Physics Letters A 379 (2015) 928–932

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Analytical investigation of the specific heat for the Cantor energy spectrum

A.A. Khamzin *, R.R. Nigmatullin, D.E. Groshev

Institute of Physics, Kazan (Volga Region) Federal University, Kremlevskaya str. 18, Kazan, 420008, Russia

ARTICLE INFO

Article history:
Received 20 October 2014
Received in revised form 27 January 2015
Accepted 28 January 2015
Available online 30 January 2015
Communicated by C.R. Doering

ABSTRACT

For the energy spectrum obtained for monoscale Cantor set the correct analytical calculations of the specific heat in the frame of the Boltzmann–Maxwell statistics have been performed. These evaluations were realized with the help of Mellin's transform. The accurate analytical expressions for the specific heat in all temperature range were obtained. They demonstrate the log-periodic behavior in low-temperature and non-oscillatory behavior in high-temperature regions, accordingly. The accurate value of the limiting temperature determining the boundary between these two regions was found and evaluated.

1. Introduction

The experimental discovery of quasi-crystals by Shechtman et al. [1] produced a great interest in the understanding of the properties of these systems, as was shown later by the great amount of theoretical and experimental work that followed. The fact that they are in some sense midway between disorder (many of their physical properties exhibit an erratic appearance) and order (their definition, and construction, follows purely deterministic rules) makes them attractive objects of research. Since their first experimental realization in quasi-periodic GaAs–AlAs heterostructures in 1985 by Merlin and collaborators [2], their interest has only increased. More specifically, the Molecular Beam Epitaxy technique has produced and driven a multiplication of possible such structures (Fibonacci, Thue–Morse, double-period sequences; other possibilities could be Cantor sets, prime numbers, etc.). The behavior of a variety of particles and quasi-particles (electrons, photons, plasmon-polaritons, magnons) in quasi-periodic sequences has been and is currently being studied [3–11]. Now, there is a common feature which can be considered as the basic signature of such structures, and this is a fractal energy spectrum. These spectra tend, however, to be quite complex. In order to enlighten the thermodynamic consequences of fractal energy spectra, in Refs. [12,13] and [14], one- and multiscale fractal energy spectra were studied within Boltzmann statistics. It was shown that the scale invariance of the spectrum has strong consequences on the thermodynamical quantities. In particular, the specific heat oscillates log-periodically as a function of the temperature. Moreover, general scaling arguments and a detailed analysis of the integrated density of states allowed for a quantitative prediction of the average value (which is related to the average density of states), period and amplitude of the oscillations. Moreover, these results were extended to N-particle systems described by quantum statistics [15]; it was shown that for phonons, and for bosons in general, the Boltzmann scenario survives the inclusion of quantum symmetries. The fermionic case is more delicate, however, in some special cases, log-periodic oscillations can still be observed.

The aim of this present work is to demonstrate analytical calculations of the specific heat in the frame of the Maxwell–Boltzmann statistics for fractal energy spectrum obtained from monoscale Cantor set in *all* temperature range. The possibility of realization of the correct calculations is based on the general formula for the elements of the Cantor set that has been used in papers [12,14]. In these papers for analytical proof of existence of the log-periodic oscillations that appeared in the specific heat behavior for the simplest case of monoscale energy spectrum of the Cantor set the Poisson summation formula was used. But in the frame of this approach the region of log-periodic behavior was not evaluated properly and expression for the specific heat out of this region was also not shown. But with the help of Mellin's transform it becomes possible to determine the limits of the log-periodic region and find analytical expressions for the specific heat in these two oscillation/non-oscillation regimes. In the given paper with the usage of Mellin's transformation it becomes possible to realize accurate analytical calculations for the specific heat associated with more general monoscale Cantor set in comparison with results considered earlier in paper [14]. It is necessary to note that approach based on Mellin's transform is rather productive and it was applied with success for the solution of the problems related to the anomalous dielectric relaxation [16,17], thermodynamics of spin

* Corresponding author.
E-mail address: airat.khamzin@rambler.ru (A.A. Khamzin).

http://dx.doi.org/10.1016/j.physleta.2015.01.035
0375-9601/

systems with hierarchy-subordinated dynamics [18] and calculation of the moment of inertia of the heated finite Fermi-systems [19] for analytical extraction of the oscillating components of the desired physical values.

2. Monoscale cantor sets thermodynamics

In paper [16] for construction of the fractal energy spectrum the generalized Cantor set model is suggested, which, in turn, can be monoscale or multiscale. The essence of the suggested constructions is reduced to the following scheme. On the initial step $n = 0$ we have the continuous segment $[0, 1]$. On the step $n = 1$ the set $\mathfrak{S}(l, m)$ is generated by division of the initial segment on j (l-accepts the integer values) equal segments of the length l^{-1} numbered from 0 up to $l - 1$. Then $l - m$ segments are excluded and as the result in the Cantor spectrum m segments are remained ($m < l$). In order to keep the spectrum width $\Delta = 1$ the segments with numbers 0 and $l-1$ cannot be excluded. We should note that at the given l and m, $\binom{m-2}{l-2}$ different combinations of m segments exist (and, hence, $\binom{m-2}{l-2}$ of different sets $\mathfrak{S}(l, m)$). The sets are differentiated from each other with the help of notation $\mathfrak{S}(l, m; b_1, b_2, \ldots, b_m)$, where the combination $\{b_1, b_2, \ldots, b_m\}$ defines a set containing numbers of non-excluded segments. In accordance with limitations imposed above they should satisfy to condition $0 = b_1 < b_2 < \ldots < b_m = l - 1$. After selection of one of possible combinations it should be kept during the whole construction process that is necessary for conservation of the given fractal structure. For the general monoscale $\mathfrak{S}(l, m; b_1, b_2, \ldots, b_m)$ the fractal dimension (box-counting dimension) is equaled to $d_{box} = \ln m/\ln l$.

It is necessary also to reproduce an expression for the energy spectrum that is obtained from monoscale Cantor set of the general form $\mathfrak{S}(l, m; b_1, b_2, \ldots, b_m)$ in a discrete case

$$\mathfrak{S}_n(l, m; b_1, b_2, \ldots, b_m) = E_n^- \cup E_n^+, \tag{1}$$

where

$$E_n^- = \left\{\sum_{k=1}^{n} \frac{b_k}{l^k}\right\}, \qquad E_n^+ = \left\{\sum_{k=1}^{n} \frac{b_k}{l^k} + \frac{1}{l^n}\right\}, \tag{2}$$

expressions define the smallest and the highest energies of any interval of the generalized Cantor set, correspondingly. For a monoscale Cantor set, analytical expressions can be derived for the thermodynamic functions. Regarding to the energy spectra given by Eq. (1), the partition function for the monoscale $\mathfrak{S}(l, m; b_1, b_2, \ldots, b_m)$ in the nth step of the generation process can be obtained as [14]

$$Z_n(T) = \left[1 + \exp\left(-\frac{\beta}{l^n}\right)\right]\prod_{k=1}^{n}\left[\sum_{b_k}\exp\left(-\frac{\beta b_k}{l^k}\right)\right]. \tag{3}$$

From this expression for the partition function, using the equation $C_n(T) = -\beta^2\partial^2 \ln Z_n(T)/\partial\beta^2$, the specific heat $C_n(T)$ can be obtained [16]

$$C_n(T) = \left[\frac{2l^n}{\beta}\cosh\left(\frac{\beta}{2l^n}\right)\right]^{-2} + \beta^2\sum_{k=1}^{n}\frac{\sum_{b_k}\exp(-\frac{\beta b_k}{l^k})\sum_{b_k} b_k^2\exp(-\frac{\beta b_k}{l^k}) - [\sum_{b_k}\exp(\frac{\beta b_k}{l^k})]^2}{[l^k\sum_{b_k}\exp(-\frac{\beta b_k}{l^k})]^2}. \tag{4}$$

In the $n \to \infty$ limit, only the second term is survived and then we have

$$C_n(T) = \beta^2 \times \sum_{k=1}^{\infty}\frac{\sum_{b_k}\exp(-\frac{\beta b_k}{l^k})\sum_{b_k} b_k^2\exp(-\frac{\beta b_k}{l^k}) - [\sum_{b_k}\exp(\frac{\beta b_k}{l^k})]^2}{[l^k\sum_{b_k}\exp(-\frac{\beta b_k}{l^k})]^2}. \tag{5}$$

This equation looks rather complicated and so it should be evaluated numerically for obtaining the desired temperature dependence of the specific heat. The standard way is in construction of the desired spectra and then the numerical differentiation of the partition function obtained. But an attentive analysis shows that for some specific cases expression (5) can be considerably simplified analytically.

In [14] the simplest case related to consideration of a monoscale Cantor set was considered. This approach is based on division the spectral branch into l subsegments and after that only the first and the last one are selected, i.e. $\mathfrak{S}(l, 2; 0, l - 1)$. But the triadic Cantor set represents itself the simplest case when it becomes possible to perform the summation $\sum_{b_k}$ and thereby simplify Eq. (5). Finally, we have

$$C_\infty(T) = \sum_{k=1}^{\infty}\left[\frac{2l^kT}{l-1}\cosh\left(\frac{l-1}{2l^kT}\right)\right]^{-2}. \tag{6}$$

From this expression with the help of the Poisson summation formula it becomes possible to obtain the analytical expression for the specific heat [16], which proves its log-periodic behavior. But we want to stress here that this case is not a unique example which admits the rigorous analytical results.

In the given work we suggest the following monoscale set admitting some simplifications of the general expression (5) for the specific heat. Really, we consider the modified set when after division of the spectral branch on l segments we keep also segments with numbers $c_k = p(k - 1)$, $k = 1, 2, \ldots, m$, where p accepts the integer numbers and $l - 1$ must be multiply to the number p except $m = (l - 1)/p + 1$. In the result of these manipulations we obtain monoscale Cantor set of the type $\mathfrak{S}(l, m; 0, p, 2p, .., l - 1)$. In this case the analytical evaluation of summation $\sum_{b_k}$ remains possible and after its realization it allows to simplify expression for the specific heat (5)

$$C_\infty(T) = \sum_{k=1}^{\infty}\left(\frac{p\beta}{2l^k}\right)^2\left[\frac{1}{\sinh^2(p\beta/2l^k)} - \frac{m^2}{\sinh^2(mp\beta/2l^k)}\right]. \tag{7}$$

In Fig. 1 we demonstrate two finite approximations of the specific heat behavior for monoscale Cantor sets in log-scale. We want to stress three basic points specifying this dependence. Firstly, the behavior of the specific heat at low T represents itself an oscillating function and the number of the oscillations which are controlled by the length of the chosen step participating in the generation process. With increasing of number of periods a new period appears in the low temperature region. The function $C_n(T)$ oscillates around a particular value that is defined by the fractal dimensionality of the set considered. In the cases shown in Fig. 1, we have $d_1 = \ln 6/\ln 11, d_2 = \ln 2/\ln 11$. Thirdly, note that in the oscillating regime, $C_n(T)$ represents itself a log-periodic function. Below, based on exact analytical calculations we are going to prove of the periodicity phenomenon of such type. The log-periodic phenomenon is appeared only in the oscillating regime.

We want to note that the main reason of the log-periodic oscillations in temperature dependence of the specific heat is related to the usage of the fractal model for the energy spectrum. Really, fractals have the property of a discrete scaling invariance, which is a lower symmetry than the scaling invariance [20]. It means that the functional equation for the observed physical value $\Phi(x)$

$$\Phi(\lambda x) = \gamma\Phi(x), \tag{8}$$

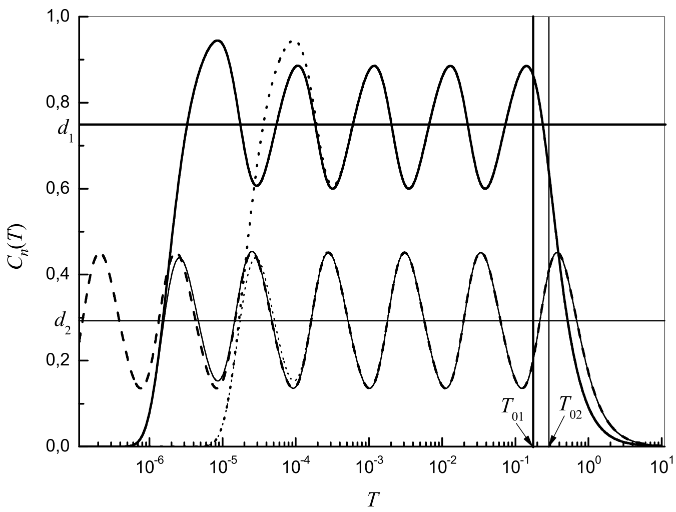

Fig. 1. Finite approximations of the specific heat for the sets $\mathfrak{S}(11, 6; 0, 2, 4, 6, 8, 10)$ (thick lines) and $\mathfrak{S}(11, 2; 0, 10)$ (thin lines). In both cases, the dotted line corresponds to $n = 4$ and the solid one to the values $n = 5$. The horizontal lines represent the corresponding fractal dimensionalities. The vertical lines represent the corresponding temperatures T_0. The dashed line represents the functions $C_\infty^{<}(T)$ $(T < T_{02})$ and $C_\infty^{>}(T)$ $(T \geq T_{02})$ for the set $\mathfrak{S}(11, 2; 0, 10)$.

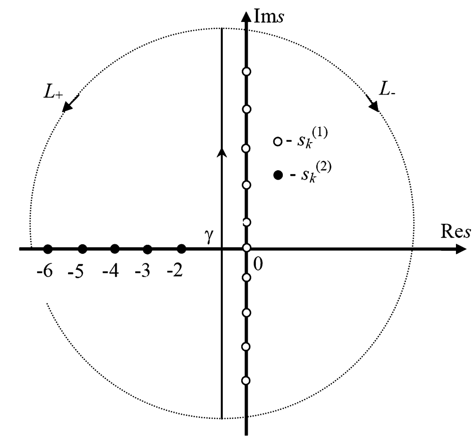

Fig. 2. Schematic of the pole positions for the function (10) and the contours $L_\pm$ in the complex plane.

determining the scale invariance property of the considered system, is satisfied only for a certain special choice of λ (and also of γ), namely, on an infinite set of discrete values $\lambda_n = \lambda^n$, where λ is the fundamental scaling parameter. In the result of this property on the power-law dependence $\Phi(x) = Ax^\nu$, where A is some constant and the exponent is $\nu = \ln\gamma / \ln\lambda$, being a solution of the functional equation (8), the perturbation is imposed. This perturbation is periodic in the logarithmic scale [20]

$$\Phi(x) = x^\nu W(\ln x / \ln\lambda). \tag{9}$$

Here $W(y)$ is a periodic function having the unit period $(W(y + 1) = W(y))$. Function $W(y)$ can be represented as a Fourier series

$$W(y) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi k i y},$$

here, the c_k are the Fourier series coefficients, whose specific expressions depend on the choice of the function $\Phi(x)$.

Expression for the specific heat (7) at low temperatures satisfies to the functional equation (8) and has the form

$$C_\infty(l \cdot T) = C_\infty(T).$$

So, in accordance with (9), expression (7) can be presented in the form

$$C_\infty(T) = \sum_{k=-\infty}^{\infty} c_k T^{2\pi i k / \ln l}.$$

In the next section we will determine the decomposition Fourier coefficients c_k.

3. Analytical calculation of the specific heat

For the proving of the log-periodic behavior of the specific heat one can subject by Mellin's transformation equation (7):

$$C_\infty(\beta) \overset{MT}{=} \hat{C}_\infty(s) = \int_0^\infty C_\infty(\beta)\beta^{s-1} d\beta. \tag{10}$$

Using the known expressions

$$\frac{x^2}{\sinh^2(x)} \overset{MT}{=} 2^{-s}\Gamma(s+2)\zeta(s+1), \quad \mathrm{Re}\, s > 0, \tag{11}$$

where $\Gamma(z)$ is conventional gamma-function, $\zeta(z)$ is Riemann's zeta-function and the properties of Mellin's transform $f(ax) \overset{MT}{=} a^{-s}\hat{f}(s)$ one can obtain from (7)

$$\hat{C}_\infty(s) = p^{-s}\left(1 - m^{-s}\right)\Gamma(s+2)\zeta(s+1)\sum_{k=1}^{\infty} l^{ks} = \frac{\hat{G}(s)}{l^{-s} - 1},$$
$$\hat{G}(s) = p^{-s}\left(1 - m^{-s}\right)\Gamma(s+2)\zeta(s+1), \quad -2 < \gamma = \mathrm{Re}\, s < 0. \tag{12}$$

Here we took into account the fact that existence domain of Mellin's image of the function $x^2\sinh^{-2}(x) - (mx)^2\sinh^{-2}(mx)$ is different from Mellin's image of the function $x^2\sinh^{-2}(x)$ (see Eq. (11)) and satisfies to condition: $\mathrm{Re}\, s > -2$.

Mellin's image $\hat{C}_\infty(s)$ has simple poles which can be divided on two groups. The first group of the poles coincides with zeros of denominator of expression (12). They are determined as

$$l^{-s} = 1 \quad \Rightarrow \quad \exp(-s\ln l) = \exp(-2\pi i k)$$
$$\Rightarrow \quad s_k^{(1)} = i\Omega k, \quad k = 0, \pm 1, \pm 2, \ldots, \Omega = \frac{2\pi}{\ln l}. \tag{13}$$

The second group is formed by the poles of the gamma-function entering to expression $\hat{G}(s)$

$$s + 2 = -k \quad \Rightarrow \quad s_k^{(2)} = -k - 2, k = 0, 1, 2, \ldots. \tag{14}$$

The poles $\{s_k^{(1)}\}$ and $\{s_k^{(2)}\}$ (see Fig. 2) form two independent sets and so the total set of poles of the function $\hat{C}_\infty(s)$ is equaled to $\{s_k\} = \{s_k^{(1)}\} \cup \{s_k^{(2)}\}$.

Then we should come back and realize the inverse Mellin's transform. Adding to the integration line the left or right semicircles that form the closed integration contour one can use the residue theorem. Finally, we obtain

$$C_\infty(\beta) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{C}_\infty(s)\beta^{-s} ds = \pm\sum_k \mathrm{Res}_{s_{s_k}}\left[\hat{C}_\infty(s)\beta^{-s}\right]. \tag{15}$$

Here the signes "+" and "−" correspond the contour L_+ and L_-, accordingly (see Fig. 2). As one can notice from Fig. 2 the poles of the first group $\{s_k^{(1)}\}$ are located inside the contour L_-, while the poles of the second group $\{s_k^{(2)}\}$ are located inside L_+.

Choosing the contour L_+, we obtain

$$C_\infty^{>}(T)=\sum_{k=0}^{\infty}\mathrm{Re}\,s_{-k-2}\left[\frac{\hat{G}(s)}{l^{-s}-1}\beta^{-s}\right]$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k(1-m^{k+2})(l-1)^{k+2}\zeta(-k-1)}{(l^{k+2}-1)k!}(p\beta)^{k+2}, \qquad (16)$$

where we use the residue values $\mathrm{Re}\,s_{-k-2}[\Gamma(s+2)]=(-1)^k/k!$. Taking into account also the values

$$\zeta(-2n)=0,\ \zeta\big(-(2n-1)\big)=-B_{2n}/2n,\quad n=1,2,3,\ldots,$$

where the constants B_{2n} coincide with Bernoulli's numbers, we obtain finally from the last expression (16)

$$C_\infty^{>}(T)=\sum_{n=1}^{\infty}\frac{m^{2n}-1}{(l^{2n}-1)(2n-2)!}\frac{B_{2n}}{2n}(p\beta)^{2n}. \qquad (17)$$

Additional investigation associated with the convergence of the series (17) based on asymptotic behavior of Bernoulli's numbers $B_{2n}\approx 2(-1)^{n+1}(2n)!/(2\pi)^{2n}$ $(n\gg 1)$ leads us to conclusion that this series is convergent at $T\geq T_0$, where

$$T_0=\frac{mp}{2\pi l}=\frac{1}{2\pi}\left(1+\frac{p-1}{l}\right),\quad \frac{1}{2\pi}<T_0<\frac{1}{\pi}. \qquad (18)$$

So, expression (17) for the specific heat is correct for the temperature range $T>T_0$ which defines non-oscillatory regime. We want to remark also that for the set $\mathfrak{S}(l,2;0,l-1)$ which coincides with the partial case at $m=2$, $p=l-1$, $T_0=(1-l^{-1})\pi^{-1}$.

If in denominator of expression (17) one replaces $l^{2n}-1$ for l^{2n} then it becomes possible to realize this summation and receive a good approximate expression for the specific heat $C_\infty^{>}(T)$

$$C_\infty^{>}(T)\simeq\left(\frac{p}{2Tl}\right)^2\left[\frac{1}{\sinh^2(p/(2Tl))}-\frac{m^2}{\sinh^2(mp/(2Tl))}\right]. \qquad (19)$$

If we choose contour L_-, containing poles $\{s_k^{(1)}\}$ then formulae (15) leads to the following result

$$C_\infty^{<}(T)=-\sum_{k=-\infty}^{\infty}\mathrm{Re}\,s_{i\Omega k}\left[\frac{\hat{G}(s)}{l^{-s}-1}\beta^{-s}\right]$$
$$=-\mathrm{Re}\,s_0\left[\frac{\hat{G}(s)}{l^{-s}-1}\beta^{-s}\right]-\sum_{k=1}^{\infty}\left\{\mathrm{Re}\,s_{i\Omega k}\left[\frac{\hat{G}(s)}{l^{-s}-1}\beta^{-s}\right]\right.$$
$$\left.+\mathrm{Re}\,s_{-i\Omega k}\left[\frac{\hat{G}(s)}{l^{-s}-1}\beta^{-s}\right]\right\}. \qquad (20)$$

Taking into account that $\lim_{s\to 0}[s\zeta(s+1)]=1$, $\lim_{s\to 0}\frac{1-m^{-s}}{l^{-s}-1}=-\frac{\ln m}{\ln l}$, $\mathrm{Re}\,s_{\pm i\Omega k}\frac{\hat{G}(s)\beta^{-s}}{l^{-s}-1}=\frac{1}{\ln l}\hat{G}(\pm i\Omega k)\beta^{\mp i\Omega k}$ one can obtain from (20) the following expression

$$C_\infty^{<}(T)=\frac{\ln m}{\ln l}+\frac{1}{\ln l}\sum_{k=1}^{\infty}\left\{\hat{G}(i\Omega k)\beta^{-i\Omega k}+\hat{G}(-i\Omega k)\beta^{i\Omega k}\right\}. \qquad (21)$$

Extracting the real and imaginary part of $\hat{G}(\pm i\Omega k)$ one can obtain finally

$$C_\infty^{<}(T)=\frac{\ln m}{\ln l}+\frac{2}{\ln l}\sum_{k=1}^{\infty}\left\{\mathrm{Re}\big[\hat{G}(i\Omega k)\big]\cdot\cos(k\Omega\ln T)\right.$$
$$\left.-\mathrm{Im}\big[\hat{G}(i\Omega k)\big]\cdot\sin(k\Omega\ln T)\right\}. \qquad (22)$$

This expression for the specific heat contains the basic information about the behavior of heat capacity for the monoscale Cantor spectrum: (a) oscillations near fractal dimension $\ln m/\ln l$ and log-periodic property of the function $C_\infty(T)$. We want to mark also that expression (22) represents itself the analytical continuation of expression (17) and, hence, it is correct for the temperature region $T<T_0$, which determines the oscillating region of the specific heat.

In Fig. 1 the boundaries dividing the oscillating and non-oscillating regions are shown by vertical lines: $T=T_{01}=6/11\pi$ for the set $\mathfrak{S}(11,6;0,2,4,6,8,10)$ and $T=T_{02}=10/11\pi$ for the set $\mathfrak{S}(11,2;0,10)$. From Fig. 1 one can notice that heat capacitance for the spectrum $\mathfrak{S}(11,6;0,2,4,6,8,10)$ at $T>T_0$ decreases monotonically with the growth of temperature while for the spectrum $\mathfrak{S}(11,2;0,10)$ at $T>T_0$ the heat capacitance loses its monotone character and has the local maximum. The temperature corresponding to the maximum point is evaluated numerically from the approximate expression for $C_\infty^{>}(T)$ (17), which increases the characteristic temperature T_0 for $m=2,3,4$.

To illustrate the properties of $C_\infty^{<}(T)$ and $C_\infty^{>}(T)$ depicted in Fig. 1 we also show for the set $\mathfrak{S}(11,2;0,10)$ the functions $C_\infty^{<}(T)$ and $C_\infty^{>}(T)$ given by Eqs. (22) and (17) by a dashed line.

4. Summary

In the given paper we demonstrated the accurate prove of the log-periodic behavior of the specific heat behavior associated with quasi-periodic system with energy spectrum obtained for the monoscale Cantor set in the frame of the Maxwell–Boltzmann statistics. As the method we used Mellin's transform, which allows finding in analytic form the log-periodic behavior of the specific heat. This transformation allowed determining the boundaries of the temperature region where the log-periodic behavior takes place. In the frame of the suggested approach it became possible to evaluate the value of the boundary temperature which depends on the structural parameters of the spectrum considered and obtain the clear expression for the specific heat out of the oscillating regime, which exhibits monotone/non-monotone behavior depending of the structure of the spectrum considered.

In conclusion we should add the following remark. A possibility of an "accurate" (analytical) investigation of thermodynamics is related closely with selection of an "idealized" fractal spectrum obtained from the generalized Cantor set. But investigation of the thermodynamics for "real" spectra associated with quasi-periodic structures is possible only with numerical calculations. In paper [14], for example, the thermodynamics of tight-binding Fibonacci spectrum was performed. This specific spectrum represents itself an *approximate* fractal (not strictly invariant under changes of scale), and then many of the properties found previously for the Cantor sets can be considered as approximate. This fact is closely associated with the interpretation of the meaning of fractal dimension D, and in the log-periodicity behavior and amplitudes of the oscillations that were found by authors of paper [14] for the Fibonacci specific heat can be interpreted only in this sense.

Acknowledgement

This work was performed with financial support of the Ministry of Education and Science of Russia within a basic part.

References

[1] D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, Phys. Rev. Lett. 53 (1984) 1951.
[2] R. Merlin, K. Bajema, R. Clarke, F.Y. Juang, P.K. Bhatacharya, Phys. Rev. Lett. 55 (1985) 1768.
[3] M. Kohmoto, L.P. Kadanoff, C. Tang, Phys. Rev. Lett. 50 (1983) 1870.
[4] M. Kohmoto, J.R. Banavar, Phys. Rev. B 34 (1986) 563.
[5] M. Kohmoto, B. Sutherland, C. Tang, Phys. Rev. B 35 (1987) 1020.
[6] C.S. Ryu, G.Y. Oh, M.H. Lee, Phys. Rev. B 48 (1993) 132.

[7] E. Macia, F. Dominguez-Adame, A. Sanchez, Phys. Rev. B 49 (1994) 9503.
[8] P. Carpena, V. Gasparian, M. Ortuno, Phys. Rev. B 51 (1995) 12813.
[9] F. Piéchon, M. Benakli, A. Jagannathan, Phys. Rev. Lett. 74 (1995) 5248.
[10] P. Carpena, V. Gasparian, M. Ortuno, Z. Phys. B, Condens. Matter 102 (1997) 425.
[11] G.G. Naumis, Phys. Rev. B 59 (1999) 11315.
[12] C. Tsallis, L.R. da Silva, R.S. Mendes, R.O. Vallejos, A.M. Mariz, Phys. Rev. E 56 (1997) R4922.
[13] R.O. Vallejos, R.S. Mendes, L.R. da Silva, C. Tsallis, Phys. Rev. E 58 (1998) 1346.
[14] P. Carpena, A.V. Coronado, P. Bernaola-Galván, Phys. Rev. E 61 (2000) 2281.
[15] R.O. Vallejos, C. Anteneodo, Phys. Rev. E 58 (1998) 4134.
[16] A.A. Khamzin, R.R. Nigmatullin, I.I. Popov, Physica A 392 (2013) 136.
[17] A.A. Khamzin, R.R. Nigmatullin, I.I. Popov, Theor. Math. Phys. 173 (2012) 1604.
[18] A.A. Khamzin, R.R. Nigmatullin, I.I. Popov, M.P. Zhelifonov, J. Phys. Conf. Ser. 394 (2012) 012008.
[19] A.A. Khamzin, A.S. Nikitin, A.S. Sitdikov, D.A. Roganov, Theor. Math. Phys. 176 (2013) 1220.
[20] D. Sornette, Phys. Rep. 297 (1998) 239.