complex-valued functions V^2. However, it will be proved below that they are self-adjoint real operators (i.e., they map real functions to real ones). Hence, an eigenvector $\mathbf{H}$ of problem $(\mathcal{P}_4)$ can be chosen in the set of real-valued functions $[W_2^1(\Omega)]^2$, and, as a consequence, the guided waves H of the waveguide can be defined so that $H = (H_1, H_2, iH_3)$, where H_l belong to the real space $W_2^1(\Omega)$ $(l = 1, 2, 3)$.

2. EXISTENCE AND PROPERTIES OF DISPERSION CURVES

Before examining the solvability of problem $(\mathcal{P}_4)$, we establish some properties of the operators involved in it.

Lemma 1. *Suppose that, for an integer* $N \in [0, \infty]$, *any* $\mathbf{H}, \mathbf{H}' \in V^2$, *and* $p \geq 0$, *it holds that*

$$s_N(p; \mathbf{H}, \mathbf{H}') = \sum_{l=-N}^{N} \left(D_{|l|}(p) a_l(\mathbf{H}) \cdot \overline{a_l(\mathbf{H}')} + \frac{2\pi i l}{n_\infty^2} a_l(\mathbf{H}) \times \overline{a_l(\mathbf{H}')} \right).$$

Then, the following assertions hold:

(a) $s_N(p; \mathbf{H}, \mathbf{H}') \in \mathbb{R}$ *if* $\mathbf{H}$ *and* $\mathbf{H}'$ *are real functions;*

(b) $s_N(p; \mathbf{H}, \mathbf{H}) \geq 0$, *and this function does not decrease in* p;

(c) $s_N(p; \mathbf{H}, \mathbf{H}) \leq \mu(p) \|\mathbf{H}\|^2$, *where* $\mu(p) = 4\pi c_{1/2}^2 \max\{2, pR, \mathbb{K}_0(pR)/2\}/n_\infty^2$;

(d) $ds_N(p; \mathbf{H}, \mathbf{H})/dp \leq \mu'(p) \|\mathbf{H}\|^2$.

Here, $c_{1/2}$ *is the constant of the embedding* $W_2^1(\Omega) \subset W_2^{1/2}(\Gamma)$ *and* $\mu'(p)$ *is the derivative of* $\mu(p)$ *(if* $\mu(p) = \max\{a(p), b(p)\}$, *then* $\mu'(p) = \max\{a'(p), b'(p)\}$).

Proof. Since $a_{-l}(\overline{H}) = \overline{a_l(H)}$, assertions (a) and (b) are immediate consequences of the representation

$$s_N(p; \mathbf{H}, \mathbf{H}') = \mathbb{K}_0(pR) a_0(\mathbf{H}) \cdot \overline{a_0(\mathbf{H}')} + \sum_{l=1}^{N} \mathbb{K}_l(pR)(a_l(\mathbf{H}) \cdot \overline{a_l(\mathbf{H})'} + \overline{a_l(\overline{\mathbf{H}})} \cdot a_l(\overline{\mathbf{H}}'))$$

$$+ \frac{2\pi}{n_\infty^2} \sum_{l=1}^{N} (a_l(F)\overline{a_l(F')} + \overline{a_l(\overline{F})} a_l(\overline{F}'))$$

(which can be verified straightforwardly) and the properties of the functions $\mathbb{K}_l(z)$. Here, $F = H_1 - iH_2$.

The norm in the space $[W_2^{1/2}(\Gamma)]^2$ is defined by

$$\|\mathbf{H}\|^2_{1/2,\Gamma} = \sum_{l=-\infty}^{\infty} (|l|^2+1)^{1/2}|a_l(\mathbf{H})|^2$$

and $\|\mathbf{H}\|_{1/2,\Gamma} \le c_{1/2}\|\mathbf{H}\|$ for any $\mathbf{H} \in V^2$. Assertion (c) immediately follows from the simple estimates

$$s_N(p;\,\mathbf{H},\mathbf{H}) \le \sum_{l=-N}^{N} d_l(p)|a_l(\mathbf{H})|^2, \quad d_l(p) = D_{|l|} + \frac{2\pi}{n_\infty^2}|l|,$$

$\max\{d_0(p), d_l(p)/|l|, l \ge 1\} \le \mu(p)/c_{1/2}^2$:

$$s_N(p;\,\mathbf{H},\mathbf{H}) \le \mu(p)/c_{1/2}^2\|\mathbf{H}\|^2_{1/2,\Gamma} \le \mu(p)\|\mathbf{H}\|^2.$$

Assertion (d) is proved in a similar fashion as assertion (c).

Lemma 2. *The operator* $\mathbf{A}(p)$ *is self-adjoint, real, and positive definite for any* $p > 0$. *The operator function* $\mathbf{A}(p)$ *is continuously differentiable and increasing on* $\mathbb{R}_+$; *i.e., the function* $p \longrightarrow (\mathbf{A}(p)\mathbf{H}, \mathbf{H})$ *is increasing for any fixed* $\mathbf{H} \in V^2$). *Moreover,*

$$\begin{gathered}(\mathbf{A}(p)\mathbf{H},\mathbf{H}) \ge m_A(p)\|\mathbf{H}\|^2, \quad m_A(p) = \min\{1, p^2\}/n_+^2, \\ \|\mathbf{A}(p)\| \le M_A(p), \quad \|d\mathbf{A}(p)/dp\| \le M'_A(p),\end{gathered} \tag{16}$$

where $M_A(p) = (\max\{2, p^2\} + 2\pi c_{1/2}^2\max\{pR, \mathbb{K}_0(pR)\})/n_\infty^2$.

Proof. Let $H = (\mathbf{H}, 0)^{\mathrm{T}} \in V^3$ and $H_p = (\mathbf{H}_p, 0)^{\mathrm{T}} \in V^3(\mathbb{R}^2)$ be determined from H using the metaharmonic extension (in the definition of the form a_{Ω_∞} and in (1), (13), we must set $H_3 \equiv 0$). Then,

$$(\mathbf{A}(p)\mathbf{H},\mathbf{H}') = \int_{\mathbb{R}^2}\left(\frac{1}{n^2}\mathrm{curl}\,\mathbf{H}_p\mathrm{curl}\,\overline{\mathbf{H}'_p} + \frac{1}{n_\infty^2}\mathrm{div}\,\mathbf{H}_p\mathrm{div}\,\overline{\mathbf{H}'_p} + \frac{p^2}{n_\infty^2}\mathbf{H}_p\cdot\overline{\mathbf{H}'_p}\right)dx.$$

Since $n_\infty \le n(x) \le n_+$ in $\mathbb{R}^2$, then

$$(\mathbf{A}(p)\mathbf{H},\mathbf{H}) \ge \frac{1}{n_+^2}\int_{\mathbb{R}^2}(|\mathrm{curl}\,\mathbf{H}_p|^2 + |\mathrm{div}\,\mathbf{H}_p|^2 + p^2|\mathbf{H}_p|^2)dx = \frac{1}{n_+^2}\int_{\mathbb{R}^2}(|\nabla\mathbf{H}_p|^2 + p^2|\mathbf{H}_p|^2)dx \ge m_A(p)\|\mathbf{H}\|^2.$$

Here, the equality is verified using the integration by parts on the functions belonging to $[C_0^\infty(\mathbb{R}^2)]^2$, which is dense in $[W_2^1(\mathbb{R}^2)]^2$. To prove (16), we use assertion (c) in Lemma 1. We have

$$\begin{aligned}(\mathbf{A}(p)\mathbf{H},\mathbf{H}) &\le \frac{1}{n_\infty^2}\int_\Omega(|\mathrm{curl}\,\mathbf{H}|^2 + |\mathrm{div}\,\mathbf{H}|^2 + p^2|\mathbf{H}|^2)dx + s_\infty(p;\,\mathbf{H},\mathbf{H}) \\ &\le \frac{\max\{2, p^2\}}{n_\infty^2}\int_\Omega(|\nabla\mathbf{H}|^2 + |\mathbf{H}|^2)dx + s_\infty(p;\,\mathbf{H},\mathbf{H}) \le M_A(p)\|\mathbf{H}\|^2.\end{aligned}$$

Bound (d) in Lemma 1 implies

$$\left(\frac{d\mathbf{A}(p)}{dp}\mathbf{H},\mathbf{H}\right) = \frac{2p}{n_\infty^2}\int_\Omega|\mathbf{H}|^2dx + \frac{d}{dp}s_\infty(p;\,\mathbf{H},\mathbf{H}) \le M'_A(p)\|\mathbf{H}\|^2.$$

The other assertions of the lemma immediately follow from the definition of the operator $\mathbf{A}(p)$ and Lemma 1.

The following lemma is proved analogously.

Lemma 3. *The operator $L(p)$ is self-adjoint, real, and positive definite for any $p > 0$; the operator function $L(p)$ is continuously differentiable and increasing on $\mathbb{R}_+$. Moreover,*

$$(L(p)H, H) \ge m_L(p)\|H\|^2, \quad \|L(p)\| \le M_L(p), \quad \|dL(p)/dp\| \le M'_L(p),$$

where $m_L(p) = \min\{1, p^2\}/n_+^2$ and $M_L(p) = \max\{1, p^2\}/n_\infty^2 + \mu(p)$.

Lemma 4. *The operator $\mathbf{B}(p)$ is self-adjoint, real, positive semidefinite, and compact for any $p > 0$; the operator function $\mathbf{B}(p)$ is continuously differentiable and nonincreasing on $\mathbb{R}_+$. Moreover,*

$$0 \le (\mathbf{B}(p)\mathbf{H}, \mathbf{H}) < M_B|\mathbf{H}|^2, \quad M_B = \nu_+ + (\nu_+ n_+)^2, \tag{17}$$

$$\left|\left(\frac{d}{dp}\mathbf{B}(p)\mathbf{H}, \mathbf{H}\right)\right| \le M'_B(p)\|\mathbf{H}\|^2, \quad M'_B(p) = 2\nu_+^2 n_+^2/p + \mu'(p), \tag{18}$$

where $\nu_+ = \max\{\nu(x), x \in \mathbb{R}^2\} = n_\infty^{-2} - n_+^{-2} > 0$.

Proof. The lower bound in (17) follows from the representation

$$(\mathbf{B}(p)\mathbf{H}, \mathbf{H}) = (\mathbf{B}_0\mathbf{H}, \mathbf{H}) + (L^{-1}(p)\mathbf{C}_0\mathbf{H}, \mathbf{C}_0\mathbf{H}),$$

and the nonnegativity of $\mathbf{B}_0$ and $L^{-1}(p)$. Let us prove the upper bound. Define $H = L^{-1}(p)\mathbf{C}_0\mathbf{H}$. The equality $(L(p)H, H) = (\mathbf{C}_0\mathbf{H}, H)$ implies the bounds (see the definitions of the forms) $|\nabla H| < \nu_+ n_+^2|\mathbf{H}|$ and $p|H| \le \nu_+ n_+ n_\infty|\mathbf{H}|$. Furthermore,

$$(\mathbf{B}(p)\mathbf{H}, \mathbf{H}) = (\mathbf{B}_0\mathbf{H}, \mathbf{H}) + (C_0^* H, \mathbf{H}) = \int_\Omega \nu|\mathbf{H}|^2 dx + \int_\Omega \nu\nabla H \cdot \bar{\mathbf{H}} dx \le \nu_+|\mathbf{H}|^2 + \nu_+|\nabla H||\mathbf{H}| \le M_B|\mathbf{H}|^2.$$

The compactness of $\mathbf{B}(p)$ is a consequence of (17) and of the compactness of the embedding $V^2(\Omega) \subset H^2(\Omega)$. In the proof of (18), we use the following bound on the vector H defined above:

$$(d\mathbf{B}(p)/dp\mathbf{H}, \mathbf{H}) = (L^{-1}(p)dL(p)/dpL^{-1}(p)\mathbf{C}_0\mathbf{H}, \mathbf{C}_0\mathbf{H})$$

$$= (dL(p)/dpH, H) = \frac{2p}{n_\infty^2}|H|^2 + \sum_{l=-\infty}^{\infty} D'_{|l|}(p)|a_l(H)|^2 \le M'_B(p)\|\mathbf{H}\|^2.$$

The other assertions of the lemma immediately follow from the definition of the operator $\mathbf{B}(p)$ and the properties of $L(p)$. The lemma is proved.

Theorem 2. *The set of solutions of problem $(\mathcal{P}_4)$ can be represented as $\{(\beta_l(p), p, \mathbf{H}_l(p)), p > 0, l = 1, 2, \ldots\}$. Moreover, the following assertions hold:*

(a) $\beta_1(p) \le \ldots \le \beta_l(p) \le \ldots, \beta_l(p) \longrightarrow \infty$ *as* $l \longrightarrow \infty$; *any $\beta_i(p)$ has a finite multiplicity (i.e., $\beta_i(p)$ can coincide only with a finite number of $\beta_j(p), j \ge 1$).*

(b) *The functions $p \longrightarrow \beta_l(p)$ are monotonically increasing and have the Lipschitz property.*

(c) $(\mathbf{A}(p)\mathbf{H}_i(p), \mathbf{H}_j(p)) = \delta_{ij}$.

(d) $\beta_2(p) \longrightarrow 0$ *as* $p \longrightarrow 0$.

Proof. Let us represent $(\mathcal{P}_4)$ in the form

$$\mathbf{B}(p)\mathbf{H} = \lambda(p)\mathbf{A}(p)\mathbf{H}, \quad \mathbf{H} \in V^2\backslash\{0\}, \quad \lambda(p) = \beta^{-2}(p).$$

Here, $\mathbf{A}(p)$ is a self-adjoint bounded positive definite operator and $\mathbf{B}(p)$ is a self-adjoint compact positive semidefinite operator. It follows from the theory of compact operators that, for each $p > 0$, this problem has a countable number of positive eigenvalues of a finite multiplicity. We index them in descending order indicating each value with account for its multiplicity: $\lambda_1(p) \ge \ldots, \lambda_l(p) \longrightarrow 0$ as $l \longrightarrow \infty$. Thus, we have the sequence $\beta_l(p) = 1/\lambda_l(p)$, $(\beta_l(p), p) \in K$, $\beta_1(p) \le \ldots \le \beta_l(p) \le \ldots, \beta_l(p) \longrightarrow \infty$ as $l \longrightarrow \infty$. The eigenfunctions $\mathbf{H}_l(p)$ $(l = 1, 2, \ldots)$ corresponding to these eigenvalues are orthonormal: $(\mathbf{A}(p)\mathbf{H}_i(p), \mathbf{H}_j(p)) = \delta_{ij}$.

According to the Courant–Fisher principle,

$$\lambda_l(p) = \max_{V_l^2} \min_{\mathbf{H} \in V_l^2\backslash\{0\}} R(p, \mathbf{H}), \quad R(p, \mathbf{H}) = \frac{(\mathbf{B}(p)\mathbf{H}, \mathbf{H})}{(\mathbf{A}(p)\mathbf{H}, \mathbf{H})},$$

where the maximum is taken over all l-dimensional subspaces of V^2.

The monotonicity of the Rayleigh ratio in p implies that $\lambda_l(p)$ are decreasing functions and $\beta_l(p)$ are increasing functions of p ($l = 1, 2, \ldots$).

Since the operators $\mathbf{A}(p)$ and $\mathbf{B}(p)$ are differentiable, the Rayleigh ratio $R(p, \mathbf{H})$ is also differentiable and $\lambda_l(p)$ are locally Lipschitz functions for $p > 0$. Indeed, taking into account the fact that

$$\frac{d}{dp}R(p, \mathbf{H}) = \frac{([dA(p)/dp - R(p, \mathbf{H})dB(p)/dp]\mathbf{H}, \mathbf{H})}{(A(p)\mathbf{H}, \mathbf{H})},$$

and the bound $R(p, \mathbf{H}) \le \lambda_1(\text{p})$ for any $\mathbf{H} \in V^2$, we obtain

$$\left|\frac{d}{dp}R(p, \mathbf{H})\right| \le [M'_A(p) + \lambda_1(p)M'_B(p)]\lambda_1(p)/m_A(p) \equiv M'_R(p) < \infty.$$

Furthermore, let $E_l(p)$ be the linear hull of the eigenvectors $\mathbf{H}_1(p), \ldots, \mathbf{H}_l(p)$, and $\{p, p'\}$ be the segment connecting the points p and p'. We have

$$\lambda_l(p') = \max_{V_l^2} \min_{\mathbf{H} \in V_l^2\backslash\{0\}} R(p', \mathbf{H}) \ge \min_{\mathbf{H} \in V_l^2\backslash\{0\}} R(p', \mathbf{H})$$

$$\ge \min_{\mathbf{H} \in V_l^2\backslash\{0\}} R(p', \mathbf{H}) - \max_{\mathbf{H} \in E_l(p)\backslash\{0\}} |R(p', \mathbf{H}) - R(p, \mathbf{H})|$$

$$\ge \lambda_l(p) - c(p, p')|p' - p|, \quad c(p, p') = \max_{s \in \{p, p'\}} M'_R(s).$$

Interchanging p and p', we obtain $|\lambda_l(p') - \lambda_l(p)| \le c(p, p')|p' - p|$. This inequality implies that $\beta_l(p)$ are locally Lipschitz functions because $\beta_l(p) > 0$.

Define the vector fields $H_1 = (1, 0, 0)^T$ and $H_2 = (0, 1, 0)^T$, which are constant in Ω. Denote the linear hull of these fields by V_2^2. We have $V_2^2 \subset V^2$. Let $\mathbf{H} = c_1H_1 + c_2H_2$ and $|c|^2 = |c_1|^2 + |c_2|^2$. Then,

$$R(p, \mathbf{H}) \ge \frac{\nu_0|c|^2}{[\pi R^2p^2/n_\infty^2 + D_0(p)]|c|^2} \equiv \frac{1}{\theta(p)}, \quad \nu_0 = \int_\Omega \nu(x)dx > 0.$$

According to the Courant–Fisher principle, we have

$$\lambda_2(p) \ge \min_{\mathbf{H} \in V_2^2\backslash\{0\}} R(p, \mathbf{H}) \ge 1/\theta(p).$$

Therefore, $\beta_2^2(p) \le \theta(p) \longrightarrow 0$ as $p \longrightarrow 0$. This completes the proof.

The dispersion curves $\beta = \beta_l(p)$ for a homogeneous waveguide with a circular cross section are shown in Fig. 1b. Due to the symmetry of the problem, the upper curve is multiple: $\beta_1(p) = \beta_2(p) < \beta_3(p)$ for $p > 0$.

We defined the dispersion curves $\beta = \beta_l(p)$ for $p > 0$. However, it is clear that they can be continuously extended and defined at $p = 0$. The operator functions $\mathbf{A}(p)$ and $\mathbf{B}(p)$ are differentiable for $p > 0$ and uniformly bounded in p for $0 < p \le 1$ (see Theorems 2 and 4). Therefore, the self-adjoint operators $\mathbf{A}(0)$ (a bounded operator) and $\mathbf{B}(0)$ (a compact operator) can be continuously defined; for this purpose, it is sufficient to set $p = 0$ in the definition of $\mathbf{A}(p)$ and $\mathbf{B}(p)$, respectively, taking into account that $\mathbb{K}_l(0) = 0$. The problem of finding $(\beta(0), \mathbf{H}) \in \mathbb{R}_+ \times V^2\backslash\{0\}$ such that

$$\mathbf{A}(0)\mathbf{H} = \beta^2(0)\mathbf{B}(0)\mathbf{H}, \quad \mathbf{B}(0) = \mathbf{B}_0(0) + C_0^*L^{-1}(0)\mathbf{C}_0, \qquad (\mathcal{P}_4^0)$$

which is the limiting problem $(\mathcal{P}_4)$, is called the cut-off equation; the numbers $\beta_l(0)$ ($l = 1, 2, \ldots$) are called the cut-off numbers.

Theorem 3. *Problem* $(\mathcal{P}_4^0)$ *has a countable number of solutions* $(\beta_l(0), \mathbf{H}_l(0))$ ($l = 1, 2, \ldots$), *where* $\mathbf{H}_1(0) = (1, 0)^T$ *and* $\mathbf{H}_2(0) = (0, 1)^T$ *in* Ω,

$$0 = \beta_1(0) = \beta_2(0) < \beta_3(0) \le \ldots \le \beta_l(0) \le \ldots, \quad \beta_l(0) \longrightarrow \infty, \quad l \longrightarrow \infty.$$

At a fixed $\beta > 0$, *the number of solutions of problem* $(\mathcal{P}_4)$ *is*

$$m(\beta) = \max\{l : \beta_l(0) < \beta, l = 1, 2, \dots\}.$$

Proof. Consider problem $(\mathcal{P}_4)$ for $p \geq 0$. Due to the continuity of the operators $\mathbf{A}(p)$ and $\mathbf{B}(p)$ in p, we have

$$(\beta_l(0), \mathbf{H}_l(0)) = \lim_{p \to 0}(\beta_l(p), \mathbf{H}_l(p)).$$

It is easy to verify that the kernel of $\mathbf{A}(0)$ is two-dimensional, and the functions $\mathbf{H}_1(0)$ and $\mathbf{H}_2(0)$ form a basis in it. Therefore, $\beta_1(0) = \beta_2(0) = 0$ (see also assertion (d) of Theorem 2) and $\beta_3(0) > 0$. The last assertion of the theorem is a consequence of the continuity and the monotone increase of the functions $p \longrightarrow \beta_l(p)$ ($l = 1, 2, \dots, p \geq 0$).

Corollary 1. For any $\beta > 0$, $m(\beta) \geq 2$; for $\beta \leq \beta_3(0)$, $m(\beta) = 2$.

Remark 2. Note that the operator $\mathbf{B}(0) = \mathbf{B}_0 + C_0^* L^{-1}(0)\mathbf{C}_0$ is defined on the entire space V even though $L^{-1}(0)$ is not. The kernel of $L(0)$ consists of the functions that are constant on Ω. Let us define the operator $\tilde{L} : V \longrightarrow V$ by

$$(\tilde{L}H, H') = (L(0), H, H') + \alpha_0(H)\overline{a_0(H')}, \quad H, H' \in V.$$

It is easy to verify that $\tilde{L}$ is positive definite and $L^{-1}(0)\mathbf{C}_0 H = \tilde{L}^{-1}\mathbf{C}_0 H$ $\forall H \in V$. Therefore, below we assume that $L(0) = \tilde{L}$.

3. APPROXIMATE SOLUTION OF THE PROBLEM

Taking into account Remark 1, we describe a numerical method for solving problem $(\mathcal{P}_4)$. Upon discretization (which will be described later), we obtain the sparse real matrices $\mathbf{A}_h^N(p)$, $\mathbf{B}_{0h}^N$, $\mathbf{C}_{0h}^N$, and $L_h^N(p)$, which are discrete analogs of the operators $\mathbf{A}(p)$, $\mathbf{B}_0$, $\mathbf{C}_0$, and $L(p)$, respectively. These approximations depend on two parameters: the real parameter h (the characteristic size of finite elements, $h \longrightarrow 0$) and the integer parameter N, which specifies the number of the Fourier harmonics taken into account ($N \longrightarrow \infty$). Since we assume that the domain $\Omega = B_R$ (a circle of radius R) is fixed, we do not explicitly indicate the dependence on the third parameter R of the problem here.

The finite-dimensional approximation of $(\mathcal{P}_4)$ is naturally described as the generalized algebraic eigenvalue problem

$$\mathbf{A}_h^N(p)\mathbf{H}_h^N = \lambda_h^N \mathbf{B}_h^N(p)\mathbf{H}_h^N, \quad \mathbf{B}_h^N(p) = \mathbf{B}_{0h}^N + (\mathbf{C}_{0h}^N)^{\mathrm{T}}[L_h^N(p)]^{-1}\mathbf{C}_{0h}^N. \tag{19}$$

Here, $\mathbf{A}_h^N(p)$ and $\mathbf{B}_h^N(p)$ are large symmetric matrices, $\mathbf{A}_h^N(p)$ is sparse and positive definite, and $\mathbf{B}_h^N(p)$ is full (not sparse), positive semidefinite, and such that there is an efficient method for multiplying this matrix by a vector (after an LU factorization of the matrix $L_h^N(p)$). For each fixed $p \geq 0$, we need to know how to find all the eigenvalues $\lambda_h^N = \lambda_n^N(p)$ of problem (19) belonging to the given interval $(p^2/(1-(n_\infty/n_+)^2), \beta_{\max}^2)$ and the corresponding eigenvectors $H_h^N = H_h^N(p)$.

There are many methods that solve problem (19) subject to the constraints specified above. We used the Lanczos method to solve the equivalent eigenvalue problem

$$Ax = \mu x, \quad A = \mathbf{S}^{-\mathrm{T}}\mathbf{B}_h^N(p)\mathbf{S}^{-1}, \quad H_h^N = \mathbf{S}^{-1}x, \quad \lambda_h^N = 1/\mu. \tag{20}$$

This method only requires that the product Ax can be calculated. Here, $\mathbf{A}_h^N(p) = \mathbf{S}^{\mathrm{T}}\mathbf{S}$ is an LU factorization of $\mathbf{A}_h^N(p)$. For $p = 0$, we used the regularization $\mathbf{A}_h^N(p) = \mathbf{A}_h^N(0) + \varepsilon I$ with a sufficiently small ε and the identity matrix I.

To construct a discrete approximation of the problem, we used the finite element method with a numerical integration based on a conformal approximation V_h of the real Sobolev space $W_2^1(\Omega)$ (see [14, p. 47; 15]).

Let us give a short description of the construction of V_h. Let $\mathcal{T}_h$ be a family of triangular finite elements e that form an exact regular triangulation of Ω; i.e.,

$$\bar{\Omega}_h = \bigcup_{e \in \mathcal{T}_h} \bar{e} = \bar{\Omega}, \quad h = \max_{e \in \mathcal{T}_h} \operatorname{diam}(e).$$

We assume that triangular elements e of two types can be used: elements with three rectilinear sides and elements with two rectilinear and one curvilinear side (curved elements). From the practical point of view, it is natural to use curved elements only as the boundary ones when two vertices of the triangle e lie on $\partial\Omega$ or $\partial\Omega_i$. Recall that $\Omega = B_R \supset \Omega_i$, where Ω_i is the waveguide cross section.

Consider a fixed element $e \in \mathcal{T}_h$ with the vertices a_1, a_2, and a_3. We assume that its two sides a_1a_2 and a_1a_3 are rectilinear, and the third side lies on the curve Γ parameterized by the arc coordinate s: $x = \chi(s)$, $s \in [0, L]$, $a_2 = \chi(s_2)$, $a_3 = \chi(s_3)$, and $l_e = s_3 - s_2$ is the length of the arc a_2a_3. If $\hat{e}$ is a basis triangle in the plane $(\hat{x}_1\hat{x}_2)$ with the vertices $\hat{a}_1 = (0, 0)$, $\hat{a}_2 = (1, 0)$, and $\hat{a}_3 = (0, 1)$, then the mapping

$$x = x_e(\hat{x}) = a_1 + (a_2 - a_1)\hat{x}_1 + (a_3 - a_1)\hat{x}_2 + \tilde{x}_e(\hat{x}),$$

$$\tilde{x}_e(\hat{x}) = \frac{\hat{x}_1}{1 - \hat{x}_2}\{\chi(s_2 + l_e\hat{x}_2) - [a_2 + (a_3 - a_2)\hat{x}_2]\}$$

defines a transformation of $\hat{e}$ into e that preserves the orientation and satisfies the relation $a_i = x_e(\hat{a}_i)$. It is known (see [14, p. 48]) that this is a one-to-one diffeomorphism if Γ is sufficiently smooth ($\Gamma \in C^2$) and h is sufficiently small. Denote by $\hat{x} = \overset{-1}{x_e}(x)$ the inverse mapping and by $\hat{P}_q$ the set of polynomials

$$\hat{P}_q = \left\{ \sum_{|\alpha| \le q} c_\alpha \hat{x}_1^{\alpha_1} \hat{x}_2^{\alpha_2}, c_\alpha \in \mathbb{R} \right\}, \quad \dim \hat{P}_q = m \equiv (q+1)(q+2)/2,$$

of degree $q \ge 1$ on $\hat{e}$, and define the following space of functions on e:

$$P_q^e = \{(p : p(x) = \hat{p}(\overset{-1}{x_e}(x))), \hat{p} \in \hat{P}_q, x \in e\}.$$

If $\hat{\varphi}_i(\hat{x})$ $(i = 1, 2, \dots, m)$ is the Lagrange basis in $\hat{P}_q$ determined by the interpolation nodes $\hat{\omega} = \{\hat{a}_i, i = 1, 2, \dots, m\}$ (i.e., $\hat{\varphi}_i \in \hat{P}_q$, $\hat{\varphi}_i(\hat{a}_j) = \delta_{ij}$, $i, j = 1, 2, \dots, m$), then the Lagrange basis in P_q^e consists of the functions $\varphi_i^e(x) = \hat{\varphi}_i(x^{-1}(x))$, $i = 1, 2, \dots, m$. They are related to the corresponding interpolation nodes $\omega^e = \{a_i^e = x_e(\hat{a}_i), \hat{a}_i \in \hat{\omega}\}$ on the element e.

Note that if Γ is a straight segment (i.e., all the sides of e are rectilinear), then $\tilde{x}_e(\hat{x}) \equiv 0$, $x = x_e(\hat{x})$ is an affine mapping, and P_q^e is the space of polynomials of degree not greater than q. In the general case, the functions P_q^e are not polynomials; however, it can be easily verified that the restriction of an arbitrary function from P_q^e to an arbitrary side e (including curvilinear sides) is a polynomial of degree not greater than q in the arc coordinate along this side.

Now, the approximation V_h of the space $V = W_2^1(\Omega)$ is defined as $V_h = \{v_h \in C(\bar{\Omega}) : v_h|_e \in P_q^e \ \forall e \in \mathcal{T}_h\}$. This space is finite-dimensional, and the Lagrange basis in it is defined in the conventional way: if $\omega_h = \{a_i, i = 1, 2, \dots, n\}$ is a grid on Ω formed by various points from $\bigcup\{\omega^e, e \in \mathcal{T}_h\}$, then the node $a_l \in \omega_h$ is assigned the basis function $\varphi_l(x)$ such that $\varphi_l(a_j) = \delta_{lj}$ $(l, j = 1, 2, \dots, n)$. By definition,

$$u_h(x) = \sum_{i=1}^{n} c_i\varphi_i(x), \quad x \in \bar{\Omega}, \quad c_i = u_h(a_i), \quad u_h \in V_h.$$

For the numerical evaluation of integrals over the domain Ω, we use composite quadrature rules. Let $\hat{S}(\cdot)$

be a given quadrature rule on the element $\hat{e}$ with positive coefficients:

$$\int_{\hat{e}} \varphi(\hat{x})d\hat{x} \approx \sum_{i=1}^{M} \hat{c}_i \varphi(\hat{b}_i) \equiv \hat{S}(\varphi).$$

Assume that it is exact on the polynomials belonging to $\hat{P}_{2q-1}$. If $J_e(\hat{x})$ is the Jacobian of the transformation $x_e(\hat{x})$, then the following formulas specify the desired composite quadrature rule $S_h(\cdot)$

$$\int_{\Omega} \varphi(x)dx = \sum_{e \in \mathcal{T}_h} \int_e \varphi(x)dx = \sum_{e \in \mathcal{T}_h} \int_{\hat{e}} J_e(\hat{x})\varphi(x_e(\hat{x}))d\hat{x}$$

$$\approx \sum_{e \in \mathcal{T}_h} \sum_{i=1}^{M} \hat{c}_i J_e(x_e(\hat{b}_i))\varphi(x_e(\hat{b}_i)) \equiv S_h(\varphi).$$

Using the approximations V_h and $S_h(\cdot)$, we give a definition of the matrix $\mathbf{A}_h^N(p)$ that approximates $\mathbf{A}(p)$. The definitions of the other matrices $\mathbf{B}_{0h}^N$, $\mathbf{C}_{0h}^N$, and $L_h^N(p)$ are given in a similar fashion.

Let $\mathbf{H}_h = (H_{1h}, H_{2h}) \in \mathbf{V}_h = V_h \times V_h$ be an approximation of the vector $\mathbf{H}$,

$$H_{lh}(x) = \sum_{i=1}^{n} H_{li}\varphi_i(x), \quad x \in \overline{\Omega}, \quad H_{li} = H_{lh}(a_i), \quad a_i \in \omega_h, \quad l = 1, 2,$$

$\mathbb{H}_l = (H_{l1}, \dots, H_{ln})$ $(l = 1, 2)$, and $\mathbb{H} = (\mathbb{H}_1, \mathbb{H}_2)^{\mathrm{T}}$ be the vector of node parameters of the function $\mathbf{H}_h$. Then, the matrix $\mathbf{A}_h^N(p)$ is determined by its bilinear form

$$\begin{aligned}\mathbf{A}_h^N(p)\mathbb{H} \cdot \mathbb{H}' &= S_h\left(\frac{1}{n^2}\mathrm{curl}\mathbf{H}_h\mathrm{curl}\mathbf{H}'_h + \frac{1}{n_\infty^2}\mathrm{div}\mathbf{H}_h\mathrm{div}\mathbf{H}'_h + \frac{p^2}{n_\infty^2}\mathbf{H}_h \cdot \mathbf{H}'_h\right) \\ &+ \sum_{l=-N}^{N}\left(D_l(p)a_l(\mathbf{H}_h) \cdot \overline{a_l(\mathbf{H}'_h)} + \frac{2\pi i l}{n_\infty^2}a_l(\mathbf{H}_h) \times \overline{a_l(\mathbf{H}'_h)}\right).\end{aligned} \tag{21}$$

The matrix $\mathbf{A}_h^N(p)$ is the sum of two matrices $\mathbf{A}_\Omega$ and $\mathbf{A}_\Gamma$ induced by the corresponding terms on the right-hand side of (21). In practice, $\mathbf{A}_\Omega$ is calculated using the well-known technique of the finite element method.

Remark 3. When curved elements are used, there is no need for the calculation of the basis functions $\varphi_i^e(x)$. It is sufficient to know how to calculate the basis functions $\hat{\varphi}_i(\hat{x})$, the transformation $x = x_e(\hat{x})$, and their derivatives at the nodes of the quadrature $\hat{b}_j$.

The matrix $\mathbf{A}_\Gamma$ has nonzero entries only at the positions (i, j) for which the grid nodes $a_i, a_j \in \Gamma$. To calculate these entries, it is sufficient to calculate the Fourier coefficients $s_{lj} = a_l(\varphi_j)$ $(l = -N, -N+1, \dots, N, j: a_j \in \Gamma)$. Since $\varphi_j|_\Gamma$ is a piecewise polynomial function of the arc coordinate of Γ, this integral can be found exactly.

As for the accuracy of scheme (19), intuition suggests that, for sufficiently large N, the error in the eigenvalues is of the order h^{2q}, and the error in the eigenfunctions is of the order h^q (in the norm of the space V^3). We are going to devote a special paper to the investigation of the accuracy of scheme (19) and to the substantiation of the corresponding estimates. In that paper, we will prove that it is sufficient to take fairly small N of the order $\ln(1/h)$ (this is explained by the infinite smoothness of the eigenfunctions in a neighborhood of Γ).

4. NUMERICAL RESULTS

In this section, we present some numerical results that illustrate the accuracy of the algorithm described above and its universality. In the numerical experiments, we used the simplest scheme among those described above ($q = 1$, the quadrature rule with a single node at the center of gravity of the element). To

Table 1

n_Γ	h	n	β_h	err	err/h^2	err · n
18	0.35	45	3.9055	1.3e-1	1.10	6.04
36	0.17	178	3.5444	2.9e-2	0.97	5.24
54	0.12	388	3.4822	1.1e-2	0.84	4.40
72	0.087	656	3.4675	7.1e-3	0.93	4.64
108	0.058	1528	3.4535	3.0e-3	0.89	4.61
126	0.05	2149	3.4505	2.1e-3	0.86	4.60
144	0.044	2792	3.4489	1.7e-3	0.88	4.66

Table 2

n_Γ	h	n	β_h	err	err/h^2	err · n
16	0.39	78	4.2436	9.1e-002	0.59	7.1
32	0.2	151	4.1567	6.9e-002	1.8	10
48	0.13	325	3.9839	2.4e-002	1.4	7.9
64	0.098	573	3.9457	1.4e-002	1.5	8.3
72	0.087	645	3.9423	1.4e-002	1.8	8.8
80	0.079	887	3.9165	7.0e-003	1.1	6.2
96	0.065	1264	3.9081	4.8e-003	1.1	6.1
112	0.056	1516	3.9041	3.8e-003	1.2	5.8
154	0.041	3150	3.8894			

numerically investigate the accuracy of the method, we solved the problem for a waveguide with a circular cross section with a constant refractive index (in this case, the exact solution is known, see [1]). We used the following values of the parameters: the constant refractive index of the waveguide $n = \sqrt{2}$, the refractive index of the surroundings $n_\infty = 1$. The radius of the waveguide was equal to 1, the radius of the domain Ω was 1.5, and the centers of the circles Ω_i and Ω were identical. For each fixed p in the interval from 0 to 5, the first six (with account for the multiplicity) eigenvalues μ_l ($l = 1, \ldots, 6$) of problem (20) and the corresponding eigenvectors were found. The calculations were performed for the number of grid nodes in Ω in the range from 45 to 2792. Table 1 presents the results for the sixth eigenvalue μ_6 for $p = 1.2$. Here, h is the maximal diameter of the triangles in the triangulation of Ω, $\beta_h = \mu_6^{-1/2}$ is the approximate value of the propagation constant calculated for the given h, err $= |\beta - \beta_h|/|\beta|$, $\beta = 3.4431$ is the exact value of the propagation constant, n is the number of grid nodes in Ω, and n_Γ is the number of grid nodes on the boundary Ω.

The results confirm that the approximate eigenvalue converges to the exact eigenvalue and the order of convergence in h is two. In all the other cases that we examined, the convergence rate had the same order.

We also solved the problem numerically for a waveguide with a rectangular cross section with a constant refractive index. The results were compared with the experimental data presented in [16]. As in [16], the sides of the rectangle Ω_i were equal to 1.5 and 1, $n = \sqrt{2.08}$ within the rectangle, and $n_\infty = 1$. The radius of the circle Ω was 1.5, and its center coincided with the center of Ω_i. The first four eigenvalues μ_l of problem (20) and the corresponding eigenvectors were found. Figure 2 shows the dispersion curves, i.e., the plots of the functions $\beta_l = \mu_l(p)^{-1/2}$ for $l = 1, 2, 3, 4$. The solid lines show the results obtained with 2179 grid nodes in the domain Ω; the dots show the experimental data presented in [16].

To demonstrate the universality of the method proposed in this paper, we numerically solved the problem for the domain consisting of three circles of radius 0.4 tangent to each other. The center of Ω_i coincided with the center of the circle Ω of radius 1.5. The refractive index within Ω_i was $n = \sqrt{2}$, and $n_\infty = 1$. For each fixed p in the interval from 0 to 3, the first six (with account for the multiplicity) eigenvalues μ_l of problem (20) and the corresponding eigenvectors were found. The calculations were performed for the number of

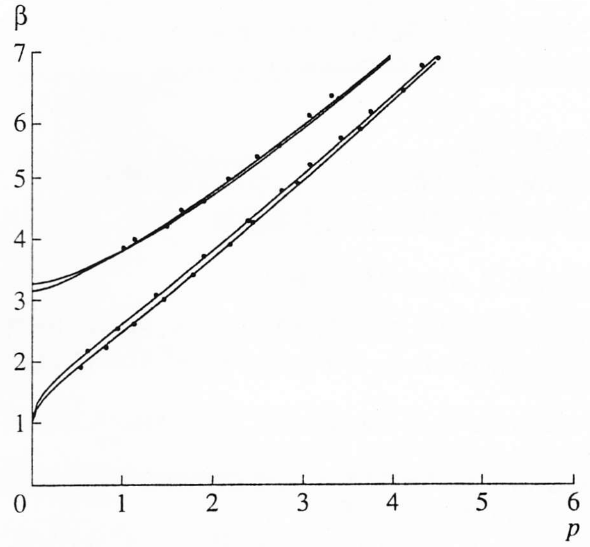

Fig. 2.

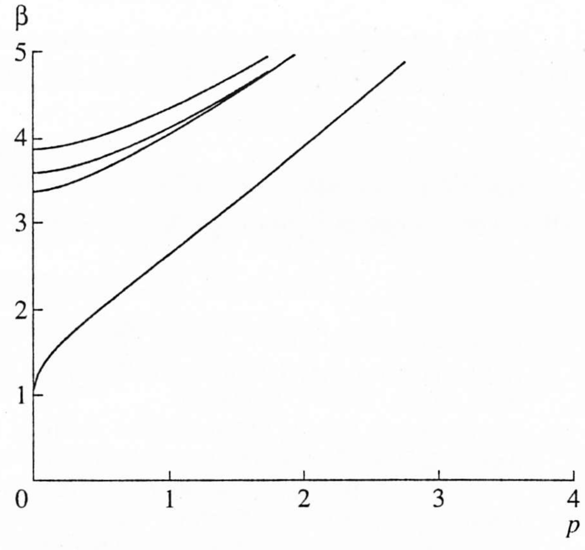

Fig. 3.

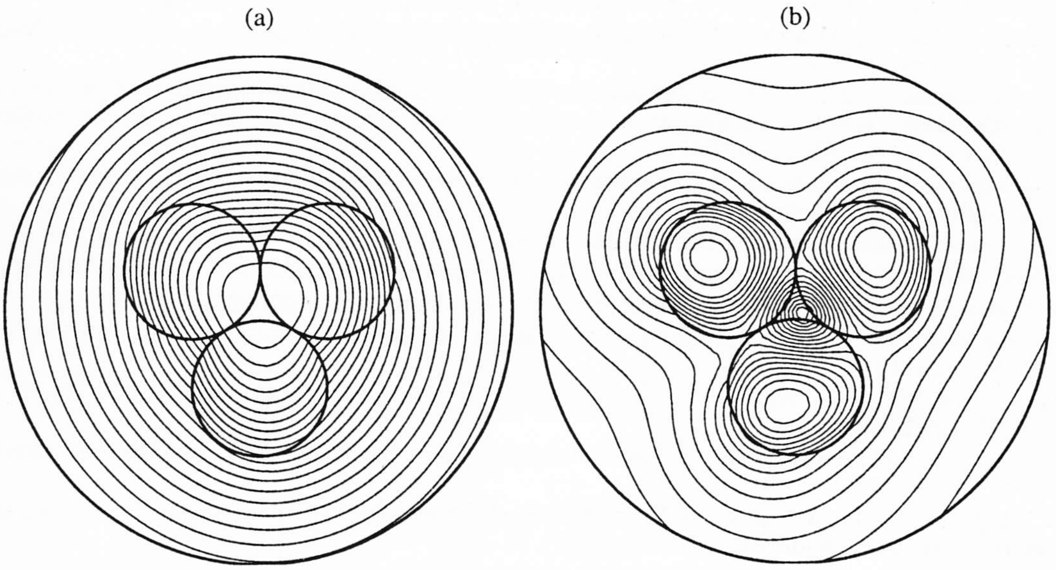

Fig. 4.

grid nodes in Ω in the range from 78 to 3150. Table 2 presents the results for the sixth eigenvalue μ_6 for p = 0.03. Here, h is the maximal diameter of the triangles in the triangulation of Ω, $\beta_h = \mu_6^{-1/2}$ is the approximate value of the propagation constant calculated for the given h, err = $|\beta - \beta_h|/|\beta|$, β = 3.8894 is the approximate value of the propagation constant obtained with 3150 grid nodes, n is the number of grid nodes in Ω, and n_Γ is the number of grid nodes on the boundary Γ. The rate of convergence in the parameter h had the same order in all the other cases that we examined.

In Fig. 3, the dispersion curves, i.e., the plots of the functions $\beta_l = \mu_l(p)^{-1/2}$ (l = 1, ..., 6) are shown. There are only four dispersion curves because the upper and the lower curves are multiple due to the symmetry of the problem: $\beta_1 = \beta_2$ and $\beta_5 = \beta_6$. Figure 4 shows, for p = 0.2, the level curves of the function $|H| = \sqrt{H \cdot \overline{H}}$ corresponding to β_1 = 1.6136 (Fig. 4a) and to β_6 = 3.9255 (Fig. 4b). The calculations were performed with 3150 grid nodes within Ω.

All the calculations described above were performed with $N = 20$. The further increase in the number of Fourier harmonics taken into account did not affect the accuracy of the calculations.

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REFERENCES

1. A. W. Snyder and D. Love, *Optical Waveguide Theory* (Chapman and Hall, London, 1983; Radio i Svyaz', 1987).
2. A. Bamberger and A. S. Bonnet, "Mathematical Analysis of the Guided Modes of in Optical Fiber," SIAM J. Math. Anal. **21** (6), 1487–1510 (1990).
3. P. Joly and C. Poirier, "Mathematical Analysis of Electromagnetic Open Waveguides," Math. Modell. Numer. Anal. **29**, 505–575 (1995).
4. A. I. Kleev, A. B. Manenkov, and A. G. Rozhnev, "Numerical Methods for the Design of Dielectric Waveguides: Universal Techniques (A Survey)," Radiotekh. Elektron., No. 11, 1938–1968 (1993).
5. P. Joly and C. Poirier, "A Numerical Method for the Computation of Electromagnetic Modes in Optical Fibres," Math. Meth. Appl. Sci. **22**, 389–447 (1999).
6. A. S. Bonnet-Ben Dhia and P. Joly, "Mathematical Analysis and Numerical Approximation of Optical Waveguides," in *Mathematical Modelling in Optical Science, SIAM Frontiers Book Series in Applied Mathematics*, Ed. by G. Bao, L. Cowsar, and W. Masters (SIAM, Philadelphia, 2001), Vol. 22, pp. 273–324.
7. A. Bossavit, "Solving Maxwell's Equations in a Closed Cavity and the Questions of Spurious Modes," IEEE Orans. Magnetics **26**, 702–705 (1990).
8. R. Z. Dautov and E. M. Karchevskii, "Solution of the Vector Eigenmode Problem for Cylindrical Dielectric Waveguides Based on a Nonlocal Boundary Condition," Zh. Vychisl. Mat. Mat. Fiz. **42**, 1051–1066 (2002) [Comput. Math. Math. Phys. 42, 1012–1027 (2002)].
9. A. S. Il'inskii, V. V. Kravtsov, and A. G. Sveshnikov, *Mathematical Models in Electrodynamics* (Vysshaya Shkola, Moscow, 1991) [in Russian].
10. D. Givoli and J. B. Keller, "Exact Non-Reflecting Boundary Conditions," J. Comput. Phys. **82**, 172–192 (1989).
11. D. Givoli, "Nonreflecting Boundary Conditions (Review Article)," J. Comput. Phys. **94**, 1–29 (1991).
12. I. N. Vekua, "On Metaharmonic Functions," Tr. Tbilisskogo Mat. Instituta **12**, 105–174 (1943).
13. E. Jahnke, F. Emde, and F. Lesch, *Tables of Functions with Formulae and Curves* (Stechert, New York, 1938; Nauka, Moscow, 1977).
14. V. G. Korneev, *Schemes of High Orders of Accuracy for the Finite Element Method* (Leningr. Gos. Univ., Leningrad, 1977) [in Russian].
15. M. Zlamal, "Curved Elements in Finite Element Method," Part I: SIAM J. Numer. Anal. **10**, 229–240 (1973); Part II: **11**, 347–362 (1974).
16. V. A. Karpenko, Yu. D. Stolyarov, and V. F. Kholomeev, "Theoretical and Experimental Study of a Rectangular Dielectric Waveguide," Radiotekh. Elektron. **25** (1), 51–57.