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CFA Institute®
CFA Program

FIXED INCOME AND DERIVATIVES

CFA® Program Curriculum
2020 • LEVEL II • VOLUME 5

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How to Use the CFA Program Curriculum

Congratulations on reaching Level II of the Chartered Financial Analyst® (CFA®) Program. This exciting and rewarding program of study reflects your desire to become a serious investment professional. You have embarked on a program noted for its high ethical standards and the breadth of knowledge, skills, and abilities (competencies) it develops. Your commitment to the CFA Program should be educationally and professionally rewarding.

The credential you seek is respected around the world as a mark of accomplishment and dedication. Each level of the program represents a distinct achievement in professional development. Successful completion of the program is rewarded with membership in a prestigious global community of investment professionals. CFA charterholders are dedicated to life-long learning and maintaining currency with the ever-changing dynamics of a challenging profession. The CFA Program represents the first step toward a career-long commitment to professional education.

The CFA examination measures your mastery of the core knowledge, skills, and abilities required to succeed as an investment professional. These core competencies are the basis for the Candidate Body of Knowledge (CBOK™). The CBOK consists of four components:

- A broad outline that lists the major topic areas covered in the CFA Program (https://www.cfainstitute.org/programs/cfa/curriculum/cbok);
- Topic area weights that indicate the relative exam weightings of the top-level topic areas (https://www.cfainstitute.org/programs/cfa/curriculum/overview);
- Learning outcome statements (LOS) that advise candidates about the specific knowledge, skills, and abilities they should acquire from readings covering a topic area (LOS are provided in candidate study sessions and at the beginning of each reading); and
- The CFA Program curriculum that candidates receive upon examination registration.

Therefore, the key to your success on the CFA examinations is studying and understanding the CBOK. The following sections provide background on the CBOK, the organization of the curriculum, features of the curriculum, and tips for designing an effective personal study program.

BACKGROUND ON THE CBOK

The CFA Program is grounded in the practice of the investment profession. Beginning with the Global Body of Investment Knowledge (GBIK), CFA Institute performs a continuous practice analysis with investment professionals around the world to determine the competencies that are relevant to the profession. Regional expert panels and targeted surveys are conducted annually to verify and reinforce the continuous feedback about the GBIK. The practice analysis process ultimately defines the CBOK. The

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CBOK reflects the competencies that are generally accepted and applied by investment professionals. These competencies are used in practice in a generalist context and are expected to be demonstrated by a recently qualified CFA charterholder.

The CFA Institute staff, in conjunction with the Education Advisory Committee and Curriculum Level Advisors that consist of practicing CFA charterholders, designs the CFA Program curriculum in order to deliver the CBOK to candidates. The examinations, also written by CFA charterholders, are designed to allow you to demonstrate your mastery of the CBOK as set forth in the CFA Program curriculum. As you structure your personal study program, you should emphasize mastery of the CBOK and the practical application of that knowledge. For more information on the practice analysis, CBOK, and development of the CFA Program curriculum, please visit www.cfainstitute.org.

ORGANIZATION OF THE CURRICULUM

The Level II CFA Program curriculum is organized into 10 topic areas. Each topic area begins with a brief statement of the material and the depth of knowledge expected. It is then divided into one or more study sessions. These study sessions—17 sessions in the Level II curriculum—should form the basic structure of your reading and preparation. Each study session includes a statement of its structure and objective and is further divided into assigned readings. An outline illustrating the organization of these 17 study sessions can be found at the front of each volume of the curriculum.

The readings are commissioned by CFA Institute and written by content experts, including investment professionals and university professors. Each reading includes LOS and the core material to be studied, often a combination of text, exhibits, and in-text examples and questions. A reading typically ends with practice problems followed by solutions to these problems to help you understand and master the material. The LOS indicate what you should be able to accomplish after studying the material. The LOS, the core material, and the practice problems are dependent on each other, with the core material and the practice problems providing context for understanding the scope of the LOS and enabling you to apply a principle or concept in a variety of scenarios.

The entire readings, including the practice problems at the end of the readings, are the basis for all examination questions and are selected or developed specifically to teach the knowledge, skills, and abilities reflected in the CBOK.

You should use the LOS to guide and focus your study because each examination question is based on one or more LOS and the core material and practice problems associated with the LOS. As a candidate, you are responsible for the entirety of the required material in a study session.

We encourage you to review the information about the LOS on our website (www.cfainstitute.org/programs/cfa/curriculum/study-sessions), including the descriptions of LOS "command words" on the candidate resources page at www.cfainstitute.org.

FEATURES OF THE CURRICULUM

OPTIONAL SEGMENT

Required vs. Optional Segments You should read all of an assigned reading. In some cases, though, we have reprinted an entire publication and marked certain parts of the reading as "optional." The CFA examination is based only on the required segments, and the optional segments are included only when it is determined that they might

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help you to better understand the required segments (by seeing the required material in its full context). When an optional segment begins, you will see an icon and a dashed vertical bar in the outside margin that will continue until the optional segment ends, accompanied by another icon. *Unless the material is specifically marked as optional, you should assume it is required.* You should rely on the required segments and the reading-specific LOS in preparing for the examination.

END OPTIONAL SEGMENT

Practice Problems/Solutions *All practice problems at the end of the readings as well as their solutions are part of the curriculum and are required material for the examination.* In addition to the in-text examples and questions, these practice problems should help demonstrate practical applications and reinforce your understanding of the concepts presented. Some of these practice problems are adapted from past CFA examinations and/or may serve as a basis for examination questions.

Glossary For your convenience, each volume includes a comprehensive glossary. Throughout the curriculum, a **bolded** word in a reading denotes a term defined in the glossary.

Note that the digital curriculum that is included in your examination registration fee is searchable for key words, including glossary terms.

LOS Self-Check We have inserted checkboxes next to each LOS that you can use to track your progress in mastering the concepts in each reading.

Source Material The CFA Institute curriculum cites textbooks, journal articles, and other publications that provide additional context and information about topics covered in the readings. As a candidate, you are not responsible for familiarity with the original source materials cited in the curriculum.

Note that some readings may contain a web address or URL. The referenced sites were live at the time the reading was written or updated but may have been deactivated since then.

Some readings in the curriculum cite articles published in the *Financial Analysts Journal*®, which is the flagship publication of CFA Institute. Since its launch in 1945, the *Financial Analysts Journal* has established itself as the leading practitioner-oriented journal in the investment management community. Over the years, it has advanced the knowledge and understanding of the practice of investment management through the publication of peer-reviewed practitioner-relevant research from leading academics and practitioners. It has also featured thought-provoking opinion pieces that advance the common level of discourse within the investment management profession. Some of the most influential research in the area of investment management has appeared in the pages of the *Financial Analysts Journal*, and several Nobel laureates have contributed articles.

Candidates are not responsible for familiarity with *Financial Analysts Journal* articles that are cited in the curriculum. But, as your time and studies allow, we strongly encourage you to begin supplementing your understanding of key investment management issues by reading this practice-oriented publication. Candidates have full online access to the *Financial Analysts Journal* and associated resources. All you need is to log in on www.cfapubs.org using your candidate credentials.

Errata The curriculum development process is rigorous and includes multiple rounds of reviews by content experts. Despite our efforts to produce a curriculum that is free of errors, there are times when we must make corrections. Curriculum errata are periodically updated and posted on the candidate resources page at www.cfainstitute.org.

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DESIGNING YOUR PERSONAL STUDY PROGRAM

Create a Schedule An orderly, systematic approach to examination preparation is critical. You should dedicate a consistent block of time every week to reading and studying. Complete all assigned readings and the associated problems and solutions in each study session. Review the LOS both before and after you study each reading to ensure that you have mastered the applicable content and can demonstrate the knowledge, skills, and abilities described by the LOS and the assigned reading. Use the LOS self-check to track your progress and highlight areas of weakness for later review.

Successful candidates report an average of more than 300 hours preparing for each examination. Your preparation time will vary based on your prior education and experience, and you will probably spend more time on some study sessions than on others. As the Level II curriculum includes 17 study sessions, a good plan is to devote 15–20 hours per week for 17 weeks to studying the material and use the final four to six weeks before the examination to review what you have learned and practice with practice questions and mock examinations. This recommendation, however, may underestimate the hours needed for appropriate examination preparation depending on your individual circumstances, relevant experience, and academic background. You will undoubtedly adjust your study time to conform to your own strengths and weaknesses and to your educational and professional background.

You should allow ample time for both in-depth study of all topic areas and additional concentration on those topic areas for which you feel the least prepared.

As part of the supplemental study tools that are included in your examination registration fee, you have access to a study planner to help you plan your study time. The study planner calculates your study progress and pace based on the time remaining until examination. For more information on the study planner and other supplemental study tools, please visit www.cfainstitute.org.

As you prepare for your examination, we will e-mail you important examination updates, testing policies, and study tips. Be sure to read these carefully.

CFA Institute Practice Questions Your examination registration fee includes digital access to hundreds of practice questions that are additional to the practice problems at the end of the readings. These practice questions are intended to help you assess your mastery of individual topic areas as you progress through your studies. After each practice question, you will be able to receive immediate feedback noting the correct responses and indicating the relevant assigned reading so you can identify areas of weakness for further study. For more information on the practice questions, please visit www.cfainstitute.org.

CFA Institute Mock Examinations Your examination registration fee also includes digital access to three-hour mock examinations that simulate the morning and afternoon sessions of the actual CFA examination. These mock examinations are intended to be taken after you complete your study of the full curriculum and take practice questions so you can test your understanding of the curriculum and your readiness for the examination. You will receive feedback at the end of the mock examination, noting the correct responses and indicating the relevant assigned readings so you can assess areas of weakness for further study during your review period. We recommend that you take mock examinations during the final stages of your preparation for the actual CFA examination. For more information on the mock examinations, please visit www.cfainstitute.org.

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Preparatory Providers After you enroll in the CFA Program, you may receive numerous solicitations for preparatory courses and review materials. When considering a preparatory course, make sure the provider belongs to the CFA Institute Approved Prep Provider Program. Approved Prep Providers have committed to follow CFA Institute guidelines and high standards in their offerings and communications with candidates. For more information on the Approved Prep Providers, please visit www.cfainstitute.org/programs/cfa/exam/prep-providers.

Remember, however, that there are no shortcuts to success on the CFA examinations; reading and studying the CFA curriculum *is* the key to success on the examination. The CFA examinations reference only the CFA Institute assigned curriculum—no preparatory course or review course materials are consulted or referenced.

SUMMARY

Every question on the CFA examination is based on the content contained in the required readings and on one or more LOS. Frequently, an examination question is based on a specific example highlighted within a reading or on a specific practice problem and its solution. To make effective use of the CFA Program curriculum, please remember these key points:

1. All pages of the curriculum are required reading for the examination except for occasional sections marked as optional. You may read optional pages as background, but you will not be tested on them.
2. All questions, problems, and their solutions—found at the end of readings—are part of the curriculum and are required study material for the examination.
3. You should make appropriate use of the practice questions and mock examinations as well as other supplemental study tools and candidate resources available at www.cfainstitute.org.
4. Create a schedule and commit sufficient study time to cover the 17 study sessions using the study planner. You should also plan to review the materials and take topic tests and mock examinations.
5. Some of the concepts in the study sessions may be superseded by updated rulings and/or pronouncements issued after a reading was published. Candidates are expected to be familiar with the overall analytical framework contained in the assigned readings. Candidates are not responsible for changes that occur after the material was written.

FEEDBACK

At CFA Institute, we are committed to delivering a comprehensive and rigorous curriculum for the development of competent, ethically grounded investment professionals. We rely on candidate and investment professional comments and feedback as we work to improve the curriculum, supplemental study tools, and candidate resources.

Please send any comments or feedback to info@cfainstitute.org. You can be assured that we will review your suggestions carefully. Ongoing improvements in the curriculum will help you prepare for success on the upcoming examinations and for a lifetime of learning as a serious investment professional.

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Fixed Income

STUDY SESSIONS

Study Session 12	Fixed Income (1)
Study Session 13	Fixed Income (2)

TOPIC LEVEL LEARNING OUTCOME

The candidate should be able to estimate the risks and expected returns for fixed-income instruments, analyze the term structure of interest rates and yield spreads, and evaluate fixed-income instruments with embedded options and unique features.

Understanding interest rate dynamics including changes in the yield curve is critical for investment activities such as economic and capital market forecasting, asset allocation, and active fixed-income management. Active fixed-income managers, for instance, must identify and exploit perceived investment opportunities, manage interest rate and yield curve exposure, and report on benchmark relative performance.

Many fixed-income securities contain embedded options. Issuers use bonds with call provisions to manage interest rate exposure and interest payments. Investors may prefer bonds granting early redemption or equity conversion rights. Given their widespread use and inherent complexity, investors and issuers should understand when option exercise might occur and how to value these bonds.

Evaluating bonds for credit risk is very important. As demonstrated by the 2008 global financial crisis, systemic mispricing of risk can have wide ranging and severe consequences that extend far beyond any individual position or portfolio.

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FIXED INCOME

STUDY SESSION 12

Fixed Income (1)

This study session introduces the yield curve and key relationships underlying its composition. Traditional and modern theories and models explaining the shape of the yield curve are presented. An arbitrage-free framework using observed market prices is introduced for valuing option-free bonds. This approach also holds for more complex valuation of bonds with embedded options and other bond types.

READING ASSIGNMENTS

Reading 32	The Term Structure and Interest Rate Dynamics by Thomas S.Y. Ho, PhD, Sang Bin Lee, PhD, and Stephen E. Wilcox, PhD, CFA
Reading 33	The Arbitrage-Free Valuation Framework by Steven V. Mann, PhD

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READING

32

The Term Structure and Interest Rate Dynamics

by Thomas S.Y. Ho, PhD, Sang Bin Lee, PhD, and Stephen E. Wilcox, PhD, CFA

Thomas S.Y. Ho, PhD, is at Thomas Ho Company Ltd (USA). Sang Bin Lee, PhD, is at Hanyang University (South Korea). Stephen E. Wilcox, PhD, CFA, is at Minnesota State University, Mankato (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** describe relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve;
☐	**b.** describe the forward pricing and forward rate models and calculate forward and spot prices and rates using those models;
☐	**c.** describe how zero-coupon rates (spot rates) may be obtained from the par curve by bootstrapping;
☐	**d.** describe the assumptions concerning the evolution of spot rates in relation to forward rates implicit in active bond portfolio management;
☐	**e.** describe the strategy of riding the yield curve;
☐	**f.** explain the swap rate curve and why and how market participants use it in valuation;
☐	**g.** calculate and interpret the swap spread for a given maturity;
☐	**h.** describe the Z-spread;
☐	**i.** describe the TED and Libor–OIS spreads;
☐	**j.** explain traditional theories of the term structure of interest rates and describe the implications of each theory for forward rates and the shape of the yield curve;

(continued)

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LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	**k.** describe modern term structure models and how they are used;
☐	**l.** explain how a bond's exposure to each of the factors driving the yield curve can be measured and how these exposures can be used to manage yield curve risks;
☐	**m.** explain the maturity structure of yield volatilities and their effect on price volatility.

1 INTRODUCTION

Interest rates are both a barometer of the economy and an instrument for its control. The term structure of interest rates—market interest rates at various maturities—is a vital input into the valuation of many financial products. The goal of this reading is to explain the term structure and interest rate dynamics—that is, the process by which the yields and prices of bonds evolve over time.

A spot interest rate (in this reading, "spot rate") is a rate of interest on a security that makes a single payment at a future point in time. The forward rate is the rate of interest set today for a single-payment security to be issued at a future date. Section 2 explains the relationship between these two types of interest rates and why forward rates matter to active bond portfolio managers. Section 2 also briefly covers other important return concepts.

The swap rate curve is the name given to the swap market's equivalent of the yield curve. Section 3 describes in more detail the swap rate curve and a related concept, the swap spread, and describes their use in valuation.

Sections 4 and 5 describe traditional and modern theories of the term structure of interest rates, respectively. Traditional theories present various largely qualitative perspectives on economic forces that may affect the shape of the term structure. Modern theories model the term structure with greater rigor.

Section 6 describes yield curve factor models. The focus is a popular three-factor term structure model in which the yield curve changes are described in terms of three independent movements: level, steepness, and curvature. These factors can be extracted from the variance–covariance matrix of historical interest rate movements.

A summary of key points concludes the reading.

2 SPOT RATES AND FORWARD RATES

In this section, we will first explain the relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve. We will then discuss the assumptions made about forward rates in active bond portfolio management.

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At any point in time, the price of a risk-free single-unit payment (e.g., \$1, €1, or £1) at time T is called the **discount factor** with maturity T, denoted by $P(T)$. The yield to maturity of the payment is called a **spot rate**, denoted by $r(T)$. That is,

$$P(T) = \frac{1}{\left[1 + r(T)\right]^T} \tag{1}$$

The discount factor, $P(T)$, and the spot rate, $r(T)$, for a range of maturities in years $T > 0$ are called the **discount function** and the **spot yield curve** (or, more simply, **spot curve**), respectively. The spot curve represents the term structure of interest rates at any point in time. Note that the discount function completely identifies the spot curve and vice versa. The discount function and the spot curve contain the same set of information about the time value of money.

The spot curve shows, for various maturities, the annualized return on an option-free and default-risk-free **zero-coupon bond** (**zero** for short) with a single payment of principal at maturity. The spot rate as a yield concept avoids the complications associated with the need for a reinvestment rate assumption for coupon-paying securities. Because the spot curve depends on the market pricing of these option-free zero-coupon bonds at any point in time, the shape and level of the spot yield curve are dynamic—that is, continually changing over time.

As Equation 1 suggests, the default-risk-free spot curve is a benchmark for the time value of money received at any future point in time as determined by the market supply and demand for funds. It is viewed as the most basic term structure of interest rates because there is no reinvestment risk involved; the stated yield equals the actual realized return if the zero is held to maturity. Thus, the yield on a zero-coupon bond maturing in year T is regarded as the most accurate representation of the T-year interest rate.

A **forward rate** is an interest rate that is determined today for a loan that will be initiated in a future time period. The term structure of forward rates for a loan made on a specific initiation date is called the **forward curve**. Forward rates and forward curves can be mathematically derived from the current spot curve.

Denote the forward rate of a loan initiated T^* years from today with tenor (further maturity) of T years by $f(T^*,T)$. Consider a forward contract in which one party to the contract, the buyer, commits to pay the other party to the contract, the seller, a forward contract price, denoted by $F(T^*,T)$, at time T^* years from today for a zero-coupon bond with maturity T years and unit principal. This is only an agreement to do something in the future at the time the contract is entered into; thus, no money is exchanged between the two parties at contract initiation. At T^*, the buyer will pay the seller the contracted forward price value and will receive from the seller at time $T^* + T$ the principal payment of the bond, defined here as a single currency unit.

The **forward pricing model** describes the valuation of forward contracts. The no-arbitrage argument that is used to derive the model is frequently used in modern financial theory; the model can be adopted to value interest rate futures contracts and related instruments, such as options on interest rate futures.

The no-arbitrage principle is quite simple. It says that tradable securities with identical cash flow payments must have the same price. Otherwise, traders would be able to generate risk-free arbitrage profits. Applying this argument to value a forward contract, we consider the discount factors—in particular, the values $P(T^*)$ and $P(T^* + T)$ needed to price a forward contract, $F(T^*,T)$. This forward contract price has to follow Equation 2, which is known as the forward pricing model.

$$P(T^* + T) = P(T^*)F(T^*,T) \tag{2}$$

To understand the reasoning behind Equation 2, consider two alternative investments: (1) buying a zero-coupon bond that matures in $T^* + T$ years at a cost of $P(T^*+ T)$, and (2) entering into a forward contract valued at $F(T^*,T)$ to buy at T^* a zero-coupon bond

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with maturity T at a cost today of $P(T^*)F(T^*,T)$. The payoffs for the two investments at time $T^* + T$ are the same. For this reason, the initial costs of the investments have to be the same, and therefore, Equation 2 must hold. Otherwise, any trader could sell the overvalued investment and buy the undervalued investment with the proceeds to generate risk-free profits with zero net investment.

Working the problems in Example 1 should help confirm your understanding of discount factors and forward prices. Please note that the solutions in the examples that follow may be rounded to two or four decimal places.

EXAMPLE 1

Spot and Forward Prices and Rates (1)

Consider a two-year loan ($T = 2$) beginning in one year ($T^* = 1$). The one-year spot rate is $r(T^*) = r(1) = 7\% = 0.07$. The three-year spot rate is $r(T^* + T) = r(1 + 2) = r(3) = 9\% = 0.09$.

1 Calculate the one-year discount factor: $P(T^*) = P(1)$.
2 Calculate the three-year discount factor: $P(T^* + T) = P(1 + 2) = P(3)$.
3 Calculate the forward price of a two-year bond to be issued in one year: $F(T^*,T) = F(1,2)$.
4 Interpret your answer to Problem 3.

Solution to 1:

Using Equation 1,

$$P(1) = \frac{1}{(1 + 0.07)^1} = 0.9346$$

Solution to 2:

$$P(3) = \frac{1}{(1 + 0.09)^3} = 0.7722$$

Solution to 3:

Using Equation 2,

$0.7722 = 0.9346 \times F(1,2)$.

$F(1,2) = 0.7722 \div 0.9346 = 0.8262$.

Solution to 4:

The forward contract price of $F(1,2) = 0.8262$ is the price, agreed on today, that would be paid one year from today for a bond with a two-year maturity and a risk-free unit-principal payment (e.g., \$1, €1, or £1) at maturity. As shown in the solution to 3, it is calculated as the three-year discount factor, $P(3) = 0.7722$, divided by the one-year discount factor, $P(1) = 0.9346$.

2.1 The Forward Rate Model

This section uses the forward rate model to establish that when the spot curve is upward sloping, the forward curve will lie above the spot curve, and that when the spot curve is downward sloping, the forward curve will lie below the spot curve.

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The forward rate $f(T^*,T)$ is the discount rate for a risk-free unit-principal payment $T^* + T$ years from today, valued at time T^*, such that the present value equals the forward contract price, $F(T^*,T)$. Then, by definition,

$$F(T^*,T) = \frac{1}{\left[1 + f(T^*,T)\right]^T} \quad (3)$$

By substituting Equations 1 and 3 into Equation 2, the forward pricing model can be expressed in terms of rates as noted by Equation 4, which is the **forward rate model**:

$$\left[1 + r(T^* + T)\right]^{(T^*+T)} = \left[1 + r(T^*)\right]^{T^*}\left[1 + f(T^*,T)\right]^T \quad (4)$$

Thus, the spot rate for $T^* + T$, which is $r(T^* + T)$, and the spot rate for T^*, which is $r(T^*)$, imply a value for the T-year forward rate at T^*, $f(T^*,T)$. Equation 4 is important because it shows how forward rates can be extrapolated from spot rates; that is, they are implicit in the spot rates at any given point in time.[1]

Equation 4 suggests two interpretations or ways to look at forward rates. For example, suppose $f(7,1)$, the rate agreed on today for a one-year loan to be made seven years from today, is 3%. Then 3% is the

- reinvestment rate that would make an investor indifferent between buying an eight-year zero-coupon bond or investing in a seven-year zero-coupon bond and at maturity reinvesting the proceeds for one year. In this sense, the forward rate can be viewed as a type of breakeven interest rate.
- one-year rate that can be locked in today by buying an eight-year zero-coupon bond rather than investing in a seven-year zero-coupon bond and, when it matures, reinvesting the proceeds in a zero-coupon instrument that matures in one year. In this sense, the forward rate can be viewed as a rate that can be locked in by extending maturity by one year.

Example 2 addresses forward rates and the relationship between spot and forward rates.

EXAMPLE 2

Spot and Forward Prices and Rates (2)

The spot rates for three hypothetical zero-coupon bonds (zeros) with maturities of one, two, and three years are given in the following table.

Maturity (T)	**1**	**2**	**3**
Spot rates	$r(1)$ = 9%	$r(2)$ = 10%	$r(3)$ = 11%

1. Calculate the forward rate for a one-year zero issued one year from today, $f(1,1)$.
2. Calculate the forward rate for a one-year zero issued two years from today, $f(2,1)$.
3. Calculate the forward rate for a two-year zero issued one year from today, $f(1,2)$.
4. Based on your answers to 1 and 2, describe the relationship between the spot rates and the implied one-year forward rates.

1 An approximation formula that is based on taking logs of both sides of Equation 4 and using the approximation $\ln(1 + x) \approx x$ for small x is $f(T^*,T) \approx [(T^* + T)r(T^* + T) - T^*r(T^*)]/T$. For example, $f(1,2)$ in Example 2 could be approximated as (3 × 11% – 1 × 9%)/2 = 12%, which is very close to 12.01%.

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Solution to 1:

$f(1,1)$ is calculated as follows (using Equation 4):

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$

$$(1 + 0.10)^2 = (1 + 0.09)^1[1 + f(1,1)]^1$$

$$f(1,1) = \frac{(1.10)^2}{1.09} - 1 = 11.01\%$$

Solution to 2:

$f(2,1)$ is calculated as follows:

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$

$$(1 + 0.11)^3 = (1 + 0.10)^2[1 + f(2,1)]^1$$

$$f(2,1) = \frac{(1.11)^3}{(1.10)^2} - 1 = 13.03\%$$

Solution to 3:

$f(1,2)$ is calculated as follows:

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

$$(1 + 0.11)^3 = (1 + 0.09)^1[1 + f(1,2)]^2$$

$$f(1,2) = \sqrt[2]{\frac{(1.11)^3}{1.09}} - 1 = 12.01\%$$

Solution to 4:

The upward-sloping zero-coupon yield curve is associated with an upward-sloping forward curve (a series of increasing one-year forward rates because 13.03% is greater than 11.01%). This point is explained further in the following paragraphs.

The analysis of the relationship between spot rates and one-period forward rates can be established by using the forward rate model and successive substitution, resulting in Equations 5a and 5b:

$$[1 + r(T)]^T = [1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)] \quad \textbf{(5a)}$$

$$r(T) = \{[1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)]\}^{(1/T)} - 1 \quad \textbf{(5b)}$$

Equation 5b shows that the spot rate for a security with a maturity of $T > 1$ can be expressed as a geometric mean of the spot rate for a security with a maturity of $T = 1$ and a series of $T - 1$ forward rates.

Whether the relationship in Equation 5b holds in practice is an important consideration for active portfolio management. If an active trader can identify a series of short-term bonds whose actual returns will exceed today's quoted forward rates, then the total return over his or her investment horizon would exceed the return on a maturity-matching, buy-and-hold strategy. Later, we will use this same concept to discuss dynamic hedging strategies and the local expectations theory.

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Examples 3 and 4 explore the relationship between spot and forward rates.

EXAMPLE 3

Spot and Forward Prices and Rates (3)

Given the data and conclusions for $r(1)$, $f(1,1)$, and $f(2,1)$ from Example 2:

$r(1) = 9\%$

$f(1,1) = 11.01\%$

$f(2,1) = 13.03\%$

Show that the two-year spot rate of $r(2) = 10\%$ and the three-year spot rate of $r(3) = 11\%$ are geometric averages of the one-year spot rate and the forward rates.

Solution:

Using Equation 5a,

$$\left[1 + r(2)\right]^2 = \left[1 + r(1)\right]\left[1 + f(1,1)\right]$$

$$r(2) = \sqrt[2]{(1 + 0.09)(1 + 0.1101)} - 1 \approx 10\%$$

$$\left[1 + r(3)\right]^3 = \left[1 + r(1)\right]\left[1 + f(1,1)\right]\left[1 + f(2,1)\right]$$

$$r(3) = \sqrt[3]{(1 + 0.09)(1 + 0.1101)(1 + 0.1303)} - 1 \approx 11\%$$

We can now consolidate our knowledge of spot and forward rates to explain important relationships between the spot and forward rate curves. The forward rate model (Equation 4) can also be expressed as Equation 6.

$$\left\{\frac{\left[1 + r(T^* + T)\right]}{\left[1 + r(T^*)\right]}\right\}^{\frac{T^*}{T}}\left[1 + r(T^* + T)\right] = \left[1 + f(T^*,T)\right] \tag{6}$$

To illustrate, suppose $T^* = 1$, $T = 4$, $r(1) = 2\%$, and $r(5) = 3\%$; the left-hand side of Equation 6 is

$$\left(\frac{1.03}{1.02}\right)^{\frac{1}{4}}(1.03) = (1.0024)(1.03) = 1.0325$$

so $f(1,4) = 3.25\%$. Given that the yield curve is upward sloping—so, $r(T^* + T) > r(T^*)$—Equation 6 implies that the forward rate from T^* to T is greater than the long-term $(T^* + T)$ spot rate: $f(T^*,T) > r(T^* + T)$. In the example given, 3.25% > 3%. Conversely, when the yield curve is downward sloping, then $r(T^* + T) < r(T^*)$ and the forward rate from T^* to T is lower than the long-term spot rate: $f(T^*,T) < r(T^* + T)$. Equation 6 also shows that if the spot curve is flat, all one-period forward rates are equal to the spot rate. For an upward-sloping yield curve—$r(T^* + T) > r(T^*)$—the forward rate rises as T^* increases. For a downward-sloping yield curve—$r(T^* + T) < r(T^*)$—the forward rate declines as T^* increases.

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EXAMPLE 4

Spot and Forward Prices and Rates (4)

Given the spot rates $r(1) = 9\%$, $r(2) = 10\%$, and $r(3) = 11\%$, as in Examples 2 and 3:

1 Determine whether the forward rate $f(1,2)$ is greater than or less than the long-term rate, $r(3)$.

2 Determine whether forward rates rise or fall as the initiation date, T^*, for the forward rate is increased.

Solution to 1:

The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$, or in general, $r(T^* + T) > r(T^*)$. Thus, the forward rate will be greater than the long-term rate, or $f(T^*,T) > r(T^* + T)$. Note from Example 2 that $f(1,2) = 12.01\% > r(1 + 2) = r(3) = 11\%$.

Solution to 2:

The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$. Thus, the forward rates will rise with increasing T^*. This relationship was shown in Example 2, in which $f(1,1) = 11.01\%$ and $f(2,1) = 13.03\%$.

These relationships are illustrated in Exhibit 1, using actual data. The spot rates for US Treasuries as of 31 July 2013 are represented by the lowest curve in the exhibit, which was constructed using interpolation between the data points, shown in the table following the exhibit. Note that the spot curve is upward sloping. The spot curve and the forward curves for the end of July 2014, July 2015, July 2016, and July 2017 are also presented in Exhibit 1. Because the yield curve is upward sloping, the forward curves lie above the spot curve and increasing the initiation date results in progressively higher forward curves. The highest forward curve is that for July 2017. Note that the forward curves in Exhibit 1 are progressively flatter at later start dates because the spot curve flattens at the longer maturities.

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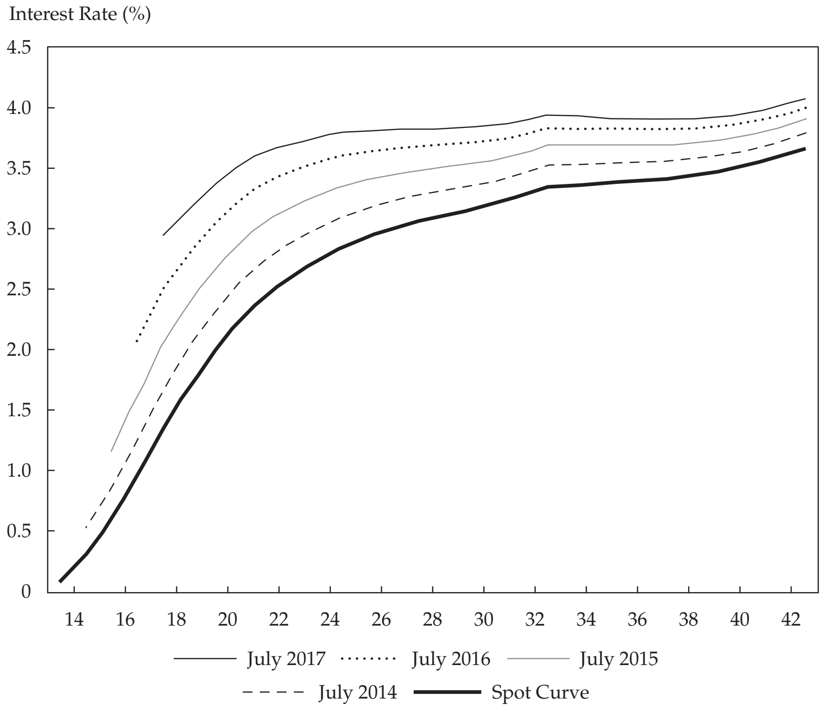
Exhibit 1 Spot Curve vs. Forward Curves, 31 July 2013



Maturity (years)	1	2	3	5	7	10	20	30
Spot rate (%)	0.11	0.33	0.61	1.37	2.00	2.61	3.35	3.66

When the spot yield curve is downward sloping, the forward yield curve will be below the spot yield curve. Spot rates for US Treasuries as of 31 December 2006 are presented in the table following Exhibit 2. We used linear interpolation to construct the spot curve based on these data points. The yield curve data were also somewhat modified to make the yield curve more downward sloping for illustrative purposes. The spot curve and the forward curves for the end of December 2007, 2008, 2009, and 2010 are presented in Exhibit 2.

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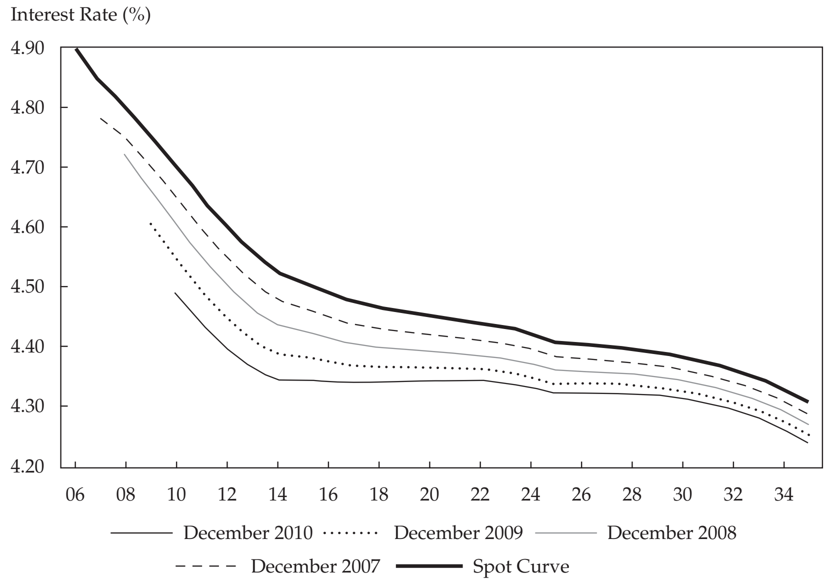
Exhibit 2 Spot Curve vs. Forward Curves, 31 December 2006 (Modified for Illustrative Purposes)



Maturity (years)	1	2	3	5	7	10	20	30
Spot rate (%)	4.90	4.82	4.74	4.70	4.60	4.51	4.41	4.31

The highest curve is the spot yield curve, and it is downward sloping. The results show that the forward curves are lower than the spot curve. Postponing the initiation date results in progressively lower forward curves. The lowest forward curve is that dated December 2010.

An important point that can be inferred from Exhibit 1 and Exhibit 2 is that forward rates do not extend any further than the furthest maturity on today's yield curve. For example, if yields extend to 30 years on today's yield curve, then three years hence, the most we can model prospectively is a bond with 27 years to final maturity. Similarly, four years hence, the longest maturity forward rate would be $f(4,26)$.

In summary, when the spot curve is upward sloping, the forward curve will lie above the spot curve. Conversely, when the spot curve is downward sloping, the forward curve will lie below the spot curve. This relationship is a reflection of the basic mathematical truth that when the average is rising (falling), the marginal data point must be above (below) the average. In this case, the spot curve represents an average over a whole time period and the forward rates represent the marginal changes between future time periods.[2]

We have thus far discussed the spot curve and the forward curve. Another curve important in practice is the government par curve. The **par curve** represents the yields to maturity on coupon-paying government bonds, priced at par, over a range of maturities. In practice, recently issued ("on the run") bonds are typically used to create the par curve because new issues are typically priced at or close to par.

2 Extending this discussion, one can also conclude that when a spot curve rises and then falls, the forward curves will also rise and then fall.

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The par curve is important for valuation in that it can be used to construct a zero-coupon yield curve. The process makes use of the fact that a coupon-paying bond can be viewed as a portfolio of zero-coupon bonds. The zero-coupon rates are determined by using the par yields and solving for the zero-coupon rates one by one, in order from earliest to latest maturities, via a process of forward substitution known as **bootstrapping**.

WHAT IS BOOTSTRAPPING?

The practical details of deriving the zero-coupon yield are outside the scope of this reading. But the meaning of bootstrapping cannot be grasped without a numerical illustration. Suppose the following yields are observed for annual coupon sovereign debt:

Par Rates:

One-year par rate = 5%, Two-year par rate = 5.97%, Three-year par rate = 6.91%, Four-year par rate = 7.81%. From these we can bootstrap zero-coupon rates.

Zero-Coupon Rates:

The one-year zero-coupon rate is the same as the one-year par rate because, under the assumption of annual coupons, it is effectively a one-year pure discount instrument. However, the two-year bond and later-maturity bonds have coupon payments before maturity and are distinct from zero-coupon instruments.

The process of deriving zero-coupon rates begins with the two-year maturity. The two-year zero-coupon rate is determined by solving the following equation in terms of one monetary unit of current market value, using the information that $r(1)$ = 5%:

$$1 = \frac{0.0597}{(1.05)} + \frac{1 + 0.0597}{\left[1 + r(2)\right]^2}$$

In the equation, 0.0597 and 1.0597 represent payments from interest and principal and interest, respectively, per one unit of principal value. The equation implies that $r(2)$ = 6%. We have bootstrapped the two-year spot rate. Continuing with forward substitution, the three-year zero-coupon rate can be bootstrapped by solving the following equation, using the known values of the one-year and two-year spot rates of 5% and 6%:

$$1 = \frac{0.0691}{(1.05)} + \frac{0.0691}{(1.06)^2} + \frac{1 + 0.0691}{\left[1 + r(3)\right]^3}$$

Thus, $r(3)$ = 7%. Finally the four-year zero-coupon rate is determined to be 8% by using

$$1 = \frac{0.0781}{(1.05)} + \frac{0.0781}{(1.06)^2} + \frac{0.0781}{(1.07)^3} + \frac{1 + 0.0781}{\left[1 + r(4)\right]^4}$$

In summary, $r(1)$ = 5%, $r(2)$ = 6%, $r(3)$ = 7%, and $r(4)$ = 8%.

In the preceding discussion, we considered an upward-sloping (spot) yield curve (Exhibit 1) and an inverted or downward-sloping (spot) yield curve (Exhibit 2). In developed markets, yield curves are most commonly upward sloping with diminishing marginal increases in yield for identical changes in maturity; that is, the yield curve "flattens" at longer maturities. Because nominal yields incorporate a premium for expected inflation, an upward-sloping yield curve is generally interpreted as reflecting a market expectation of increasing or at least level future inflation (associated with relatively strong economic growth). The existence of risk premiums (e.g., for the greater interest rate risk of longer-maturity bonds) also contributes to a positive slope.

An inverted yield curve (Exhibit 2) is somewhat uncommon. Such a term structure may reflect a market expectation of declining future inflation rates (because a nominal yield incorporates a premium for expected inflation) from a relatively high current

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level. Expectations of declining economic activity may be one reason that inflation might be anticipated to decline, and a downward-sloping yield curve has frequently been observed before recessions.[3] A flat yield curve typically occurs briefly in the transition from an upward-sloping to a downward-sloping yield curve, or vice versa. A humped yield curve, which is relatively rare, occurs when intermediate-term interest rates are higher than short- and long-term rates.

2.2 Yield to Maturity in Relation to Spot Rates and Expected and Realized Returns on Bonds

Yield to maturity (YTM) is perhaps the most familiar pricing concept in bond markets. In this section, our goal is to clarify how it is related to spot rates and a bond's expected and realized returns.

How is the yield to maturity related to spot rates? In bond markets, most bonds outstanding have coupon payments and many have various options, such as a call provision. The YTM of these bonds with maturity T would not be the same as the spot rate at T. But, the YTM should be mathematically related to the spot curve. Because the principle of no arbitrage shows that a bond's value is the sum of the present values of payments discounted by their corresponding spot rates, the YTM of the bond should be some weighted average of spot rates used in the valuation of the bond.

Example 5 addresses the relationship between spot rates and yield to maturity.

EXAMPLE 5

Spot Rate and Yield to Maturity

Recall from earlier examples the spot rates $r(1) = 9\%$, $r(2) = 10\%$, and $r(3) = 11\%$. Let $y(T)$ be the yield to maturity.

1 Calculate the price of a two-year annual coupon bond using the spot rates. Assume the coupon rate is 6% and the face value is \$1,000. Next, state the formula for determining the price of the bond in terms of its yield to maturity. Is $r(2)$ greater than or less than $y(2)$? Why?

2 Calculate the price of a three-year annual coupon-paying bond using the spot rates. Assume the coupon rate is 5% and the face value is £100. Next, write a formula for determining the price of the bond using the yield to maturity. Is $r(3)$ greater or less than $y(3)$? Why?

Solution to 1:

Using the spot rates,

$$\text{Price} = \frac{\$60}{(1+0.09)^1} + \frac{\$1{,}060}{(1+0.10)^2} = \$931.08$$

Using the yield to maturity,

$$\text{Price} = \frac{\$60}{\left[1+y(2)\right]^1} + \frac{\$1{,}060}{\left[1+y(2)\right]^2} = \$931.08$$

3 The US Treasury yield curve inverted in August 2006, more than a year before the recession that began in December 2007. See Haubrich (2006).

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Note that $y(2)$ is used to discount both the first- and second-year cash flows. Because the bond can have only one price, it follows that $r(1) < y(2) < r(2)$ because $y(2)$ is a weighted average of $r(1)$ and $r(2)$ and the yield curve is upward sloping. Using a calculator, one can calculate the yield to maturity $y(2) = 9.97\%$, which is less than $r(2) = 10\%$ and greater than $r(1) = 9\%$, just as we would expect. Note that $y(2)$ is much closer to $r(2)$ than to $r(1)$ because the bond's largest cash flow occurs in Year 2, thereby giving $r(2)$ a greater weight than $r(1)$ in the determination of $y(2)$.

Solution to 2:

Using the spot rates,

$$\text{Price} = \frac{£5}{(1 + 0.09)^1} + \frac{£5}{(1 + 0.10)^2} + \frac{£105}{(1 + 0.11)^3} = £85.49$$

Using the yield to maturity,

$$\text{Price} = \frac{£5}{\left[1 + y(3)\right]^1} + \frac{£5}{\left[1 + y(3)\right]^2} + \frac{£105}{\left[1 + y(3)\right]^3} = £85.49$$

Note that $y(3)$ is used to discount all three cash flows. Because the bond can have only one price, $y(3)$ must be a weighted average of $r(1)$, $r(2)$, and $r(3)$. Given that the yield curve is upward sloping in this example, $y(3) < r(3)$. Using a calculator to compute yield to maturity, $y(3) = 10.93\%$, which is less than $r(3) = 11\%$ and greater than $r(1) = 9\%$, just as we would expect because the weighted yield to maturity must lie between the highest and lowest spot rates. Note that $y(3)$ is much closer to $r(3)$ than it is to $r(2)$ or $r(1)$ because the bond's largest cash flow occurs in Year 3, thereby giving $r(3)$ a greater weight than $r(2)$ and $r(1)$ in the determination of $y(3)$.

Is the yield to maturity the expected return on a bond? In general, it is not, except under extremely restrictive assumptions. The expected rate of return is the return one anticipates earning on an investment. The YTM is the expected rate of return for a bond that is held until its maturity, assuming that all coupon and principal payments are made in full when due and that coupons are reinvested at the original YTM. However, the assumption regarding reinvestment of coupons at the original yield to maturity typically does not hold. The YTM can provide a poor estimate of expected return if (1) interest rates are volatile; (2) the yield curve is steeply sloped, either upward or downward; (3) there is significant risk of default; or (4) the bond has one or more embedded options (e.g., put, call, or conversion). If either (1) or (2) is the case, reinvestment of coupons would not be expected to be at the assumed rate (YTM). Case (3) implies that actual cash flows may differ from those assumed in the YTM calculation, and in case (4), the exercise of an embedded option would, in general, result in a holding period that is shorter than the bond's original maturity.

The realized return is the actual return on the bond during the time an investor holds the bond. It is based on actual reinvestment rates and the yield curve at the end of the holding period. With perfect foresight, the expected bond return would equal the realized bond return.

To illustrate these concepts, assume that $r(1) = 5\%$, $r(2) = 6\%$, $r(3) = 7\%$, $r(4) = 8\%$, and $r(5) = 9\%$. Consider a five-year annual coupon bond with a coupon rate of 10%. The forward rates extrapolated from the spot rates are $f(1,1) = 7.0\%$, $f(2,1) = 9.0\%$, $f(3,1) = 11.1\%$, and $f(4,1) = 13.1\%$. The price, determined as a percentage of par, is 105.43.

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The yield to maturity of 8.62% can be determined using a calculator or by solving

$$105.43 = \frac{10}{\left[1 + y(5)\right]} + \frac{10}{\left[1 + y(5)\right]^2} + \cdots + \frac{110}{\left[1 + y(5)\right]^5}$$

The yield to maturity of 8.62% is the bond's expected return assuming no default, a holding period of five years, and a reinvestment rate of 8.62%. But what if the forward rates are assumed to be the future spot rates?

Using the forward rates as the expected reinvestment rates results in the following expected cash flow at the end of Year 5:

$$10(1 + 0.07)(1 + 0.09)(1 + 0.111)(1 + 0.131) + 10(1 + 0.09)(1 + 0.011)(1 + 0.131) + 10(1 + 0.111)(1 + 0.131) + 10(1 + 0.131) + 110 \approx 162.22$$

Therefore, the expected bond return is (162.22 – 105.43)/105.43 = 53.87% and the expected annualized rate of return is 9.00% [solve $(1 + x)^5 = 1 + 0.5387$].

From this example, we can see that the expected rate of return is not equal to the YTM even if we make the generally unrealistic assumption that the forward rates are the future spot rates. Implicit in the determination of the yield to maturity as a potentially realistic estimate of expected return is a flat yield curve; note that in the formula just used, every cash flow was discounted at 8.62% regardless of its maturity.

Example 6 will reinforce your understanding of various yield and return concepts.

EXAMPLE 6

Yield and Return Concepts

1 When the spot curve is upward sloping, the forward curve:

- **A** lies above the spot curve.
- **B** lies below the spot curve.
- **C** is coincident with the spot curve.

2 Which of the following statements concerning the yield to maturity of a default-risk-free bond is *most* accurate? The yield to maturity of such a bond:

- **A** equals the expected return on the bond if the bond is held to maturity.
- **B** can be viewed as a weighted average of the spot rates applying to its cash flows.
- **C** will be closer to the realized return if the spot curve is upward sloping rather than flat through the life of the bond.

3 When the spot curve is downward sloping, an increase in the initiation date results in a forward curve that is:

- **A** closer to the spot curve.
- **B** a greater distance above the spot curve.
- **C** a greater distance below the spot curve.

Solution to 1:

A is correct. Points on a spot curve can be viewed as an average of single-period rates over given maturities whereas forward rates reflect the marginal changes between future time periods.

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Solution to 2:

B is correct. The YTM is the discount rate that, when applied to a bond's promised cash flows, equates those cash flows to the bond's market price and the fact that the market price should reflect discounting promised cash flows at appropriate spot rates.

Solution to 3:

C is correct. This answer follows from the forward rate model as expressed in Equation 6. If the spot curve is downward sloping (upward sloping), increasing the initiation date (T^*) will result in a forward curve that is a greater distance below (above) the spot curve. See Exhibit 1 and Exhibit 2.

2.3 Yield Curve Movement and the Forward Curve

This section establishes several important results concerning forward prices and the spot yield curve in anticipation of discussing the relevance of the forward curve to active bond investors.

The first observation is that the forward contract price remains unchanged as long as future spot rates evolve as predicted by today's forward curve. Therefore, a change in the forward price reflects a deviation of the spot curve from that predicted by today's forward curve. Thus, if a trader expects that the future spot rate will be lower than what is predicted by the prevailing forward rate, the forward contract value is expected to increase. To capitalize on this expectation, the trader would buy the forward contract. Conversely, if the trader expects the future spot rate to be higher than what is predicted by the existing forward rate, then the forward contract value is expected to decrease. In this case, the trader would sell the forward contract.

Using the forward pricing model defined by Equation 2, we can determine the forward contract price that delivers a T-year-maturity bond at time T^*, $F(T^*,T)$ using Equation 7 (which is Equation 2 solved for the forward price):

$$F(T^*,T) = \frac{P(T^* + T)}{P(T^*)} \tag{7}$$

Now suppose that after time t, the new discount function is the same as the forward discount function implied by today's discount function, as shown by Equation 8.

$$P^*(T) = \frac{P(t + T)}{P(t)} \tag{8}$$

Next, after a lapse of time t, the time to expiration of the contract is $T^* - t$, and the forward contract price at time t is $F^*(t,T^*,T)$. Equation 7 can be rewritten as Equation 9:

$$F^*(t,T^*,T) = \frac{P^*(T^* + T - t)}{P^*(T^* - t)} \tag{9}$$

Substituting Equation 8 into Equation 9 and adjusting for the lapse of time t results in Equation 10:

$$F^*(t,T^*,T) = \frac{\dfrac{P(t + T^* + T - t)}{P(t)}}{\dfrac{P(t + T^* - t)}{P(t)}} = \frac{P(T^* + T)}{P(T^*)} = F(T^*,T) \tag{10}$$

Equation 10 shows that the forward contract price remains unchanged as long as future spot rates are equal to what is predicted by today's forward curve. Therefore, a change in the forward price is the result of a deviation of the spot curve from what is predicted by today's forward curve.

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To make these observations concrete, consider a flat yield curve for which the interest rate is 4%. Using Equation 1, the discount factors for the one-year, two-year, and three-year terms are, to four decimal places,

$$P(1) = \frac{1}{(1 + 0.04)^1} = 0.9615$$

$$P(2) = \frac{1}{(1 + 0.04)^2} = 0.9246$$

$$P(3) = \frac{1}{(1 + 0.04)^3} = 0.8890$$

Therefore, using Equation 7, the forward contract price that delivers a one-year bond at Year 2 is

$$F(2,1) = \frac{P(2+1)}{P(2)} = \frac{P(3)}{P(2)} = \frac{0.8890}{0.9246} = 0.9615$$

Suppose the future discount function at Year 1 is the same as the forward discount function implied by the Year 0 spot curve. The lapse of time is $t = 1$. Using Equation 8, the discount factors for the one-year and two-year terms one year from today are

$$P^*(1) = \frac{P(1+1)}{P(1)} = \frac{P(2)}{P(1)} = \frac{0.9246}{0.9615} = 0.9616$$

$$P^*(2) = \frac{P(1+2)}{P(1)} = \frac{P(3)}{P(1)} = \frac{0.8890}{0.9615} = 0.9246$$

Using Equation 9, the price of the forward contract one year from today is

$$F^*(1,2,1) = \frac{P^*(2+1-1)}{P^*(2-1)} = \frac{P^*(2)}{P^*(1)} = \frac{0.9246}{0.9616} = 0.9615$$

The price of the forward contract has not changed. This will be the case as long as future discount functions are the same as those based on today's forward curve.

From this numerical example, we can see that if the spot rate curve is unchanged, then each bond "rolls down" the curve and earns the forward rate. Specifically, when one year passes, a three-year bond will return (0.9246 − 0.8890)/0.8890 = 4%, which is equal to the spot rate. Furthermore, if another year passes, the bond will return (0.9615 − 0.9246)/0.9246 = 4%, which is equal to the implied forward rate for a one-year security one year from today.

2.4 Active Bond Portfolio Management

One way active bond portfolio managers attempt to outperform the bond market's return is by anticipating changes in interest rates relative to the projected evolution of spot rates reflected in today's forward curves.

Some insight into these issues is provided by the forward rate model (Equation 4). By re-arranging terms in Equation 4 and letting the time horizon be one period, $T^* = 1$, we get

$$\frac{[1 + r(T+1)]^{T+1}}{[1 + f(1,T)]^{T}} = [1 + r(1)] \qquad (11)$$

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The numerator of the left hand side of Equation 11 is for a bond with an initial maturity of $T + 1$ and a remaining maturity of T after one period passes. Suppose the prevailing spot yield curve after one period is the current forward curve; then, Equation 11 shows that the total return on the bond is the one-period risk-free rate. The following sidebar shows that the return of bonds of varying tenor over a one-year period is always the one-year rate (the risk-free rate over the one-year period) if the spot rates evolve as implied by the current forward curve at the end of the first year.

WHEN SPOT RATES EVOLVE AS IMPLIED BY THE CURRENT FORWARD CURVE

As in earlier examples, assume the following:

$r(1) = 9\%$

$r(2) = 10\%$

$r(3) = 11\%$

$f(1,1) = 11.01\%$

$f(1,2) = 12.01\%$

If the spot curve one year from today reflects the current forward curve, the return on a zero-coupon bond for the one-year holding period is 9%, regardless of the maturity of the bond. The computations below assume a par amount of 100 and represent the percentage change in price. Given the rounding of price and the forward rates to the nearest hundredth, the returns all approximate 9%. However, with no rounding, all answers would be precisely 9%.

The return of the one-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 91.74 and is worth the par amount of 100 at maturity.

$$\left(100 \div \frac{100}{1 + r(1)}\right) - 1 = \left(100 \div \frac{100}{1 + 0.09}\right) - 1 = \frac{100}{91.74} - 1 = 9\%$$

The return of the two-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 82.64. One year from today, the two-year bond has a remaining maturity of one year. Its price one year from today is 90.08, determined as the par amount divided by 1 plus the forward rate for a one-year bond issued one year from today.

$$\left(\frac{100}{1 + f(1,1)} \div \frac{100}{[1 + r(2)]^2}\right) - 1 = \left(\frac{100}{1 + 0.1101} \div \frac{100}{(1 + 0.10)^2}\right) - 1$$

$$= \frac{90.08}{82.64} - 1 = 9\%$$

The return of the three-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 73.12. One year from today, the three-year bond has a remaining maturity of two years. Its price one year from today of 79.71 reflects the forward rate for a two-year bond issued one year from today.

$$\left(\frac{100}{[1 + f(1,2)]^2} \div \frac{100}{[1 + r(3)]^3}\right) - 1 =$$

$$\left(\frac{100}{(1 + 0.1201)^2} \div \frac{100}{(1 + 0.11)^3}\right) - 1 = \frac{79.71}{73.12} - 1 \cong 9\%$$

This numerical example shows that the return of a bond over a one-year period is always the one-year rate (the risk-free rate over the one period) if the spot rates evolve as implied by the current forward curve.

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But if the spot curve one year from today differs from today's forward curve, the returns on each bond for the one-year holding period will not all be 9%. To show that the returns on the two-year and three-year bonds over the one-year holding period are not 9%, we assume that the spot rate curve at Year 1 is flat with yields of 10% for all maturities.

The return on a one-year zero-coupon bond over the one-year holding period is

$$\left(100 \div \frac{100}{1+0.09}\right) - 1 = 9\%$$

The return on a two-year zero-coupon bond over the one-year holding period is

$$\left(\frac{100}{1+0.10} \div \frac{100}{(1+0.10)^2}\right) - 1 = 10\%$$

The return on a three-year zero-coupon bond over the one-year holding period is

$$\left(\frac{100}{(1+0.10)^2} \div \frac{100}{(1+0.11)^3}\right) - 1 = 13.03\%$$

The bond returns are 9%, 10%, and 13.03%. The returns on the two-year and three-year bonds differ from the one-year risk-free interest rate of 9%.

Equation 11 provides a total return investor with a means to evaluate the cheapness or expensiveness of a bond of a certain maturity. If any one of the investor's expected future spot rates is lower than a quoted forward rate for the same maturity, then (all else being equal) the investor would perceive the bond to be undervalued in the sense that the market is effectively discounting the bond's payments at a higher rate than the investor is and the bond's market price is below the intrinsic value perceived by the investor.

Another example will reinforce the point that if a portfolio manager's projected spot curve is above (below) the forward curve and his or her expectation turns out to be true, the return will be less (more) than the one-period risk-free interest rate.

For the sake of simplicity, assume a flat yield curve of 8% and that a trader holds a three-year bond paying annual coupons based on a 8% coupon rate. Assuming a par value of 100, the current market price is also 100. If today's forward curve turns out to be the spot curve one year from today, the trader will earn an 8% return.

If the trader projects that the spot curve one year from today is above today's forward curve—for example, a flat yield curve of 9%—the trader's expected rate of return is 6.24%, which is less than 8%:

$$\frac{8 + \dfrac{8}{1+0.09} + \dfrac{108}{(1+0.09)^2}}{100} - 1 = 6.24\%$$

If the trader predicts a flat yield curve of 7%, the trader's expected return is 9.81%, which is greater than 8%:

$$\frac{8 + \dfrac{8}{1+0.07} + \dfrac{108}{(1+0.07)^2}}{100} - 1 = 9.81\%$$

As the gap between the projected future spot rate and the forward rate widens, so too will the difference between the trader's expected return and the original yield to maturity of 8%.

This logic is the basis for a popular yield curve trade called **riding the yield curve** or **rolling down the yield curve**. As we have noted, when a yield curve is upward sloping, the forward curve is always above the current spot curve. If the trader does not believe that the yield curve will change its level and shape over an investment

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horizon, then buying bonds with a maturity longer than the investment horizon would provide a total return greater than the return on a maturity-matching strategy. The total return of the bond will depend on the spread between the forward rate and the spot rate as well as the maturity of the bond. The longer the bond's maturity, the more sensitive its total return is to the spread.

In the years following the 2008 financial crisis, many central banks around the world acted to keep short-term interest rates very low. As a result, yield curves subsequently had a steep upward slope (see Exhibit 1). For active management, this provided a big incentive for traders to access short-term funding and invest in long-term bonds. Of course, this trade is subject to significant interest rate risk, especially the risk of an unexpected increase in future spot rates (e.g., as a result of a spike in inflation). Yet, such a carry trade is often made by traders in an upward-sloping yield curve environment.[4]

In summary, when the yield curve slopes upward, as a bond approaches maturity or "rolls down the yield curve," it is valued at successively lower yields and higher prices. Using this strategy, a bond can be held for a period of time as it appreciates in price and then sold before maturity to realize a higher return. As long as interest rates remain stable and the yield curve retains an upward slope, this strategy can continuously add to the total return of a bond portfolio.

Example 7 address how the preceding analysis relates to active bond portfolio management.

EXAMPLE 7

Active Bond Portfolio Management

1 The "riding the yield curve" strategy is executed by buying bonds whose maturities are:

- **A** equal to the investor's investment horizon.
- **B** longer than the investor's investment horizon.
- **C** shorter than the investor's investment horizon.

2 A bond will be overvalued if the expected spot rate is:

- **A** equal to the current forward rate.
- **B** lower than the current forward rate.
- **C** higher than the current forward rate.

3 Assume a flat yield curve of 6%. A three-year £100 bond is issued at par paying an annual coupon of 6%. What is the portfolio manager's expected return if she predicts that the yield curve one year from today will be a flat 7%?

- **A** 4.19%
- **B** 6.00%
- **C** 8.83%

4 A forward contract price will increase if:

- **A** future spot rates evolve as predicted by current forward rates.

4 Carry trades can take many forms. Here, we refer to a maturity spread carry trade in which the trader borrows short and lends long in the same currency. The maturity spread carry trade is used frequently by hedge funds. There are also cross-currency and credit spread carry trades. Essentially, a carry trade involves simultaneously borrowing and lending to take advantage of what a trader views as being a favorable interest rate differential.

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B future spot rates are lower than what is predicted by current forward rates.

C future spot rates are higher than what is predicted by current forward rates.

Solution to 1:

B is correct. A bond with a longer maturity than the investor's investment horizon is purchased but then sold prior to maturity at the end of the investment horizon. If the yield curve is upward sloping and yields do not change, the bond will be valued at successively lower yields and higher prices over time. The bond's total return will exceed that of a bond whose maturity is equal to the investment horizon.

Solution to 2:

C is correct. If the expected discount rate is higher than the forward rate, then the bond will be overvalued. The expected price of the bond is lower than the price obtained from discounting using the forward rate.

Solution to 3:

A is correct. Expected return will be less than the current yield to maturity of 6% if yields increase to 7%. The expected return of 4.19% is computed as follows:

$$\frac{6 + \dfrac{6}{1 + 0.07} + \dfrac{106}{(1 + 0.07)^2}}{100} - 1 \approx 4.19\%$$

Solution to 4:

B is correct. The forward rate model can be used to show that a change in the forward contract price requires a deviation of the spot curve from that predicted by today's forward curve. If the future spot rate is lower than what is predicted by the prevailing forward rate, the forward contract price will increase because it is discounted at an interest rate that is lower than the originally anticipated rate.

3 THE SWAP RATE CURVE

Section 2 described the spot rate curve of default-risk-free bonds as a measure of the time value of money. The swap rate curve, or swap curve for short, is another important representation of the time value of money used in the international fixed-income markets. In this section, we will discuss how the swap curve is used in valuation.

3.1 The Swap Rate Curve

Interest rate swaps are an integral part of the fixed-income market. These derivative contracts, which typically exchange, or swap, fixed-rate interest payments for floating-rate interest payments, are an essential tool for investors who use them to speculate or modify risk. The size of the payments reflects the floating and fixed rates, the amount of principal—called the notional amount, or notional—and the maturity of the swap. The interest rate for the fixed-rate leg of an interest rate swap is known as the **swap rate**. The level of the swap rate is such that the swap has zero value at the initiation of the swap agreement. Floating rates are based on some short-term reference interest rate, such as three-month or six-month dollar Libor (London Interbank Offered

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Rate); other reference rates include euro-denominated Euribor (European Interbank Offered Rate) and yen-denominated Tibor (Tokyo Interbank Offered Rate). Note that the risk inherent in various floating reference rates varies according to the risk of the banks surveyed; for example, the spread between Tibor and yen Libor was positive as of October 2013, reflecting the greater risk of the banks surveyed for Tibor. The yield curve of swap rates is called the **swap rate curve**, or, more simply, the **swap curve**. Because it is based on so-called **par swaps**, in which the fixed rates are set so that no money is exchanged at contract initiation—the present values of the fixed-rate and benchmark floating-rate legs being equal— the swap curve is a type of par curve. When we refer to the "par curve' in this reading, the reference is to the government par yield curve, however.

The swap market is a highly liquid market for two reasons. First, unlike bonds, a swap does not have multiple borrowers or lenders, only counterparties who exchange cash flows. Such arrangements offer significant flexibility and customization in the swap contract's design. Second, swaps provide one of the most efficient ways to hedge interest rate risk. The Bank for International Settlements (BIS) estimated that the notional amount outstanding on interest rate swaps was about US$370 trillion in December 2012.[5]

Many countries do not have a liquid government bond market with maturities longer than one year. The swap curve is a necessary market benchmark for interest rates in these countries. In countries in which the private sector is much bigger than the public sector, the swap curve is a far more relevant measure of the time value of money than is the government's cost of borrowing.

In Asia, the swap markets and the government bond markets have developed in parallel, and both are used in valuation in credit and loan markets. In South Korea, the swap market is active out to a maturity of 10 years, whereas the Japanese swap market is active out to a maturity of 30 years. The reason for the longer maturity in the Japanese government market is that the market has been in existence for much longer than the South Korean market.

According to the *2013 CIA World Fact Book*, the size of the government bond market relative to GDP is 214.3% for Japan but only 46.9% for South Korea. For the United States and Germany, the numbers are 73.6% and 81.7%, and the world average is 64%. Even though the interest rate swap market in Japan is very active, the US interest rate swap market is almost three times larger than the Japanese interest rate swap market, based on outstanding amounts.

3.2 Why Do Market Participants Use Swap Rates When Valuing Bonds?

Government spot curves and swap rate curves are the chief reference curves in fixed-income valuation. The choice between them can depend on multiple factors, including the relative liquidity of these two markets. In the United States, where there is both an active Treasury security market and a swap market, the choice of a benchmark for the time value of money often depends on the business operations of the institution using the benchmark. On the one hand, wholesale banks frequently use the swap curve to value assets and liabilities because these organizations hedge many items on their balance sheet with swaps. On the other hand, retail banks with little exposure to the swap market are more likely to use the government spot curve as their benchmark.

5 Because the amount outstanding relates to notional values, it represents far less than $370 trillion of default exposure.

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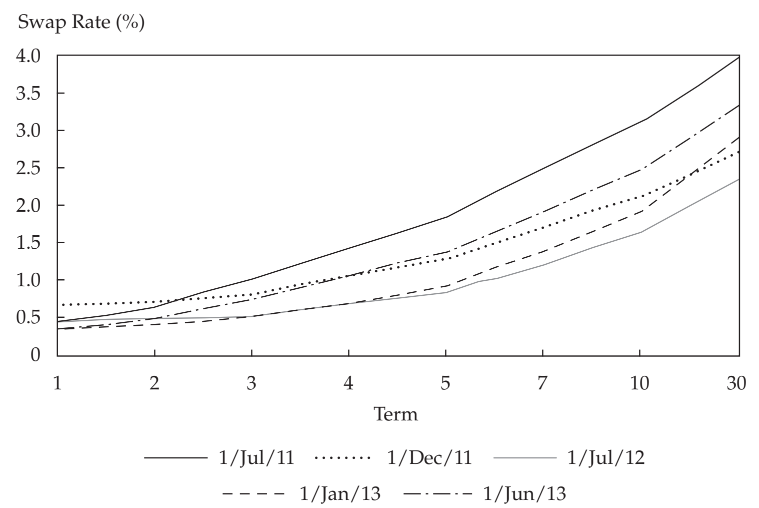
Let us illustrate how a financial institution uses the swap market for its internal operations. Consider the case of a bank raising funds using a certificate of deposit (CD). Assume the bank can borrow $10 million in the form of a CD that bears interest of 1.5% for a two-year term. Another $10 million CD offers 1.70% for a three-year term. The bank can arrange two swaps: (1) The bank receives 1.50% fixed and pays three-month Libor minus 10 bps with a two-year term and $10 million notional, and (2) the bank receives 1.70% fixed and pays three-month Libor minus 15 bps with a three-year term and a notional amount of $10 million. After issuing the two CDs and committing to the two swaps, the bank has raised $20 million with an annual funding cost for the first two years of three-month Libor minus 12.5 bps applied to the total notional amount of $20 million. The fixed interest payments received from the counterparty to the swap are paid to the CD investors; in effect, fixed-rate liabilities have been converted to floating-rate liabilities. The margins on the floating rates become the standard by which value is measured in assessing the total funding cost for the bank.

By using the swap curve as a benchmark for the time value of money, the investor can adjust the swap spread so that the swap would be fairly priced given the spread. Conversely, given a swap spread, the investor can determine a fair price for the bond. We will use the swap spread in the following section to determine the value of a bond.

3.3 How Do Market Participants Use the Swap Curve in Valuation?

Swap contracts are non-standardized and are simply customized contracts between two parties in the over-the-counter market. The fixed payment can be specified by an amortization schedule or to be coupon paying with non-standardized coupon payment dates. For this section, we will focus on zero-coupon bonds. The yields on these bonds determine the swap curve, which, in turn, can be used to determine bond values. Examples of swap par curves are given in Exhibit 3.

Exhibit 3 Historical Swap Curves



Note: Horizontal axis is not drawn to scale. (Such scales are commonly used as an industry standard because most of the distinctive shape of yield curves is typically observed before 10 years.)

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Each forward date has an associated discount factor that represents the value today of a hypothetical payment that one would receive on the forward date, expressed as a fraction of the hypothetical payment. For example, if we expect to receive ₩10,000 (10,000 South Korean won) in one year and the current price of the security is ₩9,259.30, then the discount factor for one year would be 0.92593 (= ₩9,259.30/₩10,000). Note that the rate associated with this discount factor is 1/0.92593 −1 ≈ 8.00%.

To price a swap, we need to determine the present value of cash flows for each leg of the transaction. In an interest rate swap, the fixed leg is fairly straightforward because the cash flows are specified by the coupon rate set at the time of the agreement. Pricing the floating leg is more complex because, by definition, the cash flows change with future changes in interest rates. The forward rate for each floating payment date is calculated by using the forward curves.

Let $s(T)$ stand for the swap rate at time T. Because the value of a swap at origination is set to zero, the swap rates must satisfy Equation 12. Note that the swap rates can be determined from the spot rates and the spot rates can be determined from the swap rates.

$$\sum_{t=1}^{T} \frac{s(T)}{\left[1 + r(t)\right]^t} + \frac{1}{\left[1 + r(T)\right]^T} = 1 \quad (12)$$

The right side of Equation 12 is the value of the floating leg, which is always 1 at origination. The swap rate is determined by equating the value of the fixed leg, on the left-hand side, to the value of the floating leg.

Example 8 addresses the relationship between the swap rate curve and spot curve.

EXAMPLE 8

Determining the Swap Rate Curve

Suppose a government spot curve implies the following discount factors:

$P(1) = 0.9524$

$P(2) = 0.8900$

$P(3) = 0.8163$

$P(4) = 0.7350$

Given this information, determine the swap rate curve.

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Solution:

Recall from Equation 1 that $P(T) = \dfrac{1}{\left[1 + r(T)\right]^T}$. Therefore,

$$r(T) = \left\{\frac{1}{\left[P(T)\right]}\right\}^{(1/T)} - 1$$

$$r(1) = \left(\frac{1}{0.9524}\right)^{(1/1)} - 1 = 5.00\%$$

$$r(2) = \left(\frac{1}{0.8900}\right)^{(1/2)} - 1 = 6.00\%$$

$$r(3) = \left(\frac{1}{0.8163}^{(1/3)}\right) - 1 = 7.00\%$$

$$r(4) = \left(\frac{1}{0.7350}\right)^{(1/4)} - 1 = 8.00\%$$

Using Equation 12, for $T = 1$,

$$\frac{s(1)}{\left[1 + r(1)\right]^1} + \frac{1}{\left[1 + r(1)\right]^1} = \frac{s(1)}{(1 + 0.05)^1} + \frac{1}{(1 + 0.05)^1} = 1$$

Therefore, $s(1) = 5\%$.

For $T = 2$,

$$\frac{s(2)}{\left[1 + r(1)\right]^1} + \frac{s(2)}{\left[1 + r(2)\right]^2} + \frac{1}{\left[1 + r(2)\right]^2} = \frac{s(2)}{(1 + 0.05)^1} + \frac{s(2)}{(1 + 0.06)^2} + \frac{1}{(1 + 0.06)^2} = 1$$

Therefore, $s(2) = 5.97\%$.

For $T = 3$,

$$\frac{s(3)}{\left[1 + r(1)\right]^1} + \frac{s(3)}{\left[1 + r(2)\right]^2} + \frac{s(3)}{\left[1 + r(3)\right]^3} + \frac{1}{\left[1 + r(3)\right]^3} =$$

$$\frac{s(3)}{(1 + 0.05)^1} + \frac{s(3)}{(1 + 0.06)^2} + \frac{s(3)}{(1 + 0.07)^3} + \frac{1}{(1 + 0.07)^3} = 1$$

Therefore, $s(3) = 6.91\%$.

For $T = 4$,

$$\frac{s(4)}{\left[1 + r(1)\right]^1} + \frac{s(4)}{\left[1 + r(2)\right]^2} + \frac{s(4)}{\left[1 + r(3)\right]^3} + \frac{s(4)}{\left[1 + r(4)\right]^4} + \frac{1}{\left[1 + r(4)\right]^4} =$$

$$\frac{s(4)}{(1 + 0.05)^1} + \frac{s(4)}{(1 + 0.06)^2} + \frac{s(4)}{(1 + 0.07)^3} + \frac{s(4)}{(1 + 0.08)^4} + \frac{1}{(1 + 0.08)^4} = 1$$

Therefore, $s(4) = 7.81\%$.

Note that the swap rates, spot rates, and discount factors are all mathematically linked together. Having access to data for one of the series allows you to calculate the other two.

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3.4 The Swap Spread

The swap spread is a popular way to indicate credit spreads in a market. The **swap spread** is defined as the spread paid by the fixed-rate payer of an interest rate swap over the rate of the "on-the-run" (most recently issued) government security with the same maturity as the swap.[6]

Often, fixed-income prices will be quoted in SWAPS +, for which the yield is simply the yield on an equal-maturity government bond plus the swap spread. For example, if the fixed rate of a five-year fixed-for-float Libor swap is 2.00% and the five-year Treasury is yielding 1.70%, the swap spread is 2.00% − 1.70% = 0.30%, or 30 bps.

For euro-denominated swaps, the government yield used as a benchmark is most frequently bunds (German government bonds) with the same maturity. Gilts (UK government bonds) are used as a benchmark in the United Kingdom. CME Group began clearing euro-denominated interest rate swaps in 2011.

A Libor/swap curve is probably the most widely used interest rate curve because it is often viewed as reflecting the default risk of private entities at a rating of about A1/A+, roughly the equivalent of most commercial banks. (The swap curve can also be influenced by the demand and supply conditions in government debt markets, among other factors.) Another reason for the popularity of the swap market is that it is unregulated (not controlled by governments), so swap rates are more comparable across different countries. The swap market also has more maturities with which to construct a yield curve than do government bond markets. Libor is used for short-maturity yields, rates derived from eurodollar futures contracts are used for mid-maturity yields, and swap rates are used for yields with a maturity of more than one year. The swap rates used are the fixed rates that would be paid in swap agreements for which three-month Libor floating payments are received.[7]

HISTORY OF THE US SWAP SPREAD, 2008–2013

Normally, the Treasury swap spread is positive, which reflects the fact that governments generally pay less to borrow than do private entities. However, the 30-year Treasury swap spread turned negative following the collapse of Lehman Brothers Holdings Inc. in September 2008. Liquidity in many corners of the credit markets evaporated during the financial crisis, leading investors to doubt the safety and security of their counterparties in some derivatives transactions. The 30-year Treasury swap spread tumbled to a record low of −62 bps in November 2008. The 30-year Treasury swap spread again turned positive in the middle of 2013. A dramatic shift in sentiment regarding the Federal Reserve outlook since early May 2013 was a key catalyst for a selloff in most bonds. The sharp rise in Treasury yields at that time pushed up funding and hedging costs for companies, which was reflected in a rise in swap rates.

To illustrate the use of the swap spread in fixed-income pricing, consider a US$1 million investment in GE Capital (GECC) notes with a coupon rate of 1 5/8% (1.625%) that matures on 2 July 2015. Coupons are paid semiannually. The evaluation date is 12 July 2012, so the remaining maturity is 2.97 years [= 2 + (350/360)]. The Treasury rates for two-year and three-year maturities are 0.525% and 0.588%,

6 The term "swap spread" is sometimes also used as a reference to a bond's basis point spread over the interest rate swap curve and is a measure of the credit and/or liquidity risk of a bond. In its simplest form, the swap spread in this sense can be measured as the difference between the yield to maturity of the bond and the swap rate given by a straight-line interpolation of the swap curve. These spreads are frequently quoted as an I-spread, ISPRD, or interpolated spread, which is a reference to a linearly interpolated yield. In this reading, the term "swap spread" refers to an excess yield of swap rates over the yields on government bonds and I-spreads to refer to bond yields net of the swap rates of the same maturities.

7 The US dollar market uses three-month Libor, but other currencies may use one-month or six-month Libor.

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respectively. By simple interpolation between these two rates, the treasury rate for 2.97 years is 0.586% [= 0.525% + (350/360)(0.588% − 0.525%)]. If the swap spread for the same maturity is 0.918%, then the yield to maturity on the bond is 1.504% (= 0.918% + 0.586%). Given the yield to maturity, the invoice price (price including accrued interest) for US$1 million face value is

$$\frac{1{,}000{,}000\left(\dfrac{0.01625}{2}\right)}{\left(1+\dfrac{0.01504}{2}\right)^{\left(1-\frac{10}{180}\right)}}+\frac{1{,}000{,}000\left(\dfrac{0.01625}{2}\right)}{\left(1+\dfrac{0.01504}{2}\right)^{\left(2-\frac{10}{180}\right)}}+\cdots+$$

$$\frac{1{,}000{,}000\left(\dfrac{0.01625}{2}\right)}{\left(1+\dfrac{0.01504}{2}\right)^{\left(6-\frac{10}{180}\right)}}+\frac{1{,}000{,}000}{\left(1+\dfrac{0.01504}{2}\right)^{\left(6-\frac{10}{180}\right)}}=\text{US\$1,003,954.12}$$

The left side sums the present values of the semiannual coupon payments and the final principal payment of US$1,000,000. The accrued interest rate amount is US$451.39 [= 1,000,000 × (0.01625/2)(10/180)]. Therefore, the clean price (price not including accrued interest) is US$1,003,502.73 (= 1,003,954.12 – 451.39).

The swap spread helps an investor to identify the time value, credit, and liquidity components of a bond's yield to maturity. If the bond is default free, then the swap spread could provide an indication of the bond's liquidity or it could provide evidence of market mispricing. The higher the swap spread, the higher the return that investors require for credit and/or liquidity risks.

Although swap spreads provide a convenient way to measure risk, a more accurate measure of credit and liquidity is called the zero-spread (Z-spread). The **Z-spread** is the constant basis point spread that would need to be added to the implied spot yield curve so that the discounted cash flows of a bond are equal to its current market price. This spread will be more accurate than a linearly interpolated yield, particularly with steep interest rate swap curves.

USING THE Z-SPREAD IN VALUATION

Consider again the GECC semi-annual coupon note with a maturity of 2.97 years and a par value of US$1,000,000. The implied spot yield curve is

$r(0.5) = 0.16\%$

$r(1) = 0.21\%$

$r(1.5) = 0.27\%$

$r(2) = 0.33\%$

$r(2.5) = 0.37\%$

$r(3) = 0.41\%$

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The Z-spread is given as 109.6 bps. Using the spot curve and the Z-spread, the invoice price is

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.0016+0.01096}{2}\right)^{\left(1-\frac{10}{180}\right)}}+\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.00021+0.01096}{2}\right)^{\left(2-\frac{10}{180}\right)}}+\cdots+$$

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.0041+0.01096}{2}\right)^{\left(6-\frac{10}{180}\right)}}+$$

$$\frac{1{,}000{,}000}{\left(1+\frac{0.0041+0.01096}{2}\right)^{\left(6-\frac{10}{180}\right)}}=\text{US\$}1{,}003{,}954.12$$

3.5 Spreads as a Price Quotation Convention

We have discussed both Treasury curves and swap curves as benchmarks for fixed-income valuation, but they usually differ. Therefore, quoting the price of a bond using the bond yield net of either a benchmark Treasury yield or swap rate becomes a price quote convention.

The Treasury rate can differ from the swap rate for the same term for several reasons. Unlike the cash flows from US Treasury bonds, the cash flows from swaps are subject to much higher default risk. Market liquidity for any specific maturity may differ. For example, some parts of the term structure of interest rates may be more actively traded with swaps than with Treasury bonds. Finally, arbitrage between these two markets cannot be perfectly executed.

Swap spreads to the Treasury rate (as opposed to the **I-spreads**, which are bond rates net of the swap rates of the same maturities) are simply the differences between swap rates and government bond yields of a particular maturity. One problem in defining swap spreads is that, for example, a 10-year swap matures in exactly 10 years whereas there typically is no government bond with exactly 10 years of remaining maturity. By convention, therefore, the 10-year swap spread is defined as the difference between the 10-year swap rate and the 10-year on-the-run government bond. Swap spreads of other maturities are defined similarly.

To generate the curves in Exhibit 4, we used the constant-maturity Treasury note to exactly match the corresponding swap rate. The 10-year swap spread is the 10-year swap rate less the 10-year constant-maturity Treasury note yield. Because counterparty risk is reflected in the swap rate and US government debt is considered nearly free of default risk, the swap rate is usually greater than the corresponding Treasury note rate and the 10-year swap spread is usually, but not always, positive.

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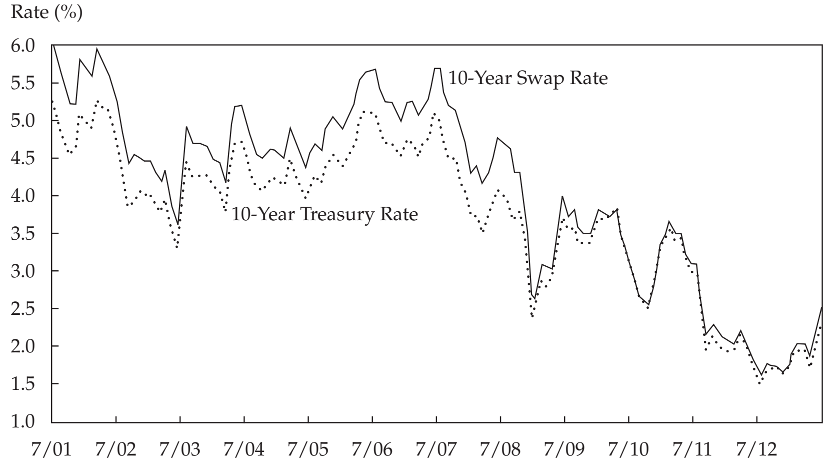
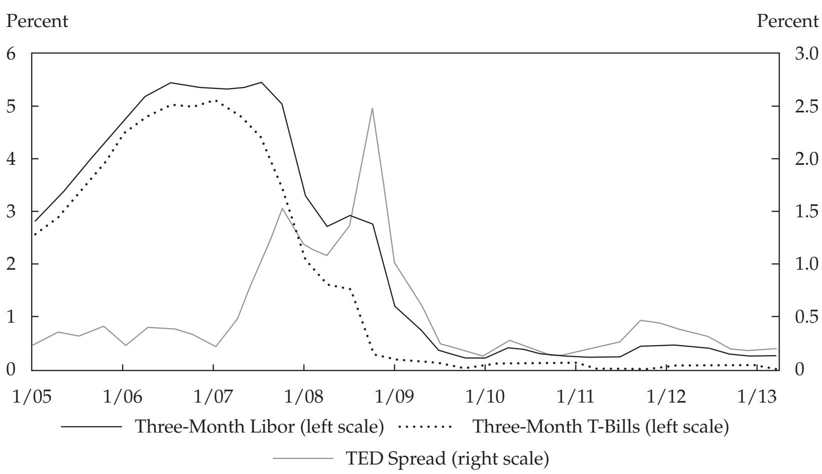
Exhibit 4 10-Year Swap Rate vs. 10-Year Treasury Rate



The **TED spread** is an indicator of perceived credit risk in the general economy. TED is an acronym formed from US T-bill and ED, the ticker symbol for the eurodollar futures contract. The TED spread is calculated as the difference between Libor and the yield on a T-bill of matching maturity. An increase (decrease) in the TED spread is a sign that lenders believe the risk of default on interbank loans is increasing (decreasing). Therefore, as it relates to the swap market, the TED spread can also be thought of as a measure of counterparty risk. Compared with the 10-year swap spread, the TED spread more accurately reflects risk in the banking system, whereas the 10-year swap spread is more often a reflection of differing supply and demand conditions.

Exhibit 5 TED Spread



Another popular measure of risk is the **Libor–OIS spread**, which is the difference between Libor and the overnight indexed swap (OIS) rate. An OIS is an interest rate swap in which the periodic floating rate of the swap is equal to the geometric average of an overnight rate (or overnight index rate) over every day of the payment period. The index rate is typically the rate for overnight unsecured lending between banks—for

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example, the federal funds rate for US dollars, Eonia (Euro OverNight Index Average) for euros, and Sonia (Sterling OverNight Index Average) for sterling. The Libor–OIS spread is considered an indicator of the risk and liquidity of money market securities.

TRADITIONAL THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

4

This section presents four traditional theories of the underlying economic factors that affect the shape of the yield curve.

4.1 Local Expectations Theory

One branch of traditional term structure theory focuses on interpreting term structure shape in terms of investors' expectations. Historically, the first such theory is known as the **unbiased expectations theory** or **pure expectations theory**. It says that the forward rate is an unbiased predictor of the future spot rate; its broadest interpretation is that bonds of any maturity are perfect substitutes for one another. For example, buying a bond with a maturity of five years and holding it for three years has the same expected return as buying a three-year bond or buying a series of three one-year bonds.

The predictions of the unbiased expectations theory are consistent with the assumption of risk neutrality. In a risk-neutral world, investors are unaffected by uncertainty and risk premiums do not exist. Every security is risk free and yields the risk-free rate for that particular maturity. Although such an assumption leads to interesting results, it clearly is in conflict with the large body of evidence that shows that investors are risk averse.

A theory that is similar but more rigorous than the unbiased expectations theory is the **local expectations theory**. Rather than asserting that every maturity strategy has the same expected return over a given investment horizon, this theory instead contends that the expected return for every bond over short time periods is the risk-free rate. This conclusion results from an assumed no-arbitrage condition in which bond pricing does not allow for traders to earn arbitrage profits.

The primary way that the local expectations theory differs from the unbiased expectations theory is that it can be extended to a world characterized by risk. Although the theory requires that risk premiums be nonexistent for very short holding periods, no such restrictions are placed on longer-term investments. Thus, the theory is applicable to both risk-free as well as risky bonds.

Using the formula for the discount factor in Equation 1 and the variation of the forward rate model in Equation 5, we can produce Equation 13, where $P(t,T)$ is the discount factor for a T-period security at time t.

$$\frac{1}{P(t,T)} = [1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)] \quad (13)$$

Using Equation 13, we can show that if the forward rates are realized, the one-period return of a long-term bond is $r(1)$, the yield on a one-period risk-free security, as shown in Equation 14.

$$\frac{P(t+1,T-1)}{P(t,T)} = 1 + r(1) \quad (14)$$

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The local expectations theory extends this equation to incorporate uncertainty while still assuming risk neutrality in the short term. When we relax the certainty assumption, then Equation 14 becomes Equation 15, where the tilde (~) represents an uncertain outcome. In other words, the one-period return of a long-term risky bond is the one-period risk-free rate.

$$\frac{E\left[\tilde{P}(t+1,T-1)\right]}{P(t,T)} = 1 + r(1) \tag{15}$$

Although the local expectations theory is economically appealing, it is often observed that short-holding-period returns on long-dated bonds do exceed those on short-dated bonds. The need for liquidity and the ability to hedge risk essentially ensure that the demand for short-term securities will exceed that for long-term securities. Thus, both the yields and the actual returns for short-dated securities are typically lower than those for long-dated securities.

4.2 Liquidity Preference Theory

Whereas the unbiased expectations theory leaves no room for risk aversion, liquidity preference theory attempts to account for it. **Liquidity preference theory** asserts that **liquidity premiums** exist to compensate investors for the added interest rate risk they face when lending long term and that these premiums increase with maturity.[8] Thus, given an expectation of unchanging short-term spot rates, liquidity preference theory predicts an upward-sloping yield curve. The forward rate provides an estimate of the expected spot rate that is biased upward by the amount of the liquidity premium, which invalidates the unbiased expectations theory.

For example, the US Treasury offers bonds that mature in 30 years. However, the majority of investors have an investment horizon that is shorter than 30 years.[9] For investors to hold these bonds, they would demand a higher return for taking the risk that the yield curve changes and that they must sell the bond prior to maturity at an uncertain price. That incrementally higher return is the liquidity premium. Note that this premium is not to be confused with a yield premium for the lack of liquidity that thinly traded bonds may bear. Rather, it is a premium applying to all long-term bonds, including those with deep markets.

Liquidity preference theory fails to offer a complete explanation of the term structure. Rather, it simply argues for the existence of liquidity premiums. For example, a downward-sloping yield curve could still be consistent with the existence of liquidity premiums if one of the factors underlying the shape of the curve is an expectation of deflation (i.e., a negative rate of inflation due to monetary or fiscal policy actions). Expectations of sharply declining spot rates may also result in a downward-sloping yield curve if the expected decline in interest rates is severe enough to offset the effect of the liquidity premiums.

In summary, liquidity preference theory claims that lenders require a liquidity premium as an incentive to lend long term. Thus, forward rates derived from the current yield curve provide an upwardly biased estimate of expected future spot rates. Although downward-sloping or hump-shaped yield curves may sometimes occur, the existence of liquidity premiums implies that the yield curve will typically be upward sloping.

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8 The wording of a technical treatment of this theory would be that these premiums increase monotonically with maturity. A sequence is said to be monotonically increasing if each term is greater than or equal to the one before it. Define $LP(T)$ as the liquidty premium at maturity T. If premiums increase monotonically with maturity, then $LP(T + t) \geq LP(T)$ for all $t > 0$.

9 This view can be confirmed by examining typical demand for long-term versus short-term Treasuries at auctions.

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4.3 Segmented Markets Theory

Unlike expectations theory and liquidity preference theory, **segmented markets theory** allows for lender and borrower preferences to influence the shape of the yield curve. The result is that yields are not a reflection of expected spot rates or liquidity premiums. Rather, they are solely a function of the supply and demand for funds of a particular maturity. That is, each maturity sector can be thought of as a segmented market in which yield is determined independently from the yields that prevail in other maturity segments.

The theory is consistent with a world where there are asset/liability management constraints, either regulatory or self-imposed. In such a world, investors might restrict their investment activity to a maturity sector that provides the best match for the maturity of their liabilities. Doing so avoids the risks associated with an asset/liability mismatch.

For example, because life insurers sell long-term liabilities against themselves in the form of life insurance contracts, they tend to be most active as buyers in the long end of the bond market. Similarly, because the liabilities of pension plans are long term, they typically invest in long-term securities. Why would they invest short term given that those returns might decline while the cost of their liabilities stays fixed? In contrast, money market funds would be limited to investing in debt with maturity of one year or less, in general.

In summary, the segmented markets theory assumes that market participants are either unwilling or unable to invest in anything other than securities of their preferred maturity. It follows that the yield of securities of a particular maturity is determined entirely by the supply and demand for funds of that particular maturity.

4.4 Preferred Habitat Theory

The **preferred habitat theory** is similar to the segmented markets theory in proposing that many borrowers and lenders have strong preferences for particular maturities but it does not assert that yields at different maturities are determined independently of each other.

However, the theory contends that if the expected additional returns to be gained become large enough, institutions will be willing to deviate from their preferred maturities or habitats. For example, if the expected returns on longer-term securities exceed those on short-term securities by a large enough margin, money market funds will lengthen the maturities of their assets. And if the excess returns expected from buying short-term securities become large enough, life insurance companies might stop limiting themselves to long-term securities and place a larger part of their portfolios in shorter-term investments.

The preferred habitat theory is based on the realistic notion that agents and institutions will accept additional risk in return for additional expected returns. In accepting elements of both the segmented markets theory and the unbiased expectations theory, yet rejecting their extreme polar positions, the preferred habitat theory moves closer to explaining real-world phenomena. In this theory, both market expectations and the institutional factors emphasized in the segmented markets theory influence the term structure of interest rates.

PREFERRED HABITAT AND QE

The term "quantitative easing" (QE) refers to an unconventional monetary policy used by central banks to increase the supply of money in an economy when central bank and/or interbank interest rates are already close to zero. The first of three QE efforts by the US Federal Reserve began in late 2008, following the establishment of a near-zero target

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range for the federal funds rate. Since then, the Federal Reserve has greatly expanded its holdings of long-term securities via a series of asset purchase programs, with the goal of putting downward pressure on long-term interest rates thereby making financial conditions even more accommodative. Exhibit 6 presents information regarding the securities held by the Federal Reserve on 20 September 2007 (when all securities held by the Fed were US Treasury issuance) and 19 September 2013 (one year after the third round of QE was launched).

Exhibit 6 Securities Held by the US Federal Reserve

(US$ millions)	20 Sept. 2007	19 Sept. 2013
Securities held outright	779,636	3,448,758
US Treasury	779,636	2,047,534
Bills	267,019	0
Notes and bonds, nominal	472,142	1,947,007
Notes and bonds, inflation indexed	35,753	87,209
Inflation compensation	4,723	13,317
Federal agency	0	63,974
Mortgage-backed securities	0	1,337,520

As Exhibit 6 shows, the Federal Reserve's security holdings on 20 September 2007 consisted entirely of US Treasury securities and about 34% of those holdings were short term in the form of T-bills. On 19 September 2013, only about 59% of the Federal Reserve's security holdings were Treasury securities and none of those holdings were T-bills. Furthermore, the Federal Reserve held well over US$1.3 trillion of mortgage-backed securities (MBS), which accounted for almost 39% of all securities held.

Prior to the QE efforts, the yield on MBS was typically in the 5%–6% range. It declined to less than 2% by the end of 2012. Concepts related to preferred habitat theory could possibly help explain that drop in yield.

The purchase of MBS by the Federal Reserve essentially reduced the supply of these securities that was available for private purchase. Assuming that many MBS investors are either unwilling or unable to withdraw from the MBS market because of their investment in gaining expertise in managing interest rate and repayment risks of MBS, MBS investing institutions would have a "preferred habitat" in the MBS market. If they were unable to meet investor demand without bidding more aggressively, these buyers would drive down yields on MBS.

The case can also be made that the Federal Reserve's purchase of MBS helped reduced prepayment risk, which also resulted in a reduction in MBS yields. If a homeowner prepays on a mortgage, the payment is sent to MBS investors on a pro-rata basis. Although investors are uncertain about when such a prepayment will be received, prepayment is more likely in a declining interest rate environment.

Use Example 9 to test your understanding of traditional term structure theories.

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EXAMPLE 9

Traditional Term Structure Theories

1 In 2010, the Committee of European Securities Regulators created guidelines that restricted weighted average life (WAL) to 120 days for short-term money market funds. The purpose of this restriction was to limit the ability of money market funds to invest in long-term, floating-rate securities. This action is *most* consistent with a belief in:

   A the preferred habitat theory.

   B the segmented markets theory.

   C the local expectations theory.

2 The term structure theory that asserts that investors cannot be induced to hold debt securities whose maturities do not match their investment horizon is *best* described as the:

   A preferred habitat theory.

   B segmented markets theory.

   C unbiased expectations theory.

3 The unbiased expectations theory assumes investors are:

   A risk averse.

   B risk neutral.

   C risk seeking.

4 Market evidence shows that forward rates are:

   A unbiased predictors of future spot rates.

   B upwardly biased predictors of future spot rates.

   C downwardly biased predictors of future spot rates.

5 Market evidence shows that short holding-period returns on short-maturity bonds *most* often are:

   A less than those on long-maturity bonds.

   B about equal to those on long-maturity bonds.

   C greater than those on long-maturity bonds.

Solution to 1:

A is correct. The preferred habitat theory asserts that investors are willing to move away from their preferred maturity if there is adequate incentive to do so. The proposed WAL guideline was the result of regulatory concern about the interest rate risk and credit risk of long-term, floating-rate securities. An inference of this regulatory action is that some money market funds must be willing to move away from more traditional short-term investments if they believe there is sufficient compensation to do so.

Solution to 2:

B is correct. Segmented markets theory contends that asset/liability management constraints force investors to buy securities whose maturities match the maturities of their liabilities. In contrast, preferred habitat theory asserts that investors are willing to deviate from their preferred maturities if yield differentials encourage the switch. The unbiased expectations theory makes no assumptions about maturity preferences. Rather, it contends that forward rates are unbiased predictors of future spot rates.

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Solution to 3:

B is correct. The unbiased expectations theory asserts that different maturity strategies, such as rollover, maturity matching, and riding the yield curve, have the same expected return. By definition, a risk-neutral party is indifferent about choices with equal expected payoffs, even if one choice is riskier. Thus, the predictions of the theory are consistent with the existence of risk-neutral investors.

Solution to 4:

B is correct. The existence of a liquidity premium ensures that the forward rate is an upwardly biased estimate of the future spot rate. Market evidence clearly shows that liquidity premiums exist, and this evidence effectively refutes the predictions of the unbiased expectations theory.

Solution to 5:

A is correct. Although the local expectations theory predicts that the short-run return for all bonds will be equal to the risk-free rate, most of the evidence refutes that claim. Returns from long-dated bonds are generally higher than those from short-dated bonds, even over relatively short investment horizons. This market evidence is consistent with the risk–expected return trade-off that is central to finance and the uncertainty surrounding future spot rates.

5 MODERN TERM STRUCTURE MODELS

Modern term structure models provide quantitatively precise descriptions of how interest rates evolve. A model provides a sometimes simplified description of a real-world phenomenon on the basis of a set of assumptions; models are often used to solve particular problems. These assumptions cannot be completely accurate in depicting the real world, but instead, the assumptions are made to explain real-world phenomena sufficiently well to solve the problem at hand.

Interest rate models attempt to capture the statistical properties of interest rate movements. The detailed description of these models depends on mathematical and statistical knowledge well outside the scope of the investment generalist's technical preparation. Yet, these models are very important in the valuation of complex fixed-income instruments and bond derivatives. Thus, we provide a broad overview of these models in this reading. Equations for the models and worked examples are given for readers who are interested.

5.1 Equilibrium Term Structure Models

Equilibrium term structure models are models that seek to describe the dynamics of the term structure using fundamental economic variables that are assumed to affect interest rates. In the modeling process, restrictions are imposed that allow for the derivation of equilibrium prices for bonds and interest rate options. These models require the specification of a drift term (explained later) and the assumption of a functional form for interest rate volatility. The best-known equilibrium models are the **Cox–Ingersoll–Ross model**[10] and the **Vasicek model**,[11] which are discussed in the next two sections.

10 Cox, Ingersoll, and Ross (1985).
11 Vasicek (1977).

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Equilibrium term structure models share several characteristics:

- *They are one-factor or multifactor models.* One-factor models assume that a single observable factor (sometimes called a state variable) drives all yield curve movements. Both the Vasicek and CIR models assume a single factor, the short-term interest rate, *r*. This approach is plausible because empirically, parallel shifts are often found to explain more than 90% of yield changes. In contrast, multifactor models may be able to model the curvature of a yield curve more accurately but at the cost of greater complexity.
- *They make assumptions about the behavior of factors.* For example, if we focus on a short-rate single-factor model, should the short rate be modeled as mean reverting? Should the short rate be modeled to exhibit jumps? How should the volatility of the short rate be modeled?
- *They are, in general, more sparing with respect to the number of parameters that must be estimated compared with arbitrage-free term structure models.* The cost of this relative economy in parameters is that arbitrage-free models can, in general, model observed yield curves more precisely.[12]

An excellent example of an equilibrium term structure model is the Cox–Ingersoll–Ross (CIR) model discussed next.

5.1.1 *The Cox–Ingersoll–Ross Model*

The CIR model assumes that every individual has to make consumption and investment decisions with their limited capital. Investing in the productive process may lead to higher consumption in the following period, but it requires sacrificing today's consumption. The individual must determine his or her optimal trade-off assuming that he or she can borrow and lend in the capital market. Ultimately, interest rates will reach a market equilibrium rate at which no one needs to borrow or lend. The CIR model can explain interest rate movements in terms of an individual's preferences for investment and consumption as well as the risks and returns of the productive processes of the economy.

As a result of this analysis, the model shows how the short-term interest rate is related to the risks facing the productive processes of the economy. Assuming that an individual requires a term premium on the long-term rate, the model shows that the short-term rate can determine the entire term structure of interest rates and the valuation of interest rate–contingent claims. The CIR model is presented in Equation 16.

In Equation 16, the terms "*dr*" and "*dt*" mean, roughly, an infinitely small increment in the (instantaneous) short-term interest rate and time, respectively; the CIR model is an instance of a so-called continuous-time finance model. The model has two parts: (1) a deterministic part (sometimes called a "drift term"), the expression in *dt*, and (2) a stochastic (i.e., random) part, the expression in *dz*, which models risk.

$$dr = a(b - r)dt + \sigma\sqrt{r}dz \tag{16}$$

The way the deterministic part, $a(b - r)dt$, is formulated in Equation 16 ensures mean reversion of the interest rate toward a long-run value *b*, with the speed of adjustment governed by the strictly positive parameter *a*. If *a* is high (low), mean reversion to the long-run rate *b* would occur quickly (slowly). In Equation 16, for simplicity of

12 Other contrasts are more technical. They include that equilibrium models use real probabilities whereas arbitrage-free models use so-called risk-neutral probabilities.

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presentation we have assumed that the **term premium** of the CIR model is equal to zero.[13] Thus, as modeled here, the CIR model assumes that the economy has a constant long-run interest rate that the short-term interest rate converges to over time.

Mean reversion is an essential characteristic of the interest rate that sets it apart from many other financial data series. Unlike stock prices, for example, interest rates cannot rise indefinitely because at very high levels, they would hamper economic activity, which would ultimately result in a decrease in interest rates. Similarly, with rare historical exceptions, nominal interest rates are non-negative. As a result, short-term interest rates tend to move in a bounded range and show a tendency to revert to a long-run value b.

Note that in Equation 16, there is only one stochastic driver, dz, of the interest rate process; very loosely, dz can be thought of as an infinitely small movement in a "random walk." The stochastic or volatility term, $\sigma\sqrt{r}dz$, follows the random normal distribution for which the mean is zero, the standard deviation is 1, and the standard deviation factor is $\sigma\sqrt{r}$. The standard deviation factor makes volatility proportional to the square root of the short-term rate, which allows for volatility to increase with the level of interest rates. It also avoids the possibility of non-positive interest rates for all positive values of a and b.[14]

Note that a, b, and σ are model parameters that have to be specified in some manner.

AN ILLUSTRATION OF THE CIR MODEL

Assume again that the current short-term rate is r = 3% and the long-run value for the short-term rate is b = 8%. As before, assume that the speed of the adjustment factor is a = 0.40 and the annual volatility is σ = 20%. Using Equation 16, the CIR model provides the following formula for the change in short-term interest rates, dr:

$$dr = 0.40(8\% - r)dt + (20\%)\sqrt{r}dz$$

Assume that a random number generator produced standard normal random error terms, dz, of 0.50, −0.10, 0.50, and −0.30. The CIR model would produce the evolution of interest rates shown in Exhibit 7. The bottom half of the exhibit shows the pricing of bonds consistent with the evolution of the short-term interest rate.

Exhibit 7 Evolution of the Short-Term Rate in the CIR Model

	Time				
Parameter	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
r	3.000%	6.732%	6.720%	9.825%	7.214%
$a(b - r) = 0.40(8\% - r)$	2.000%	0.507%	0.512%	−0.730%	
dz	0.500	−0.100	0.500	−0.300	
$\sigma\sqrt{r}dz = 20\%\sqrt{r}dz$	1.732%	−0.519%	2.592%	−1.881%	
dr	3.732%	−0.012%	3.104%	−2.611%	
$r(t + 1) = r + dr$	6.732%	6.720%	9.825%	7.214%	

YTM for Zero-Coupon Bonds Maturing in

13 Equilibrium models, but not arbitrage-free models, assume that a term premium is required on long-term interest rates. A term premium is the additional return required by lenders to invest in a bond to maturity net of the expected return from continually reinvesting at the short-term rate over that same time horizon.
14 As long as $2ab > \sigma^2$, per Yan (2001, p. 65).

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Exhibit 7 (Continued)

Parameter	Time $t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
1 Year	3.862%	6.921%	6.911%	9.456%	7.316%
2 Years	4.499%	7.023%	7.015%	9.115%	7.349%
5 Years	5.612%	7.131%	7.126%	8.390%	7.327%
10 Years	6.333%	7.165%	7.162%	7.854%	7.272%
30 Years	6.903%	7.183%	7.182%	7.415%	7.219%

The simulation of interest rates starts with an interest rate of 3%, which is well below the long-run value of 8%. Interest rates generated by the model quickly move toward this long-run value. Note that the standard normal variable *dz* is assumed to be 0.50 in time periods $t = 0$ and $t = 2$ but the volatility term, $\sigma\sqrt{r}dz$, is much higher in $t = 2$ than in $t = 0$ because volatility increases with the level of interest rates in the CIR model.

This example is stylized and intended for illustrative purposes only. The parameters used in practice typically vary significantly from those used here.

5.1.2 *The Vasicek Model*

Although not developed in the context of a general equilibrium of individuals seeking to make optimal consumption and investment decisions, as was the case for the CIR model, the Vasicek model is viewed as an equilibrium term structure model. Similar to the CIR model, the Vasicek model captures mean reversion.

Equation 17 presents the Vasicek model:

$$dr = a(b - r)dt + \sigma dz \tag{17}$$

The Vasicek model has the same drift term as the CIR model and thus tends toward mean reversion in the short rate, r. The stochastic or volatility term, σdz, follows the random normal distribution for which the mean is zero and the standard deviation is 1. Unlike the CIR Model, interest rates are calculated assuming that volatility remains constant over the period of analysis. As with the CIR model, there is only one stochastic driver, dz, of the interest rate process and a, b, and σ are model parameters that have to be specified in some manner. The main disadvantage of the Vasicek model is that it is theoretically possible for the interest rate to become negative.

AN ILLUSTRATION OF THE VASICEK MODEL

Assume that the current short-term rate is $r = 3\%$ and the long-run value for the short-term rate is $b = 8\%$. Also assume that the speed of the adjustment factor is $a = 0.40$ and the annual volatility is $\sigma = 2\%$. Using Equation 17, the Vasicek model provides the following formula for the change in short-term interest rates, *dr*:

$$dr = 0.40(8\% - r)dt + (2\%)dz$$

The stochastic term, *dz*, is typically drawn from a standard normal distribution with a mean of zero and a standard deviation of 1. Assume that a random number generator produced standard normal random error terms of 0.45, 0.18, −0.30, and 0.25. The Vasicek model would produce the evolution of interest rates shown in Exhibit 8.

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Exhibit 8 Evolution of the Short-Term Rate in the Vasicek Model

	Time				
Parameter	***t* = 0**	***t* = 1**	***t* =2**	***t* = 3**	***t* = 4**
r	3.000%	5.900%	7.100%	6.860%	7.816%
$a(b - r)$	2.000%	0.840%	0.360%	0.456%	
dz	0.450	0.180	−0.300	0.250	
σdz	0.900%	0.360%	−0.600%	0.500%	
dr	2.900%	1.200%	−0.240%	0.956%	
$r(t + 1) = r + dr$	5.900%	7.100%	6.860%	7.816%	
YTM for Zero-Coupon Bonds Maturing in					
1 Year	3.874%	6.264%	7.253%	7.055%	7.843%
2 Years	4.543%	6.539%	7.365%	7.200%	7.858%
5 Years	5.791%	7.045%	7.563%	7.460%	7.873%
10 Years	6.694%	7.405%	7.670%	7.641%	7.876%
30 Years	7.474%	7.716%	7.816%	7.796%	7.875%

Note that the simulation of interest rates starts with an interest rate of 3%, which is well below the long-run value of 8%. Interest rates generated by the model move quickly toward this long-run value despite declining in the third time period, which reflects the mean reversion built into the model via the drift term $a(b - r)dt$.

This example is stylized and intended for illustrative purposes only. The parameters used in practice typically vary significantly from those used here.

Note that because both the Vasicek model and the CIR model require the short-term rate to follow a certain process, the estimated yield curve may not match the observed yield curve. But if the parameters of the models are believed to be correct, then investors can use these models to determine mispricings.

5.2 Arbitrage-Free Models: The Ho–Lee Model

In **arbitrage-free models**, the analysis begins with the observed market prices of a reference set of financial instruments and the underlying assumption is that the reference set is correctly priced. An assumed random process with a drift term and volatility factor is used for the generation of the yield curve. The computational process that determines the term structure is such that the valuation process generates the market prices of the reference set of financial instruments. These models are called "arbitrage-free" because the prices they generate match market prices.

The ability to calibrate models to market data is a desirable feature of any model, and this fact points to one of the main drawbacks of the Vasicek and CIR models: They have only a finite number of free parameters, and so it is not possible to specify these parameter values in such a way that model prices coincide with observed market prices. This problem is overcome in arbitrage-free models by allowing the parameters to vary deterministically with time. As a result, the market yield curve can be modeled with the accuracy needed for such applications as valuing derivatives and bonds with embedded options.

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The first arbitrage-free model was introduced by Ho and Lee.[15] It uses the relative valuation concepts of the Black–Scholes–Merton option-pricing model. Thus, the valuation of interest rate contingent claims is based solely on the yield curve's shape and its movements. The model assumes that the yield curve moves in a way that is consistent with a no-arbitrage condition.

In the **Ho–Lee model**, the short rate follows a normal process, as shown in Equation 18:

$$dr_t = \theta_t dt + \sigma dz_t \tag{18}$$

The model can be calibrated to market data by inferring the form of the time-dependent drift term, θ_t, from market prices, which means the model can precisely generate the current term structure. This calibration is typically performed via a binomial lattice-based model in which at each node the yield curve can move up or down with equal probability. This probability is called the "implied risk-neutral probability." Often it is called the "risk-neutral probability," which is somewhat misleading because arbitrage-free models do not assume market professionals are risk neutral as does the local expectations theory. This is analogous to the classic Black–Scholes–Merton option model insofar as the pricing dynamics are simplified because we can price debt securities "as if" market investors were risk neutral.

To make the discussion concrete, we illustrate a two-period Ho–Lee model. Assume that the current short-term rate is 4%. The time step is monthly, and the drift terms, which are determined using market prices, are θ_1 = 1% in the first month and θ_2 = 0.80% in the second month. The annual volatility is 2%. Below, we create a two-period binomial lattice-based model for the short-term rate. In the discrete binomial model, the dz term has two possible outcomes: +1 for periods in which rates move up and –1 for periods in which rates move down. Note that the monthly volatility is

$$\sigma\sqrt{\frac{1}{t}} = 2\%\sqrt{\frac{1}{12}} = 0.5774\%$$

and the time step is

$$dt = \frac{1}{12} = 0.0833$$

$$dr_t = \theta_t dt + \sigma dz_t = \theta_t(0.0833) + (0.5774)dz_t$$

If the rate goes up in the first month,

$$r = 4\% + (1\%)(0.0833) + 0.5774\% = 4.6607\%$$

If the rate goes up in the first month and up in the second month,

$$r = 4.6607\% + (0.80\%)(0.0833) + 0.5774\% = 5.3047\%$$

If the rate goes up in the first month and down in the second month,

$$r = 4.6607\% + (0.80\%)(0.0833) - 0.5774\% = 4.1499\%$$

If the rate goes down in the first month,

$$r = 4\% + (1\%)(0.0833) - 0.5774\% = 3.5059\%$$

If the rate goes down in the first month and up in the second month,

$$r = 3.5059\% + (0.80\%)(0.0833) + 0.5774\% = 4.1499\%$$

If the rate goes down in the first month and down in the second month,

$$r = 3.5059\% + (0.80\%)(0.0833) - 0.5774\% = 2.9951\%$$

15 Ho and Lee (1986).

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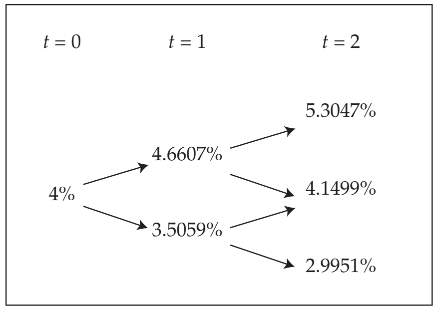

The interest rates generated by the model can be used to determine zero-coupon bond prices and the spot curve. By construction, the model output is consistent with market prices. Because of its simplicity, the Ho–Lee model is useful for illustrating most of the salient features of arbitrage-free interest rate models. Because the model generates a symmetrical ("bell-shaped" or normal) distribution of future rates, negative interest rates are possible. Note that although the volatility of the one-period rate is constant at each node point in the illustration, time-varying volatility—consistent with the historical behavior of yield curve movements—can be modeled in the Ho–Lee model because sigma (interest rate volatility) can be specified as a function of time. A more sophisticated example using a term structure of volatilities as inputs is outside the scope of this reading.

As mentioned before, models are assumptions made to describe certain phenomena and to provide solutions to problems at hand. Modern interest rate theories are proposed for the most part to value bonds with embedded options because the values of embedded options are frequently contingent on interest rates. The general equilibrium models introduced here describe yield curve movement as the movement in a single short-term rate. They are called one-factor models and, in general, seem empirically satisfactory. Arbitrage-free models do not attempt to explain the observed yield curve. Instead, these models take the yield curve as given. For this reason, they are sometimes labeled as **partial equilibrium models**.

The basic arbitrage-free concept can be used to solve much broader problems. These models can be extended to value many bond types, allowing for a term structure of volatilities, uncertain changes in the shape of the yield curve, adjustments for the credit risk of a bond, and much more. Yet, these many extensions are still based on the concept of arbitrage-free interest rate movements. For this reason, the principles of these models form a foundation for much of the modern progress made in financial modeling.

Example 10 addresses several basic points about modern term structure models.

EXAMPLE 10

Modern Term Structure Models

1 Which of the following would be expected to provide the *most* accurate modeling with respect to the observed term structure?

   A CIR model

   B Ho–Lee model

   C Vasicek model

2 Which of the following statements about the Vasicek model is *most* accurate? It has:

   A a single factor, the long rate.

   B a single factor, the short rate.

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C two factors, the short rate and the long rate.

3 The CIR model:

A assumes interest rates are not mean reverting.

B has a drift term that differs from that of the Vasicek model.

C assumes interest rate volatility increases with increases in the level of interest rates.

Solution to 1:

B is correct. The CIR model and the Vasicek model are examples of equilibrium term structure models, whereas the Ho–Lee model is an example of an arbitrage-free term structure model. A benefit of arbitrage-free term structure models is that they are calibrated to the current term structure. In other words,the starting prices ascribed to securities are those currently found in the market. In contrast, equilibrium term structure models frequently generate term structures that are inconsistent with current market data.

Solution to 2:

B is correct. Use of the Vasicek model requires assumptions for the short-term interest rate, which are usually derived from more general assumptions about the state variables that describe the overall economy. Using the assumed process for the short-term rate, one can determine the yield on longer-term bonds by looking at the expected path of interest rates over time.

Solution to 3:

C is correct. The drift term of the CIR model is identical to that of the Vasicek model, and both models assume that interest rates are mean reverting. The big difference between the two models is that the CIR model assumes that interest rate volatility increases with increases in the level of interest rates. The Vasicek model assumes that interest rate volatility is a constant.

YIELD CURVE FACTOR MODELS

6

The effect of yield volatilities on price is an important consideration in fixed-income investment, particularly for risk management and portfolio evaluation. In this section, we will describe measuring and managing the interest rate risk of bonds.

6.1 A Bond's Exposure to Yield Curve Movement

Shaping risk is defined as the sensitivity of a bond's price to the changing shape of the yield curve. The shape of the yield curve changes continually, and yield curve shifts are rarely parallel. For active bond management, a bond investor may want to base trades on a forecasted yield curve shape or may want to hedge the yield curve risk on a bond portfolio. Shaping risk also affects the value of many options, which is very important because many fixed-income instruments have embedded options.

Exhibits 9 through 11 show historical yield curve movements for US, Japanese, and South Korean government bonds from August 2005 to July 2013. The exhibits show that the shape of the yield curve changes considerably over time. In the United States and South Korea, central bank policies in response to the Great Recession led to a significant decline in short-term yields during the 2007–2009 time period.

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Long-term yields eventually followed suit, resulting in a flattening of the yield curve. Short-term and long-term Japanese yields have been low for quite some time. Note that the vertical axis values of the three exhibits differ.

Exhibit 9 Historical US Yield Curve Movements

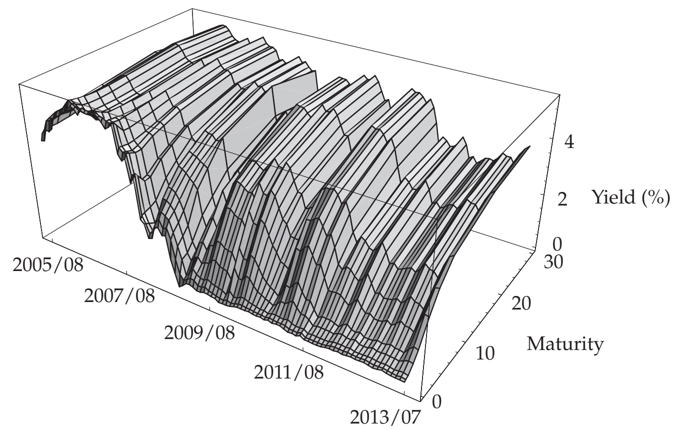

Exhibit 10 Historical Japanese Yield Curve Movements

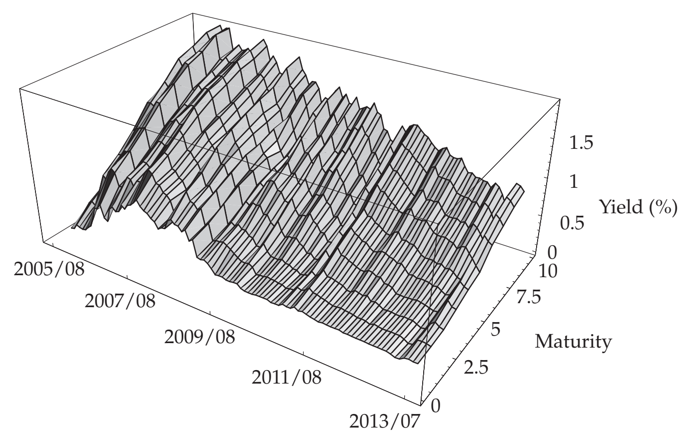

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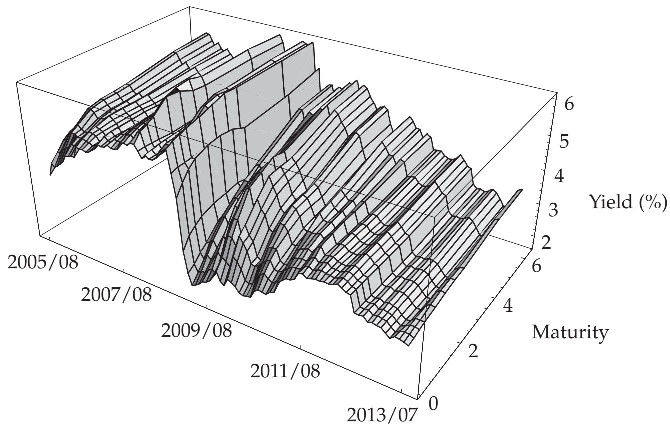
Exhibit 11 Historical Korean Yield Curve Movements



6.2 Factors Affecting the Shape of the Yield Curve

The previous section showed that the yield curve can take nearly any shape. The challenge for a fixed-income manager is to implement a process to manage the yield curve shape risk in his or her portfolio. One approach is to find a model that reduces most of the possible yield curve movements to a probabilistic combination of a few standardized yield curve movements. This section presents one of the best-known yield curve factor models.

A **yield curve factor model** is defined as a model or a description of yield curve movements that can be considered realistic when compared with historical data. Research shows that there are models that can describe these movements with some accuracy. One specific yield curve factor model is the three-factor model of Litterman and Scheinkman (1991), who found that yield curve movements are historically well described by a combination of three independent movements, which they interpreted as **level**, **steepness**, and **curvature**. The level movement refers to an upward or downward shift in the yield curve. The steepness movement refers to a non-parallel shift in the yield curve when either short-term rates change more than long-term rates or long-term rates change more than short-term rates. The curvature movement is a reference to movement in three segments of the yield curve: the short-term and long-term segments rise while the middle-term segment falls or vice versa.

The method to determine the number of factors—and their economic interpretation—begins with a measurement of the change of key rates on the yield curve, in this case 10 different points along the yield curve, as shown in Exhibits 12 and 13. The historical variance/covariance matrix of these interest rate movements is then obtained. The next step is to try to discover a number of independent factors (not to exceed the number of variables—in this case, selected points along the yield curve) that can explain the observed variance/covariance matrix. The approach that focuses on identifying the factors that best explain historical variances is known as **principal components analysis** (PCA). PCA creates a number of synthetic factors defined as (and calculated to be) statistically independent of each other; how these factors may be interpreted economically is a challenge to the researcher that can be addressed by relating movements in the factors (as we will call the principal components in this discussion) to movements in observable and easily understood variables.

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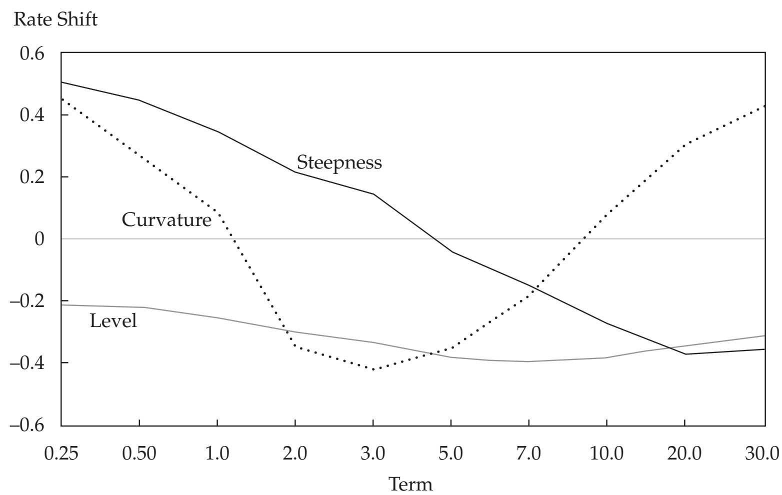
In applying this analysis to historical data for the period of August 2005–July 2013, very typical results were found, as expressed in Exhibit 12 and graphed in Exhibit 13. The first principal component explained about 77% of the total variance/covariance, and the second and third principal components (or factors) explained 17% and 3%, respectively. These percentages are more commonly recognized as R^2s, which, by the underlying assumptions of principal components analysis, can be simply summed to discover that a linear combination of the first three factors explains almost 97% of the total yield curve changes in the sample studied.

Exhibit 12 The First Three Yield Curve Factors, US Treasury Securities, August 2005–July 2013 (Entries are percents)

Time to Maturity (Years)	0.25	0.5	1	2	3	5	7	10	20	30
Factor 1 "Level"	−0.2089	−0.2199	−0.2497	−0.2977	−0.3311	−0.3756	−0.3894	−0.3779	−0.3402	−0.3102
Factor 2 "Steepness"	0.5071	0.4480	0.3485	0.2189	0.1473	−0.0371	−0.1471	−0.2680	−0.3645	−0.3514
Factor 3 "Curvature"	0.4520	0.2623	0.0878	−0.3401	−0.4144	−0.349	−0.1790	0.0801	0.3058	0.4219

Note that in Exhibit 13, the x-axis represents time to maturity in years.

Exhibit 13 The First Three Yield Curve Factors for US Treasury Securities, August 2005–July 2013



How should Exhibit 12 be interpreted? Exhibit 12 shows that for a one standard deviation positive change in the first factor (normalized to have unit standard deviation), the yield for a 0.25-year bond would decline by 0.2089%, a 0.50-year bond by 0.2199%, and so on across maturities, so that a 30-year bond would decline by 0.3102%.

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Because the responses are in the same direction and by similar magnitudes, a reasonable interpretation of the first factor is that it describes (approximately) parallel shifts up and down the entire length of the yield curve.

Examining the second factor, we notice that a unitary positive standard deviation change appears to raise rates at shorter maturities (e.g., +0.5071% for 0.25-year bonds) but lowers rates at longer maturities (e.g., −0.3645% and −0.3514% for 20- and 30-year bonds, respectively). We can reasonably interpret this factor as one that causes changes in the steepness or slope of the yield curve. We note that the R^2 associated with this factor of 17% is much less important than the 77% R^2 associated with the first factor, which we associated with parallel shifts in the yield curve.

The third factor contributes a much smaller R^2 of 3%, and we associate this factor with changes in the curvature or "twist" in the curve because a unitary positive standard deviation change in this factor leads to positive yield changes at both short and long maturities but produces declines at intermediate maturities.

PCA shows similar results when applied to other government bond markets during the August 2005–July 2013 time period. Exhibits 14 and 15 reflect the results graphically for the Japanese and South Korean markets. In these instances, results can also be well explained by factors that appear to be associated, in declining order of importance, with parallel shifts, changes in steepness, and changes in curvature. Note that in Exhibits 14 and 15, as in Exhibit 13, the x-axis represents time to maturity in years.

Exhibit 14 The First Three Yield Curve Factors for Japanese Government Securities, August 2005–July 2013

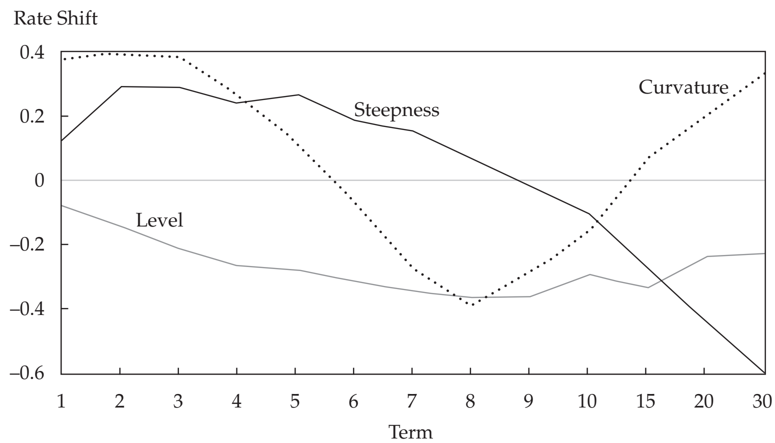

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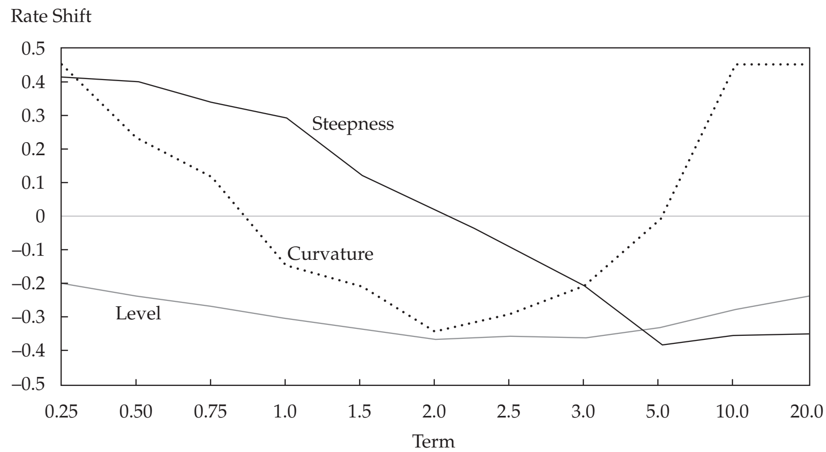
Exhibit 15 The First Three Yield Curve Factors for South Korean Government Securities, August 2005–July 2013



As in any other time series or regression model, the impact of the factors may change depending on the time period selected for study. However, if the reader selects any date within the sample period used to estimate these factors, a linear combination of the factors should explain movements of the yield curve on that date well.

6.3 The Maturity Structure of Yield Curve Volatilities

In modern fixed-income management, quantifying interest rate volatilities is important for at least two reasons. First, most fixed-income instruments and derivatives have embedded options. Option values, and hence the values of the fixed-income instrument, crucially depend on the level of interest rate volatilities. Second, fixed-income interest rate risk management is clearly an important part of any management process, and such risk management includes controlling the impact of interest rate volatilities on the instrument's price volatility.

The term structure of interest rate volatilities is a representation of the yield volatility of a zero-coupon bond for every maturity of security. This volatility curve (or "vol") or volatility term structure measures yield curve risk.

Interest rate volatility is not the same for all interest rates along the yield curve. On the basis of the typical assumption of a lognormal model, the uncertainty of an interest rate is measured by the annualized standard deviation of the proportional change in a bond yield over a specified time interval. For example, if the time interval is a one-month period, then the specified time interval equals 1/12 years. This measure is called interest rate volatility, and it is denoted $\sigma(t,T)$, which is the volatility of the rate for a security with maturity T at time t. The term structure of volatilities is given by Equation 19:

$$\sigma(t,T) = \frac{\sigma\left[\Delta r(t,T)/r(t,T)\right]}{\sqrt{\Delta t}} \tag{19}$$

In Exhibit 16, to illustrate a term structure of volatility, the data series is deliberately chosen to end before the 2008 financial crisis, which was associated with some unusual volatility magnitudes.

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Exhibit 16 Historical Volatility Term Structure: US Treasuries, August 2005–December 2007

Maturity (years)	0.25	0.50	1	2	3	5	7	10	20	30
$\sigma(t,T)$	0.3515	0.3173	0.2964	0.2713	0.2577	0.2154	0.1885	0.1621	0.1332	0.1169

For example, the 35.15% standard deviation for the three-month T-bill in Exhibit 16 is based on a monthly standard deviation of 0.1015 = 10.15%, which annualizes as

$$0.1015 \div \sqrt{\frac{1}{12}} = 0.3515 = 35.15\%$$

The volatility term structure typically shows that short-term rates are more volatile than long-term rates. Research indicates that short-term volatility is most strongly linked to uncertainty regarding monetary policy whereas long-term volatility is most strongly linked to uncertainty regarding the real economy and inflation. Furthermore, most of the co-movement between short-term and long-term volatilities appears to depend on the ever-changing correlations between these three determinants (monetary policy, the real economy, and inflation). During the period of August 2005–December 2007, long-term volatility was lower than short-term volatility, falling from 35.15% for the 0.25-year rate to 11.69% for the 30-year rate.

6.4 Managing Yield Curve Risks

Yield curve risk—risk to portfolio value arising from unanticipated changes in the yield curve—can be managed on the basis of several measures of sensitivity to yield curve movements. Management of yield curve risk involves changing the identified exposures to desired values by trades in security or derivative markets (the details fall under the rubric of fixed-income portfolio management and thus are outside the scope of this reading).

One available measure of yield curve sensitivity is effective duration, which measures the sensitivity of a bond's price to a small parallel shift in a benchmark yield curve. Another is based on key rate duration, which measures a bond's sensitivity to a small change in a benchmark yield curve at a specific maturity segment. A further measure can be developed on the basis of the factor model developed in Section 6.3. Using one of these last two measures allows identification and management of "shaping risk"—that is, sensitivity to changes in the shape of the benchmark yield curve—in addition to the risk associated with parallel yield curve changes, which is addressed adequately by effective duration.

To make the discussion more concrete, consider a portfolio of 1-year, 5-year, and 10-year zero-coupon bonds with $100 value in each position; total portfolio value is therefore $300. Also consider the hypothetical set of factor movements shown in the following table:

Year	1	5	10
Parallel	1	1	1
Steepness	–1	0	1
Curvature	1	0	1

In the table, a parallel movement or shift means that all the rates shift by an equal amount—in this case, by a unit of 1. A steepness movement means that the yield curve steepens with the long rate shifting up by one unit and the short rate shifting down by one unit. A curvature movement means that both the short rate and the long rate

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shift up by one unit whereas the medium-term rate remains unchanged. These movements need to be defined, as they are here, such that none of the movements can be a linear combination of the other two movements. Next, we address the calculation of the various yield curve sensitivity measures.

Because the bonds are zero-coupon bonds, the effective duration of each bond is the same as the maturity of the bonds.[16] The portfolio's effective duration is the weighted sum of the effective duration of each bond position; for this equally weighted portfolio, effective duration is 0.333(1 + 5 + 10) = 5.333.

To calculate **key rate durations**, consider various yield curve movements. First, suppose that the one-year rate changes by 100 bps while the other rates remain the same; the sensitivity of the portfolio to that shift is 1/[(300)(0.01)] = 0.3333. We conclude that the key rate duration of the portfolio to the one-year rate, denoted D_1, is 0.3333. Likewise, the key rate durations of the portfolio to the 5-year rate, D_5, and the 10-year rate, D_{10}, are 1.6667 and 3.3333, respectively. Note that the sum of the key rate durations is 5.333, which is the same as the effective duration of the portfolio. This fact can be explained intuitively. Key rate duration measures the portfolio risk exposure to each key rate. If all the key rates move by the same amount, then the yield curve has made a parallel shift, and as a result, the proportional change in value has to be consistent with effective duration. The related model for yield curve risk based on key rate durations would be

$$\begin{aligned}\left(\frac{\Delta P}{P}\right) &\approx -D_1\Delta r_1 - D_5\Delta r_5 - D_{10}\Delta r_{10} \\ &= -0.3333\Delta r_1 - 1.6667\Delta r_5 - 3.3333\Delta r_{10}\end{aligned} \tag{20}$$

Next, we can calculate a measure based on the decomposition of yield curve movements into parallel, steepness, and curvature movements made in Section 6.3. Define D_L, D_S, and D_C as the sensitivities of portfolio value to small changes in the level, steepness, and curvature factors, respectively. Based on this factor model, Equation 21 shows the proportional change in portfolio value that would result from a small change in the level factor (Δx_L), the steepness factor (Δx_S), and the curvature factor (Δx_C).

$$\left(\frac{\Delta P}{P}\right) \approx -D_L\Delta x_L - D_S\Delta x_S - D_C\Delta x_C \tag{21}$$

Because D_L is by definition sensitivity to a parallel shift, the proportional change in the portfolio value per unit shift (the line for a parallel movement in the table) is 5.3333 = 16/[(300)(0.01)]. The sensitivity for steepness movement can be calculated as follows (see the line for steepness movement in the table). When the steepness makes an upward shift of 100 bps, it would result in a downward shift of 100 bps for the 1-year rate, resulting in a gain of $1, and an upward shift for the 10-year rate, resulting in a loss of $10. The change in value is therefore (1 – 10). D_S is the negative of the proportional change in price per unit change in this movement and in this case is 3.0 = −(1 – 10)/[(300)(0.01)]. Considering the line for curvature movement in the table, D_C = 3.6667 = (1 + 10)/[(300)(0.01)]. Thus, for our hypothetical bond portfolio, we can analyze the portfolio's yield curve risk using

$$\left(\frac{\Delta P}{p}\right) \approx -5.3333\Delta x_L - 3.0\Delta x_S - 3.6667\Delta x_C \tag{22}$$

16 Exactly so under continuous compounding.

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For example, if $\Delta x_L = -0.0050$, $\Delta x_S = 0.002$, and $\Delta x_C = 0.001$, the predicted change in portfolio value would be +1.7%. It can be shown that key rate durations are directly related to level, steepness, and curvature in this example and that one set of sensitivities can be derived from the other. One can use the numerical example to verify that[17]

$D_L = D_1 + D_5 + D_{10}$

$D_S = -D_1 + D_{10}$

$D_C = D_1 + D_{10}$

Example 11 reviews concepts from this section and the preceding sections.

EXAMPLE 11

Term Structure Dynamics

1 The most important factor in explaining changes in the yield curve has been found to be:

A level.

B curvature.

C steepnesss.

2 A movement of the yield curve in which the short rate decreases by 150 bps and the long rate decreases by 50 bps would *best* be described as a:

A flattening of the yield curve resulting from changes in level and steepness.

B steepening of the yield curve resulting from changes in level and steepness.

C steepening of the yield curve resulting from changes in steepness and curvature.

3 A movement of the yield curve in which the short- and long-maturity sectors increase by 100 bps and 75 bps, respectively, but the intermediate-maturity sector increases by 10 bps, is *best* described as involving a change in:

A level only.

B curvature only.

C level and curvature.

4 Typically, short-term interest rates:

A are less volatile than long-term interest rates.

B are more volatile than long-term interest rates.

C have about the same volatility as long-term rates.

17 To see this, decompose Δr_1, Δr_5, and Δr_{10} into three factors—parallel, steepness, and curvature—based on the hypothetical movements in the table.

$\Delta r_1 = \Delta x_L - \Delta x_s + \Delta x_C$

$\Delta r_5 = \Delta x_L$

$\Delta r_{10} = \Delta x_L + \Delta x_S + \Delta x_C$

When we plug these equations into the expression for portfolio change based on key rate duration and simplify, we get

$$\frac{\Delta P}{P} = -D_1(\Delta x_L - \Delta x_S + \Delta x_C) - D_5(\Delta x_L) - D_{10}(\Delta x_L + \Delta x_S + \Delta x_C)$$
$$= -(D_1 + D_5 + D_{10})\Delta x_L - (-D_1 + D_{10})\Delta x_S - (D_1 + D_{10})\Delta x_C$$

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5 Suppose for a given portfolio that key rate changes are considered to be changes in the yield on 1-year, 5-year, and 10-year securities. Estimated key rate durations are $D_1 = 0.50$, $D_2 = 0.70$, and $D_3 = 0.90$. What is the percentage change in the value of the portfolio if a parallel shift in the yield curve results in all yields declining by 50 bps?

A −1.05%.

B +1.05%.

C +2.10%.

Solution to 1:

A is correct. Research shows that upward and downward shifts in the yield curve explain more than 75% of the total change in the yield curve.

Solution to 2:

B is correct. Both the short-term and long-term rates have declined, indicating a change in the level of the yield curve. Short-term rates have declined more than long-term rates, indicating a change in the steepness of the yield curve.

Solution to 3:

C is correct. Both the short-term and long-term rates have increased, indicating a change in the level of the yield curve. However, intermediate rates have increased less than both short-term and long-term rates, indicating a change in curvature.

Solution to 4:

B is correct. A possible explanation is that expectations for long-term inflation and real economic activity affecting longer-term interest rates are slower to change than those related to shorter-term interest rates.

Solution to 5:

B is correct. A decline in interest rates would lead to an increase in bond portfolio value: $-0.50(-0.005) - 0.70(-0.005) - 0.90(-0.005) = 0.0105 = 1.05\%$.

SUMMARY

- The spot rate for a given maturity can be expressed as a geometric average of the short-term rate and a series of forward rates.
- Forward rates are above (below) spot rates when the spot curve is upward (downward) sloping, whereas forward rates are equal to spot rates when the spot curve is flat.
- If forward rates are realized, then all bonds, regardless of maturity, will have the same one-period realized return, which is the first-period spot rate.
- If the spot rate curve is upward sloping and is unchanged, then each bond “rolls down” the curve and earns the forward rate that rolls out of its pricing (i.e., a T^*-period zero-coupon bond earns the T^*-period forward rate as it rolls down to be a $T^* - 1$ period security). This implies an expected return in excess of short-maturity bonds (i.e., a term premium) for longer-maturity bonds if the yield curve is upward sloping.

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- Active bond portfolio management is consistent with the expectation that today's forward curve does not accurately reflect future spot rates.
- The swap curve provides another measure of the time value of money.
- The swap markets are significant internationally because swaps are frequently used to hedge interest rate risk exposure.
- The swap spread, the I-spread, and the Z-spread are bond quoting conventions that can be used to determine a bond's price.
- Swap curves and Treasury curves can differ because of differences in their credit exposures, liquidity, and other supply/demand factors.
- The local expectations theory, liquidity preference theory, segmented markets theory, and preferred habitat theory provide traditional explanations for the shape of the yield curve.
- Modern finance seeks to provide models for the shape of the yield curve and the use of the yield curve to value bonds (including those with embedded options) and bond-related derivatives. General equilibrium and arbitrage-free models are the two major types of such models.
- Arbitrage-free models are frequently used to value bonds with embedded options. Unlike equilibrium models, arbitrage-free models begin with the observed market prices of a reference set of financial instruments, and the underlying assumption is that the reference set is correctly priced.
- Historical yield curve movements suggest that they can be explained by a linear combination of three principal movements: level, steepness, and curvature.
- The volatility term structure can be measured using historical data and depicts yield curve risk.
- The sensitivity of a bond value to yield curve changes may make use of effective duration, key rate durations, or sensitivities to parallel, steepness, and curvature movements. Using key rate durations or sensitivities to parallel, steepness, and curvature movements allows one to measure and manage shaping risk.

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PRACTICE PROBLEMS

1 Given spot rates for one-, two-, and three-year zero coupon bonds, how many forward rates can be calculated?

2 Give two interpretations for the following forward rate: The two-year forward rate one year from now is 2%.

3 Describe the relationship between forward rates and spot rates if the yield curve is flat.

4 A Define the yield to maturity for a coupon bond.

B Is it possible for a coupon bond to earn less than the yield to maturity if held to maturity?

5 If a bond trader believes that current forward rates overstate future spot rates, how might he or she profit from that conclusion?

6 Explain the strategy of riding the yield curve.

7 What are the advantages of using the swap curve as a benchmark of interest rates relative to a government bond yield curve?

8 Describe how the Z-spread can be used to price a bond.

9 What is the TED spread and what type of risk does it measure?

10 According to the local expectations theory, what would be the difference in the one-month total return if an investor purchased a five-year zero-coupon bond versus a two-year zero-coupon bond?

11 Compare the segmented market and the preferred habitat term structure theories.

12 A List the three factors that have empirically been observed to affect Treasury security returns and explain how each of these factors affects returns on Treasury securities.

B What has been observed to be the most important factor in affecting Treasury returns?

C Which measures of yield curve risk can measure shaping risk?

13 Which forward rate cannot be computed from the one-, two-, three-, and four-year spot rates? The rate for a:

A one-year loan beginning in two years.

B two-year loan beginning in two years.

C three-year loan beginning in two years.

14 Consider spot rates for three zero-coupon bonds: $r(1)$ = 3%, $r(2)$ = 4%, and $r(3)$ = 5%. Which statement is correct? The forward rate for a one-year loan beginning in one year will be:

A less than the forward rate for a one-year loan beginning in two-years.

B greater than the forward rate for a two-year loan beginning in one-year.

C greater than the forward rate for a one-year loan beginning in two-years.

15 If one-period forward rates are decreasing with maturity, the yield curve is *most likely*:

A flat.

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B upward-sloping.

C downward sloping.

The following information relates to Questions 16–29

A one-year zero-coupon bond yields 4.0%. The two- and three-year zero-coupon bonds yield 5.0% and 6.0% respectively.

16 The rate for a one-year loan beginning in one year is *closest* to:

A 4.5%.

B 5.0%.

C 6.0%.

17 The forward rate for a two-year loan beginning in one year is *closest* to:

A 5.0%.

B 6.0%.

C 7.0%.

18 The forward rate for a one-year loan beginning in two years is *closest* to:

A 6.0%.

B 7.0%.

C 8.0%.

19 The five-year spot rate is not given above; however, the forward price for a two-year zero-coupon bond beginning in three years is known to be 0.8479. The price today of a five-year zero-coupon bond is *closest* to:

A 0.7119.

B 0.7835.

C 0.9524.

20 The one-year spot rate $r(1)$ = 4%, the forward rate for a one-year loan beginning in one year is 6%, and the forward rate for a one-year loan beginning in two years is 8%. Which of the following rates is *closest* to the three-year spot rate?

A 4.0%

B 6.0%

C 8.0%

21 The one-year spot rate $r(1)$ = 5% and the forward price for a one-year zero-coupon bond beginning in one year is 0.9346. The spot price of a two-year zero-coupon bond is *closest* to:

A 0.87.

B 0.89.

C 0.93.

22 In a typical interest rate swap contract, the swap rate is *best* described as the interest rate for the:

A fixed-rate leg of the swap.

B floating-rate leg of the swap.

C difference between the fixed and floating legs of the swap.

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23 A two-year fixed-for-floating Libor swap is 1.00% and the two-year US Treasury bond is yielding 0.63%. The swap spread is *closest* to:

A 37 bps.

B 100 bps.

C 163 bps.

24 The swap spread is quoted as 50 bps. If the five-year US Treasury bond is yielding 2%, the rate paid by the fixed payer in a five-year interest rate swap is *closest* to:

A 0.50%.

B 1.50%.

C 2.50%.

25 If the three-month T-bill rate drops and the Libor rate remains the same, the relevant TED spread:

A increases.

B decreases.

C does not change.

26 Given the yield curve for US Treasury zero-coupon bonds, which spread is *most* helpful pricing a corporate bond? The:

A Z-Spread.

B TED spread.

C Libor–OIS spread.

27 A four-year corporate bond with a 7% coupon has a Z-spread of 200 bps. Assume a flat yield curve with an interest rate for all maturities of 5% and annual compounding. The bond will *most likely* sell:

A close to par.

B at a premium to par.

C at a discount to par.

28 The Z-spread of Bond A is 1.05% and the Z-spread of Bond B is 1.53%. All else equal, which statement *best* describes the relationship between the two bonds?

A Bond B is safer and will sell at a lower price.

B Bond B is riskier and will sell at a lower price.

C Bond A is riskier and will sell at a higher price.

29 Which term structure model can be calibrated to closely fit an observed yield curve?

A The Ho–Lee Model

B The Vasicek Model

C The Cox–Ingersoll–Ross Model

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The following information relates to Questions 30–36

Jane Nguyen is a senior bond trader and Christine Alexander is a junior bond trader for an investment bank. Nguyen is responsible for her own trading activities and also for providing assignments to Alexander that will develop her skills and create profitable trade ideas. Exhibit 1 presents the current par and spot rates.

Exhibit 1 Current Par and Spot Rates

Maturity	Par Rate	Spot Rate
One year	2.50%	2.50%
Two years	2.99%	3.00%
Three years	3.48%	3.50%
Four years	3.95%	4.00%
Five years	4.37%	

Note: Par and spot rates are based on annual-coupon sovereign bonds.

Nguyen gives Alexander two assignments that involve researching various questions:

Assignment 1 What is the yield to maturity of the option-free, default risk–free bond presented in Exhibit 2? Assume that the bond is held to maturity, and use the rates shown in Exhibit 1.

Exhibit 2 Selected Data for $1,000 Par Bond

Bond Name	Maturity (*T*)	Coupon
Bond Z	Three years	6.00%

Note: Terms are today for a *T*-year loan.

Assignment 2 Assuming that the projected spot curve two years from today will be below the current forward curve, is Bond Z fairly valued, undervalued, or overvalued?

After completing her assignments, Alexander asks about Nguyen's current trading activities. Nguyen states that she has a two-year investment horizon and will purchase Bond Z as part of a strategy to ride the yield curve. Exhibit 1 shows Nguyen's yield curve assumptions implied by the spot rates.

30 Based on Exhibit 1, the five-year spot rate is *closest to*:

A 4.40%.

B 4.45%.

C 4.50%.

31 Based on Exhibit 1, the market is *most likely* expecting:

A deflation.

B inflation.

C no risk premiums.

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32 Based on Exhibit 1, the forward rate of a one-year loan beginning in three years is *closest to*:

A 4.17%.

B 4.50%.

C 5.51%.

33 Based on Exhibit 1, which of the following forward rates can be computed?

A A one-year loan beginning in five years

B A three-year loan beginning in three years

C A four-year loan beginning in one year

34 For Assignment 1, the yield to maturity for Bond Z is *closest* to the:

A one-year spot rate.

B two-year spot rate.

C three-year spot rate.

35 For Assignment 2, Alexander should conclude that Bond Z is currently:

A undervalued.

B fairly valued.

C overvalued.

36 By choosing to buy Bond Z, Nguyen is *most likely* making which of the following assumptions?

A Bond Z will be held to maturity.

B The three-year forward curve is above the spot curve.

C Future spot rates do not accurately reflect future inflation.

The following information relates to Questions 37–41

Laura Mathews recently hired Robert Smith, an investment adviser at Shire Gate Advisers, to assist her in investing. Mathews states that her investment time horizon is short, approximately two years or less. Smith gathers information on spot rates for on-the-run annual-coupon government securities and swap spreads, as presented in Exhibit 1. Shire Gate Advisers recently published a report for its clients stating its belief that, based on the weakness in the financial markets, interest rates will remain stable, the yield curve will not change its level or shape for the next two years, and swap spreads will also remain unchanged.

Exhibit 1 Government Spot Rates and Swap Spreads

	Maturity (years)			
	1	2	3	4
Government spot rate	2.25%	2.70%	3.30%	4.05%
Swap spread	0.25%	0.30%	0.45%	0.70%

Smith decides to examine the following three investment options for Mathews:

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Investment 1:	Buy a government security that would have an annualized return that is nearly risk free. Smith is considering two possible implementations: a two-year investment or a combination of two one-year investments.
Investment 2:	Buy a four-year, zero-coupon corporate bond and then sell it after two years. Smith illustrates the returns from this strategy using the swap rate as a proxy for corporate yields.
Investment 3:	Buy a lower-quality, two-year corporate bond with a coupon rate of 4.15% and a Z-spread of 65 bps.

When Smith meets with Mathews to present these choices, Mathews tells him that she is somewhat confused by the various spread measures. She is curious to know whether there is one spread measure that could be used as a good indicator of the risk and liquidity of money market securities during the recent past.

37 In his presentation of Investment 1, Smith could show that under the no-arbitrage principle, the forward price of a one-year government bond to be issued in one year is *closest* to:

- **A** 0.9662.
- **B** 0.9694.
- **C** 0.9780.

38 In presenting Investment 1, using Shire Gate Advisers' interest rate outlook, Smith could show that riding the yield curve provides a total return that is *most likely*:

- **A** lower than the return on a maturity-matching strategy.
- **B** equal to the return on a maturity-matching strategy.
- **C** higher than the return on a maturity-matching strategy.

39 In presenting Investment 2, Smith should show a total return *closest* to:

- **A** 4.31%.
- **B** 5.42%.
- **C** 6.53%.

40 The bond in Investment 3 is *most likely* trading at a price of:

- **A** 100.97.
- **B** 101.54.
- **C** 104.09.

41 The *most* appropriate response to Mathews question regarding a spread measure is the:

- **A** Z-spread.
- **B** Treasury–Eurodollar (TED) spread.
- **C** Libor–OIS (overnight indexed swap) spread.

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The following information relates to Questions 42–48

Rowan Madison is a junior analyst at Cardinal Capital. Sage Winter, a senior portfolio manager and Madison's supervisor, meets with Madison to discuss interest rates and review two bond positions in the firm's fixed-income portfolio.

Winter begins the meeting by asking Madison to state her views on the term structure of interest rates. Madison responds:

> "Yields are a reflection of expected spot rates and risk premiums. Investors demand risk premiums for holding long-term bonds, and these risk premiums increase with maturity."

Winter next asks Madison to describe features of equilibrium and arbitrage-free term structure models. Madison responds by making the following statements:

Statement 1	"Equilibrium term structure models are factor models that use the observed market prices of a reference set of financial instruments, assumed to be correctly priced, to model the market yield curve."
Statement 2	"In contrast, arbitrage-free term structure models seek to describe the dynamics of the term structure by using fundamental economic variables that are assumed to affect interest rates."

Winter asks Madison about her preferences concerning term structure models. Madison states:

> "I prefer arbitrage-free models. Even though equilibrium models require fewer parameters to be estimated relative to arbitrage-free models, arbitrage-free models allow for time-varying parameters. In general, this allowance leads to arbitrage-free models being able to model the market yield curve more precisely than equilibrium models."

Winter tells Madison that, based on recent changes in spreads, she is concerned about a perceived increase in counterparty risk in the economy and its effect on the portfolio. Madison asks Winter:

> "Which spread measure should we use to assess changes in counterparty risk in the economy?"

Winter is also worried about the effect of yield volatility on the portfolio. She asks Madison to identify the economic factors that affect short-term and long-term rate volatility. Madison responds:

> "Short-term rate volatility is mostly linked to uncertainty regarding monetary policy, whereas long-term rate volatility is mostly linked to uncertainty regarding the real economy and inflation."

Finally, Winter asks Madison to analyze the interest rate risk portfolio positions in a 5-year and a 20-year bond. Winter requests that the analysis be based on level, slope, and curvature as term structure factors. Madison presents her analysis in Exhibit 1.

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Exhibit 1 Three-Factor Model of Term Structure

	Time to Maturity (years)	
Factor	**5**	**20**
Level	–0.4352%	–0.5128%
Steepness	–0.0515%	–0.3015%
Curvature	0.3963%	0.5227%

Note: Entries indicate how yields would change for a one standard deviation increase in a factor.

Winter asks Madison to perform two analyses:

Analysis 1: Calculate the expected change in yield on the 20-year bond resulting from a two standard deviation increase in the steepness factor.

Analysis 2: Calculate the expected change in yield on the five-year bond resulting from a one standard deviation decrease in the level factor and a one standard deviation decrease in the curvature factor.

42 Madison's views on the term structure of interest rates are *most* consistent with the:

A local expectations theory.

B segmented markets theory.

C liquidity preference theory.

43 Which of Madison's statement(s) regarding equilibrium and arbitrage-free term structure models is *incorrect*?

A Statement 1 only

B Statement 2 only

C Both Statement 1 and Statement 2

44 Is Madison correct in describing key differences in equilibrium and arbitrage-free models as they relate to the number of parameters and model accuracy?

A Yes.

B No, she is incorrect about which type of model requires fewer parameter estimates.

C No, she is incorrect about which type of model is more precise at modeling market yield curves.

45 The *most appropriate* response to Madison's question regarding the spread measure is the:

A Z-spread.

B Treasury–Eurodollar (TED) spread.

C Libor–OIS (overnight indexed swap) spread.

46 Is Madison's response regarding the factors that affect short-term and long-term rate volatility correct?

A Yes.

B No, she is incorrect regarding factors linked to long-term rate volatility.

C No, she is incorrect regarding factors linked to short-term rate volatility.

47 Based on Exhibit 1, the results of Analysis 1 should show the yield on the 20-year bond decreasing by:

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A 0.3015%.

B 0.6030%.

C 0.8946%.

48 Based on Exhibit 1, the results of Analysis 2 should show the yield on the five-year bond:

A decreasing by 0.8315%.

B decreasing by 0.0389%.

C increasing by 0.0389%.

The following information relates to Questions 49–57

Liz Tyo is a fund manager for an actively managed global fixed-income fund that buys bonds issued in Countries A, B, and C. She and her assistant are preparing the quarterly markets update. Tyo begins the meeting by distributing the daily rates sheet, which includes the current government spot rates for Countries A, B, and C as shown in Exhibit 1.

Exhibit 1 Today's Government Spot Rates

Maturity	Country A	Country B	Country C
One year	0.40%	–0.22%	14.00%
Two years	0.70	–0.20	12.40
Three years	1.00	–0.12	11.80
Four years	1.30	–0.02	11.00
Five years	1.50	0.13	10.70

Tyo asks her assistant how these spot rates were obtained. The assistant replies, "Spot rates are determined through the process of bootstrapping. It entails backward substitution using par yields to solve for zero-coupon rates one by one, in order from latest to earliest maturities."

Tyo then provides a review of the fund's performance during the last year and comments, "The choice of an appropriate benchmark depends on the country's characteristics. For example, although Countries A and B have both an active government bond market and a swap market, Country C's private sector is much bigger than its public sector, and its government bond market lacks liquidity."

Tyo further points out, "The fund's results were mixed; returns did not benefit from taking on additional risk. We are especially monitoring the riskiness of the corporate bond holdings. For example, our largest holdings consist of three four-year corporate bonds (Bonds 1, 2, and 3) with identical maturities, coupon rates, and other contract terms. These bonds have Z-spreads of 0.55%, 1.52%, and 1.76%, respectively."

Tyo continues, "We also look at risk in terms of the swap spread. We considered historical three-year swap spreads for Country B, which reflect that market's credit and liquidity risks, at three different points in time." Tyo provides the information in Exhibit 2.

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Exhibit 2 Selected Historical Three-Year Rates for Country B

Period	Government Bond Yield (%)	Fixed-for-Floating Libor Swap (%)
1 Month ago	–0.10	0.16
6 Months ago	–0.08	0.01
12 Months ago	–0.07	0.71

Tyo then suggests that the firm was able to add return by riding the yield curve. The fund plans to continue to use this strategy but only in markets with an attractive yield curve for this strategy.

She moves on to present her market views on the respective yield curves for a five-year investment horizon.

> **Country A:** "The government yield curve has changed little in terms of its level and shape during the last few years, and I expect this trend to continue. We assume that future spot rates reflect the current forward curve for all maturities."
>
> **Country B:** "Because of recent economic trends, I expect a reversal in the slope of the current yield curve. We assume that future spot rates will be higher than current forward rates for all maturities."
>
> **Country C:** "To improve liquidity, Country C's central bank is expected to intervene, leading to a reversal in the slope of the existing yield curve. We assume that future spot rates will be lower than today's forward rates for all maturities."

Tyo's assistant asks, "Assuming investors require liquidity premiums, how can a yield curve slope downward? What does this imply about forward rates?"

Tyo answers, "Even if investors require compensation for holding longer-term bonds, the yield curve can slope downward—for example, if there is an expectation of severe deflation. Regarding forward rates, it can be helpful to understand yield curve dynamics by calculating implied forward rates. To see what I mean, we can use Exhibit 1 to calculate the forward rate for a two-year Country C loan beginning in three years."

49 Did Tyo's assistant accurately describe the process of bootstrapping?

- **A** Yes.
- **B** No, with respect to par yields.
- **C** No, with respect to backward substitution.

50 The swap curve is a better benchmark than the government spot curve for:

- **A** Country A.
- **B** Country B.
- **C** Country C.

51 Based on the given *Z*-spreads for Bonds 1, 2, and 3, which bond has the greatest credit and liquidity risk?

- **A** Bond 1
- **B** Bond 2
- **C** Bond 3

52 Based on Exhibit 2, the implied credit and liquidity risks as indicated by the historical three-year swap spreads for Country B were the lowest:

- **A** 1 month ago.

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B 6 months ago.

C 12 months ago.

53 Based on Exhibit 1 and Tyo's expectations, which country's term structure is currently best for traders seeking to ride the yield curve?

A Country A

B Country B

C Country C

54 Based on Exhibit 1 and assuming Tyo's market views on yield curve changes are realized, the forward curve of which country will lie below its spot curve?

A Country A

B Country B

C Country C

55 Based on Exhibit 1 and Tyo's expectations for the yield curves, Tyo *most likely* perceives the bonds of which country to be fairly valued?

A Country A

B Country B

C Country C

56 With respect to their discussion of yield curves, Tyo and her assistant are *most likely* discussing which term structure theory?

A Pure expectations theory

B Local expectations theory

C Liquidity preference theory

57 Tyo's assistant should calculate a forward rate *closest* to:

A 9.07%.

B 9.58%.

C 9.97%.

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SOLUTIONS

1 Three forward rates can be calculated from the one-, two- and three-year spot rates. The rate on a one-year loan that begins at the end of Year 1 can be calculated using the one- and two-year spot rates; in the following equation one would solve for $f(1,1)$:

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$

The rate on a one-year loan that starts at the end of Year 2 can be calculated from the two- and three-year spot rates; in the following equation one would solve for $f(2,1)$:

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$

Additionally, the rate on a two-year loan that begins at the end of Year 1 can be computed from the one- and three-year spot rates; in the following equation one would solve for $f(1,2)$:

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

2 For the two-year forward rate one year from now of 2%, the two interpretations are as follows:

- 2% is the rate that will make an investor indifferent between buying a three-year zero-coupon bond or investing in a one-year zero-coupon bond and when it matures reinvesting in a zero-coupon bond that matures in two years.
- 2% is the rate that can be locked in today by buying a three-year zero-coupon bond rather than investing in a one-year zero-coupon bond and when it matures reinvesting in a zero-coupon bond that matures in two years.

3 A flat yield curve implies that all spot interest rates are the same. When the spot rate is the same for every maturity, successive applications of the forward rate model will show all the forward rates will also be the same and equal to the spot rate.

4 **A** The yield to maturity of a coupon bond is the expected rate of return on a bond if the bond is held to maturity, there is no default, and the bond and all coupons are reinvested at the original yield to maturity.

B Yes, it is possible. For example, if reinvestment rates for the future coupons are lower than the initial yield to maturity, a bond holder may experience lower realized returns.

5 If forward rates are higher than expected future spot rates the market price of the bond will be lower than the intrinsic value. This is because, everything else held constant, the market is currently discounting the bonds cash flows at a higher rate than the investor's expected future spot rates. The investor can capitalize on this by purchasing the undervalued bond. If expected future spot rates are realized, then bond prices should rise, thus generating gains for the investor.

6 The strategy of riding the yield curve is one in which a bond trader attempts to generate a total return over a given investment horizon that exceeds the return to bond with maturity matched to the horizon. The strategy involves buying a bond with maturity more distant than the investment horizon. Assuming an upward sloping yield curve, if the yield curve does not change level or shape, as

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the bond approaches maturity (or rolls down the yield curve) it will be priced at successively lower yields. So as long as the bond is held for a period less than maturity, it should generate higher returns because of price gains.

7 Some countries do not have active government bond markets with trading at all maturities. For those countries without a liquid government bond market but with an active swap market, there are typically more points available to construct a swap curve than a government bond yield curve. For those markets, the swap curve may be a superior benchmark.

8 The Z-spread is the constant basis point spread added to the default-free spot curve to correctly price a risky bond. A Z-spread of 100bps for a particular bond would imply that adding a fixed spread of 100bps to the points along the spot yield curve will correctly price the bond. A higher Z-spread would imply a riskier bond.

9 The TED spread is the difference between a Libor rate and the US T-Bill rate of matching maturity. It is an indicator of perceived credit risk in the general economy. I particular, because sovereign debt instruments are typically the benchmark for the lowest default risk instruments in a given market, and loans between banks (often at Libor) have some counterparty risk, the TED spread is considered to at least in part reflect default (or counterparty) risk in the banking sector.

10 The local expectations theory asserts that the total return over a one-month horizon for a five-year zero-coupon bond would be the same as for a two-year zero-coupon bond.

11 Both theories attempt to explain the shape of any yield curve in terms of supply and demand for bonds. In segmented market theory, bond market participants are limited to purchase of maturities that match the timing of their liabilities. In the preferred habitat theory, participants have a preferred maturity for asset purchases, but may deviate from it if they feel returns in other maturities offer sufficient compensation for leaving their preferred maturity segment.

12 A Studies have shown that there have been three factors that affect Treasury returns: (1) changes in the level of the yield curve, (2) changes in the slope of the yield curve, and (3) changes in the curvature of the yield curve. Changes in the level refer to upward or downward shifts in the yield curve. For example, an upward shift in the yield curve is likely to result in lower returns across all maturities. Changes in the slope of the yield curve relate to the steepness of the yield curve. Thus, if the yield curve steepens it is likely to result in higher returns for short maturity bonds and lower returns for long maturity bonds. An example of a change in the curvature of the yield curve is a situation where rates fall at the short and long end of the yield curve while rising for intermediate maturities. In this situation returns on short and long maturities are likely to rise to rise while declining for intermediate maturity bonds.

B Empirically, the most important factor is the change in the level of interest rates.

C Key rate durations and a measure based on sensitivities to level, slope, and curvature movements can address shaping risk, but effective duration cannot.

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13 C is correct. There is no spot rate information to provide rates for a loan that terminates in five years. That is $f(2,3)$ is calculated as follows:

$$f(2,3) = \sqrt[3]{\frac{[1 + r(5)]^5}{[1 + r(2)]^2}} - 1$$

The equation above indicates that in order to calculate the rate for a three-year loan beginning at the end of two years you need the five year spot rate $r(5)$ and the two-year spot rate $r(2)$. However $r(5)$ is not provided.

14 A is correct. The forward rate for a one-year loan beginning in one-year $f(1,1)$ is $1.04^2/1.03 - 1 = 5\%$. The rate for a one-year loan beginning in two-years $f(2,1)$ is $1.05^3/1.04^2 - 1 = 7\%$. This confirms that an upward sloping yield curve is consistent with an upward sloping forward curve.

15 C is correct. If one-period forward rates are decreasing with maturity then the forward curve is downward sloping. This turn implies a downward sloping yield curve where longer term spot rates $r(T + T^*)$ are less than shorter term spot rates $r(T)$.

16 C is correct. From the forward rate model, we have

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$

Using the one- and two-year spot rates, we have

$$(1 + .05)^2 = (1 + .04)^1[1 + f(1,1)]^1, \text{ so } \frac{(1 + .05)^2}{(1 + .04)^1} - 1 = f(1,1) = 6.010\%$$

17 C is correct. From the forward rate model,

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

Using the one and three-year spot rates, we find

$$(1 + 0.06)^3 = (1 + 0.04)^1[1 + f(1,2)]^2, \text{ so } \sqrt{\frac{(1 + 0.06)^3}{(1 + 0.04)^1}} - 1 = f(1,2) = 7.014\%$$

18 C is correct. From the forward rate model,

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$

Using the two and three-year spot rates, we find

$$(1 + 0.06)^3 = (1 + 0.05)^2[1 + f(2,1)]^1, \text{ so } \frac{(1 + 0.06)^3}{(1 + 0.05)^2} - 1 = f(2,1) = 8.029\%$$

19 A is correct. We can convert spot rates to spot prices to find $P(3) = \frac{1}{(1.06)^3} =$ 0.8396. The forward pricing model can be used to find the price of the five-year zero as $P(T^* + T) = P(T^*)F(T^*,T)$, so $P(5) = P(3)F(3,2) = 0.8396 \times 0.8479 =$ 0.7119.

20 B is correct. Applying the forward rate model, we find

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,1)]^1[1 + f(2,1)]^1$$

So $[1 + r(3)]^3 = (1 + 0.04)^1(1 + 0.06)^1(1 + 0.08)^1$, $\sqrt[3]{1.1906} - 1 = r(3) = 5.987\%$.

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21 B is correct. We can convert spot rates to spot prices and use the forward pricing model, so have $P(1) = \dfrac{1}{(1.05)^1} = 0.9524$. The forward pricing model is $P(T^* + T) = P(T^*)F(T^*,T)$ so $P(2) = P(1)F(1,1) = 0.9524 \times 0.9346 = 0.8901$.

22 A is correct. The swap rate is the interest rate for the fixed-rate leg of an interest rate swap.

23 A is correct. The swap spread = 1.00% – 0.63% = 0.37% or 37 bps.

24 C is correct. The fixed leg of the five-year fixed-for-floating swap will be equal to the five-year Treasury rate plus the swap spread: 2% + 0.5% = 2.5%.

25 A is correct. The TED spread is the difference between the three-month Libor rate and the three-month Treasury bill rate. If the T-bill rate falls and Libor does not change, the TED spread will increase.

26 A is correct. The Z-spread is the single rate which, when added to the rates of the spot yield curve, will provide the correct discount rates to price a particular risky bond.

27 A is correct. The 200bps Z-spread can be added to the 5% rates from the yield curve to price the bond. The resulting 7% discount rate will be the same for all of the bond's cash-flows, since the yield curve is flat. A 7% coupon bond yielding 7% will be priced at par.

28 B is correct. The higher Z-spread for Bond B implies it is riskier than Bond A. The higher discount rate will make the price of Bond B lower than Bond A.

29 A is correct. The Ho–Lee model is arbitrage-free and can be calibrated to closely match the observed term structure.

30 B is correct. The five-year spot rate is determined by using forward substitution and using the known values of the one-year, two-year, three-year, and four-year spot rates as follows:

$$1 = \frac{0.0437}{(1.025)} + \frac{0.0437}{(1.030)^2} + \frac{0.0437}{(1.035)^3} + \frac{0.0437}{(1.040)^4} + \frac{1 + 0.0437}{\left[1 + r(5)\right]^5}$$

$$r(5) = \sqrt[5]{\frac{1.0437}{0.8394}} - 1 = 4.453\%$$

31 B is correct. The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$. Because nominal yields incorporate a premium for expected inflation, an upward-sloping yield curve is generally interpreted as reflecting a market expectation of increasing, or at least level, future inflation (associated with relatively strong economic growth).

32 C is correct. A one-year loan beginning in three years, or $f(3,1)$, is calculated as follows:

$$\left[1 + r(3 + 1)\right]^{(3+1)} = \left[1 + r(3)\right]^3\left[1 + f(3,1)\right]^1$$

$$\left[1.040\right]^4 = \left[1.035\right]^3\left[1 + f(3,1)\right]^1$$

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$$f(3,1) = \frac{(1.04)^4}{(1.035)^3} - 1 = 5.514\%$$

33 C is correct. Exhibit 1 provides five years of par rates, from which the spot rates for $r(1)$, $r(2)$, $r(3)$, $r(4)$, and $r(5)$ can be derived. Thus the forward rate $f(1,4)$ can be calculated as follows:

$$f(1,4) = \sqrt[4]{\frac{[1 + r(5)]^5}{[1 + r(1)]}} - 1$$

34 C is correct. The yield to maturity, $y(3)$, of Bond Z should be a weighted average of the spot rates used in the valuation of the bond. Because the bond's largest cash flow occurs in Year 3, $r(3)$ will have a greater weight than $r(1)$ and $r(2)$ in determining $y(3)$.

Using the spot rates:

$$\text{Price} = \frac{\$60}{(1.025)^1} + \frac{\$60}{(1.030)^2} + \frac{\$1,060}{(1.035)^3} = \$1,071.16$$

Using the yield to maturity:

$$\text{Price} = \frac{\$60}{[1 + y(3)]^1} + \frac{\$60}{[1 + y(3)]^2} + \frac{\$1,060}{[1 + y(3)]^3} = \$1,071.16$$

Using a calculator, the compute result is $y(3) = 3.46\%$, which is closest to the three-year spot rate of 3.50%.

35 A is correct. Alexander projects that the spot curve two years from today will be below the current forward curve, which implies that her expected future spot rates beyond two years will be lower than the quoted forward rates. Alexander would perceive Bond Z to be undervalued in the sense that the market is effectively discounting the bond's payments at a higher rate than she would and the bond's market price is below her estimate of intrinsic value.

36 B is correct. Nguyen's strategy is to ride the yield curve, which is appropriate when the yield curve is upward sloping. The yield curve implied by Exhibit 1 is upward sloping, which implies that the three-year forward curve is above the current spot curve. When the yield curve slopes upward, as a bond approaches maturity or "rolls down the yield curve," the bond is valued at successively lower yields and higher prices.

37 B is correct. The forward pricing model is based on the no-arbitrage principle and is used to calculate a bond's forward price based on the spot yield curve. The spot curve is constructed by using annualized rates from option-free and default risk–free zero-coupon bonds.

Equation 2: $P(T^* + T) = P(T^*)F(T^*,T)$; we need to solve for $F(1,1)$.

$P(1) = 1/(1 + 0.0225)^1$ and $P(2) = 1/(1 + 0.0270)^2$,

$F(1,1) = P(2)/P(1) = 0.9481/0.9780 = 0.9694$.

38 C is correct. When the spot curve is upward sloping and its level and shape are expected to remain constant over an investment horizon (Shire Gate Advisers' view), buying bonds with a maturity longer than the investment horizon (i.e., riding the yield curve) will provide a total return greater than the return on a maturity-matching strategy.

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39 C is correct. The swap spread is a common way to indicate credit spreads in a market. The four-year swap rate (fixed leg of an interest rate swap) can be used as an indication of the four-year corporate yield. Riding the yield curve by purchasing a four-year zero-coupon bond with a yield of 4.75% {i.e., 4.05% + 0.70%, [$P_4 = 100/(1 + 0.0475)^4 = 83.058$]} and then selling it when it becomes a two-year zero-coupon bond with a yield of 3.00% {i.e., 2.70% + 0.30%, [$P_2 = 100/(1 + 0.0300)^2 = 94.260$]} produces an annual return of 6.53%: $(94.260/83.058)^{0.5} - 1.0 = 0.0653$.

40 B is correct. The Z-spread is the constant basis point spread that is added to the default-free spot curve to price a risky bond. A Z-spread of 65 bps for a particular bond would imply adding a fixed spread of 65 bps to maturities along the spot curve to correctly price the bond. Therefore, for the two-year bond, $r(1)$ = 2.90% (i.e., 2.25% + 0.65%), $r(2)$ = 3.35% (i.e., 2.70% + 0.65%), and the price of the bond with an annual coupon of 4.15% is as follows:

$$P = 4.15/(1 + 0.029)^1 + 4.15/(1 + 0.0335)^2 + 100/(1 + 0.0335)^2,$$

$$P = 101.54.$$

41 C is correct. The Libor–OIS spread is considered an indicator of the risk and liquidity of money market securities. This spread measures the difference between Libor and the OIS rate.

42 C is correct. Liquidity preference theory asserts that investors demand a risk premium, in the form of a liquidity premium, to compensate them for the added interest rate risk they face when buying long-maturity bonds. The theory also states that the liquidity premium increases with maturity.

43 C is correct. Both statements are incorrect because Madison incorrectly describes both types of models. Equilibrium term structure models are factor models that seek to describe the dynamics of the term structure by using fundamental economic variables that are assumed to affect interest rates. Arbitrage-free term structure models use observed market prices of a reference set of financial instruments, assumed to be correctly priced, to model the market yield curve.

44 A is correct. Consistent with Madison's statement, equilibrium term structure models require fewer parameters to be estimated relative to arbitrage-free models, and arbitrage-free models allow for time-varying parameters. Consequently, arbitrage-free models can model the market yield curve more precisely than equilibrium models.

45 B is correct. The TED spread, calculated as the difference between Libor and the yield on a T-bill of matching maturity, is an indicator of perceived credit risk in the general economy. An increase (decrease) in the TED spread signals that lenders believe the risk of default on interbank loans is increasing (decreasing). Therefore, the TED spread can be thought of as a measure of counterparty risk.

46 A is correct. Madison's response is correct; research indicates that short-term rate volatility is mostly linked to uncertainty regarding monetary policy, whereas long-term rate volatility is mostly linked to uncertainty regarding the real economy and inflation.

47 B is correct. Because the factors in Exhibit 1 have been standardized to have unit standard deviations, a two standard deviation increase in the steepness factor will lead to the yield on the 20-year bond decreasing by 0.6030%, calculated as follows:

Change in 20-year bond yield = –0.3015% × 2 = –0.6030%.

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48 C is correct. Because the factors in Exhibit 1 have been standardized to have unit standard deviations, a one standard deviation decrease in both the level factor and the curvature factor will lead to the yield on the five-year bond increasing by 0.0389%, calculated as follows:

Change in five-year bond yield = 0.4352% – 0.3963% = 0.0389%.

49 C is correct. The assistant states that bootstrapping entails *backward* substitution using par yields to solve for zero-coupon rates one by one, in order from latest to earliest maturities. Bootstrapping entails *forward* substitution, however, using par yields to solve for zero-coupon rates one by one, in order from earliest to latest maturities.

50 C is correct. Country C's private sector is much bigger than the public sector, and the government bond market in Country C currently lacks liquidity. Under such circumstances, the swap curve is a more relevant benchmark for interest rates.

51 C is correct. Although swap spreads provide a convenient way to measure risk, a more accurate measure of credit and liquidity risk is called the zero-spread (*Z*-spread). It is the constant spread that, added to the implied spot yield curve, makes the discounted cash flows of a bond equal to its current market price. Bonds 1, 2, and 3 are otherwise similar but have *Z*-spreads of 0.55%, 1.52%, and 1.76%, respectively. Bond 3 has the highest *Z*-spread, implying that this bond has the greatest credit and liquidity risk.

52 B is correct. The historical three-year swap spread for Country B was the lowest six months ago. Swap spread is defined as the spread paid by the fixed-rate payer of an interest rate swap over the rate of the "on the run" (most recently issued) government bond security with the same maturity as the swap. The lower (higher) the swap spread, the lower (higher) the return that investors require for credit and/or liquidity risks.

The fixed rate of the three-year fixed-for-floating Libor swap was 0.01% six months ago, and the three-year government bond yield was –0.08% six months ago. Thus the swap spread six months ago was 0.01% – (–0.08%) = 0.09%.

One month ago, the fixed rate of the three-year fixed-for-floating Libor swap was 0.16%, and the three-year government bond yield was –0.10%. Thus the swap spread one month ago was 0.16% – (–0.10%) = 0.26%.

Twelve months ago, the fixed rate of the three-year fixed-for-floating Libor swap was 0.71%, and the three-year government bond yield was –0.07%. Thus, the swap spread 12 months ago was 0.71% – (–0.07%) = 0.78%.

53 A is correct. Country A's yield curve is upward sloping—a condition for the strategy—and more so than Country B's.

54 B is correct. The yield curve for Country B is currently upward sloping, but Tyo expects a reversal in the slope of the current yield curve. This means she expects the resulting yield curve for Country B to slope downward, which implies that the resulting forward curve would lie below the spot yield curve. The forward curve lies below the spot curve in scenarios in which the spot curve is downward sloping; the forward curve lies above the spot curve in scenarios in which the spot curve is upward sloping.

A is incorrect because the yield curve for Country A is currently upward sloping and Tyo expects that the yield curve will maintain its shape and level. That expectation implies that the resulting forward curve would be above the spot yield curve.

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C is incorrect because the yield curve for Country C is currently downward sloping and Tyo expects a reversal in the slope of the current yield curve. This means she expects the resulting yield curve for Country C to slope upward, which implies that the resulting forward curve would be above the spot yield curve.

55 A is correct. Tyo's projected spot curve assumes that future spot rates reflect, or will be equal to, the current forward rates for all respective maturities. This assumption implies that the bonds for Country A are fairly valued because the market is effectively discounting the bond's payments at spot rates that match those projected by Tyo.

B and C are incorrect because Tyo's projected spot curves for the two countries do not match the current forward rates for all respective maturities. In the case of Country B, she expects future spot rates to be higher (than the current forward rates that the market is using to discount the bond's payments). For Country C, she expects future spot rates to be lower (than the current forward rates). Hence, she perceives the Country B bond to be currently overvalued and the Country C bond to be undervalued.

56 C is correct. Liquidity preference theory suggests that liquidity premiums exist to compensate investors for the added interest rate risk that they face when lending long term and that these premiums increase with maturity. Tyo and her assistant are assuming that liquidity premiums exist.

57 A is correct. From the forward rate model, $f(3,2)$, is found as follows:

$$[1 + r(5)]^5 = [1 + r(3)]^3[1 + f(3,2)]^2$$

Using the three-year and five-year spot rates, we find

$$(1 + 0.107)^5 = (1 + 0.118)^3[1 + f(3,2)]^2, \text{ so}$$

$$\sqrt{\frac{(1 + 0.107)^5}{(1 + 0.118)^3}} - 1 = f(3,2) = 9.07\%$$

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READING

33

The Arbitrage-Free Valuation Framework

by Steven V. Mann, PhD

Steven V. Mann, PhD, is at the University of South Carolina (USA).

LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	**a.** explain what is meant by arbitrage-free valuation of a fixed-income instrument;
☐	**b.** calculate the arbitrage-free value of an option-free, fixed-rate coupon bond;
☐	**c.** describe a binomial interest rate tree framework;
☐	**d.** describe the backward induction valuation methodology and calculate the value of a fixed-income instrument given its cash flow at each node;
☐	**e.** describe the process of calibrating a binomial interest rate tree to match a specific term structure;
☐	**f.** compare pricing using the zero-coupon yield curve with pricing using an arbitrage-free binomial lattice;
☐	**g.** describe pathwise valuation in a binomial interest rate framework and calculate the value of a fixed-income instrument given its cash flows along each path;
☐	**h.** describe a Monte Carlo forward-rate simulation and its application.

INTRODUCTION

1

The idea that market prices will adjust until there are no opportunities for arbitrage underpins the valuation of fixed-income securities, derivatives, and other financial assets. It is as intuitive as it is well-known. For a given investment, if the net proceeds are zero (e.g., buying and selling the same dollar amount of stocks) and the risk is

The presentation of the binomial trees in this reading was revised to conform with other readings in 2018 by Donald J. Smith, PhD, Boston University (USA).

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zero, the return should be zero. Valuation tools must produce a value that is arbitrage free. The purpose of this reading is to develop a set of valuation tools for bonds that are consistent with this notion.

The reading is organized around the learning objectives. After this brief introduction, Section 2 defines an arbitrage opportunity and discusses the implications of no arbitrage for the valuation of fixed-income securities. Section 3 presents some essential ideas and tools from yield curve analysis needed to introduce the binomial interest rate tree. In this section, the binomial interest rate tree framework is developed and used to value an option-free bond. The process used to calibrate the interest rate tree to match the current yield curve is introduced. This step ensures that the interest rate tree is consistent with pricing using the zero-coupon (i.e., spot) curve. The final topic presented in the section is an introduction of pathwise valuation. Section 4 describes a Monte Carlo forward-rate simulation and its application. A summary of the major results is given in Section 5.

2 THE MEANING OF ARBITRAGE-FREE VALUATION

Arbitrage-free valuation refers to an approach to security valuation that determines security values that are consistent with the absence of an **arbitrage opportunity**, which is an opportunity for trades that earn riskless profits without any net investment of money. In well-functioning markets, prices adjust until there are no arbitrage opportunities, which is the **principle of no arbitrage** that underlies the practical validity of arbitrage-free valuation. This principle itself can be thought of as an implication of the idea that identical assets should sell at the same price.

These concepts will be explained in greater detail shortly, but to indicate how they arise in bond valuation, consider first an imaginary world in which financial assets are free of risk and the benchmark yield curve is flat. A flat yield curve implies that the relevant yield is the same for all cash flows regardless of when the cash flows are delivered in time.[1] Accordingly, the value of a bond is the present value of its certain future cash flows. In discounting those cash flows—determining their present value—investors would use the risk-free interest rate because the cash flows are certain; because the yield curve is assumed to be flat, one risk-free rate would exist and apply to all future cash flows. This is the simplest case of bond valuation one can envision. When we exit this imaginary world and enter more realistic environs, bonds' cash flows are risky (i.e., there is some chance the borrower will default) and the benchmark yield curve is not flat. How would our approach change?

A fundamental principle of valuation is that the value of any financial asset is equal to the present value of its expected future cash flows. This principle holds for any financial asset from zero-coupon bonds to interest rate swaps. Thus, the valuation of a financial asset involves the following three steps:

Step 1 Estimate the future cash flows.

Step 2 Determine the appropriate discount rate or discount rates that should be used to discount the cash flows.

Step 3 Calculate the present value of the expected future cash flows found in Step 1 by applying the appropriate discount rate or rates determined in Step 2.

1 The terms yield, interest rate, and discount rate will be used interchangeably.

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The traditional approach to valuing bonds is to discount all cash flows with the same discount rate as if the yield curve were flat. However, a bond is properly thought of as a package or portfolio of zero-coupon bonds. Each zero-coupon bond in such a package can be valued separately at a discount rate that depends on the shape of the yield curve and when its single cash flow is delivered in time. The term structure of these discount rates is referred to as the spot curve. Bond values derived by summing the present values of the individual zeros (cash flows) determined by such a procedure can be shown to be arbitrage free.[2] Ignoring transaction costs for the moment, if the bond's value was much less than the sum of the values of its cash flows individually, a trader would perceive an arbitrage opportunity and buy the bond while selling claims to the individual cash flows and pocketing the excess value. Although the details bear further discussion (see Section 2.3), the valuation of a bond as a portfolio of zeros based on using the spot curve is an example of arbitrage-free valuation. Regardless of the complexity of the bond, each component must have an arbitrage-free value. A bond with embedded options can be valued in parts as the sum of the arbitrage-free bond without options (that is, a bond with no embedded options) and the arbitrage-free value of each of the options.

2.1 The Law of One Price

The central idea of financial economics is that market prices will adjust until there are no opportunities for arbitrage. We will define shortly what is meant by an arbitrage opportunity, but for now think of it as "free money." Prices will adjust until there is no free money to be acquired. Arbitrage opportunities arise as a result of violations of the **law of one price**. The law of one price states that two goods that are perfect substitutes must sell for the same current price in the absence of transaction costs. Two goods that are identical, trading side by side, are priced the same. Otherwise, if it were costless to trade, one would simultaneously buy at the lower price and sell at the higher price. The riskless profit is the difference in the prices. An individual would repeat this transaction without limit until the two prices converge. An implication of these market forces is deceptively straightforward and basic. If you do not put up any of your own money and take no risk, your expected return should be zero.

2.2 Arbitrage Opportunity

With this background, let us define arbitrage opportunity more precisely. An arbitrage opportunity is a transaction that involves no cash outlay that results in a riskless profit. There are two types of arbitrage opportunities. The first type of arbitrage opportunity is often called **value additivity** or, put simply, the value of the whole equals the sum of the values of the parts. Consider two risk-free investments with payoffs one year from today and the prices today provided in Exhibit 1. Asset A is a simple risk-free zero-coupon bond that pays off one dollar and is priced today at 0.952381 (1/1.05). Asset B is a portfolio of 105 units of Asset A that pays off 105 one year from today and is priced today at 95. The portfolio does not equal the sum of the parts. The portfolio (Asset B) is cheaper than buying 105 units of Asset A at a price of 100 and then combining. An astute investor would sell 105 units of Asset A for 105 × 0.952381 = 100 while simultaneously buying one portfolio Asset B for 95. This position generates a certain 5 today (100-95) and generates net 0 one year from today because cash inflow for Asset B matches the amount for the 105 units of Asset A sold. An investor would engage in this trade over and over again until the prices adjust.

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2 A zero is a zero-coupon bond or discount instrument.

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The second type of arbitrage opportunity is often called **dominance**. A financial asset with a risk-free payoff in the future must have a positive price today. Consider two assets, C and D, that are risk-free zero-coupon bonds. Payoffs in one year and prices today are displayed in Exhibit 1. On careful review, it appears that Asset D is cheap relative to Asset C. If both assets are risk-free, they should have the same discount rate. To make money, sell two units of Asset C at a price of 200 and use the proceeds to purchase one unit of Asset D for 200. The construction of the portfolio involves no net cash outlay today. Although it requires zero dollars to construct today, the portfolio generates 10 one year from today. Asset D will generate a 220 cash inflow whereas the two units of Asset C sold will produce a cash outflow of 210.

Exhibit 1 Price Today and Payoffs in One Year for Sample Assets

Asset	Price Today	Payoff in One Year
A	0.952381	1
B	95	105
C	100	105
D	200	220

This existence of both types of arbitrage opportunities is transitory. Investors aware of this mispricing will demand the securities in question in unlimited quantities. Something must change in order to restore stability. Prices will adjust until there are no arbitrage opportunities.

EXAMPLE 1

Arbitrage Opportunities

Which of the following investment alternatives includes an arbitrage opportunity?

Bond A: The yield for a 3% coupon 10-year annual-pay bond is 2.5% in New York City. The same bond sells for \$104.376 per \$100 face value in Chicago.

Bond B: The yield for a 3% coupon 10-year annual-pay bond is 3.2% in Hong Kong SAR. The same bond sells for RMB97.220 per RMB100 face value in Shanghai.

Solution:

Bond B is correct. Bond B's arbitrage-free price is $3/1.032 + 3/1.032^2 + \ldots + 103/1.032^{10} = 98.311$, which is higher than the price in Shanghai. Therefore, an arbitrage opportunity exists. Buy bonds in Shanghai for RMB97.220 and sell them in Hong Kong SAR for RMB98.311. You make RMB1.091 per RMB100 of bonds traded.

Bond A's arbitrage-free price is $3/1.025 + 3/1.025^2 + \ldots + 103/1.025^{10} = 104.376$, which matches the price in Chicago. Therefore, no arbitrage opportunity exists in this market.

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2.3 Implications of Arbitrage-Free Valuation for Fixed-Income Securities

Using the arbitrage-free approach, any fixed-income security should be thought of as a package or portfolio of zero-coupon bonds. Thus, a five-year 2% coupon Treasury issue should be viewed as a package of eleven zero-coupon instruments (10 semiannual coupon payments, one of which is made at maturity, and one principal value payment at maturity) The market mechanism for US Treasuries that enables this approach is the dealer's ability to separate the bond's individual cash flows and trade them as zero-coupon securities. This process is called **stripping**. In addition, dealers can recombine the appropriate individual zero-coupon securities and reproduce the underlying coupon Treasury. This process is called **reconstitution**. Dealers in sovereign debt markets around the globe are free to engage in the same process.

Arbitrage profits are possible when value additivity does not hold. The arbitrage-free valuation approach does not allow a market participant to realize an arbitrage profit through stripping and reconstitution. By viewing any security as a package of zero-coupon securities, a consistent and coherent valuation framework can be developed. Viewing a security as a package of zero-coupon bonds means that two bonds with the same maturity and different coupon rates are viewed as different packages of zero-coupon bonds and valued accordingly. Moreover, two cash flows that have identical risks delivered at the same time will be valued using the same discount rate even though they are attached to two different bonds.

3 INTEREST RATE TREES AND ARBITRAGE-FREE VALUATION

The goal of this section is to develop a method to produce an arbitrage-free value for an option-free bond and to provide a framework—based on interest rate trees−that is rich enough to be applied to the valuation of bonds with embedded options.

For bonds that are option-free, the simplest approach to arbitrage-free valuation involves determining the arbitrage-free value as the sum of the present values of expected future values using the benchmark spot rates. Benchmark securities are liquid, safe securities whose yields serve as building blocks for other interest rates in a particular country or currency. Sovereign debt is the benchmark in many countries. For example, on-the-run Treasuries serve as benchmark securities in the United States. Par rates derived from the Treasury yield curve can be used to obtain spot rates by means of bootstrapping. Gilts serve as a benchmark in the United Kingdom. In markets where the sovereign debt market is not sufficiently liquid, the swaps curve is a viable alternative.

In this reading, benchmark bonds are assumed to be correctly priced by the market. The valuation model we develop will be constructed so as to reproduce exactly the prices of the benchmark bonds.

EXAMPLE 2

The Arbitrage-Free Value of an Option-Free Bond

The yield to maturity ("par rate") for a benchmark one-year annual-pay bond is 2%, for a benchmark two-year annual-pay bond is 3%, and for a benchmark three-year annual-pay bond is 4%. A three year, 5% coupon, annual-pay bond

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with the same risk and liquidity as the benchmarks is selling for 102.7751 today (time zero) to yield 4%. Is this value correct for the bond given the current term structure?

Solution:

The first step in the solution is to find the correct spot rate (zero-coupon rates) for each year's cash flow.[3] The spot rates may be determined using bootstrapping, which is an iterative process. Using the bond valuation equation below, one can solve iteratively for the spot rates, z_t (rate on a zero-coupon bond of maturity t), given the periodic payment, PMT, on the relevant benchmark bond.

$$100 = \frac{PMT}{(1 + z_1)^1} + \frac{PMT}{(1 + z_2)^2} + \cdots + \frac{PMT + 100}{(1 + z_N)^N}$$

A revised equation, which uses the par rate rather than PMT, may also be used to calculate the spot rates. The revised equation is:

$$1 = \frac{\text{Par rate}}{[1 + r(1)]^1} + \frac{\text{Par rate}}{[1 + r(2)]^2} + \cdots + \frac{\text{Par rate} + 1}{[1 + r(N)]^N}$$

where par rate is PMT divided by 100 and represents the par rate on the benchmark bond and $r(t)$ is the t-period zero-coupon rate.

In this example, the one-year spot rate, $r(1)$, is 2%, which is the same as the one-year par rate. To solve for $r(2)$:

$$1 = \frac{0.03}{[1 + r(1)]^1} + \frac{0.03 + 1}{[1 + r(2)]^2} = \frac{0.03}{(1 + 0.02)^1} + \frac{0.03 + 1}{[1 + r(2)]^2}$$

$$r(2) = 3.015\%$$

To solve for $r(3)$:

$$1 = \frac{0.04}{(1 + 0.02)^1} + \frac{0.04}{(1 + 0.03015)^2} + \frac{0.04 + 1}{[1 + r(3)]^3}$$

$$r(3) = 4.055\%$$

The spot rates are 2%, 3.015%, and 4.055%. The correct arbitrage-free price for the bond, then, is

$$P_0 = 5/1.02 + 5/1.03015^2 + 105/1.04055^3 = 102.8102$$

To be arbitrage-free, each cash flow of a bond must be discounted by the spot rate for zero-coupon bonds maturing on the same date as the cash flow. Discounting early coupons by the bond's yield to maturity gives too much discounting with an upward sloping yield curve and too little discounting for a downward sloping yield curve. The bond is mispriced by 0.0351 per 100 of par value.

For option-free bonds, performing valuation discounting with spot rates produces an arbitrage-free valuation. For bonds that have embedded options, we need a different approach. The challenge one faces when developing a framework for valuing bonds with embedded options is that their expected future cash flows are interest rate dependent. If the bonds are option-free, changes in interest rates have no impact on the size and timing of the bond's cash flows. For bonds with options attached, changes in future interest rates impact the likelihood the option will be exercised and in so doing impact the cash flows. Therefore, in order to develop a framework that values

3 Par, spot, and forward interest rates were discussed in Level I.

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both bonds without and with embedded options, we must allow interest rates to take on different potential values in the future based on some assumed level of volatility. The vehicle to portray this information is an interest rate "tree" representing possible future interest rates consistent with the assumed volatility. Because the interest rate tree resembles a lattice, these models are often called "lattice models." The interest rate tree performs two functions in the valuation process: (1) generate the cash flows that are interest rate dependent and (2) supply the interest rates used to determine the present value of the cash flows. This approach will be used in later readings when considering learning outcome statements involving callable bonds.

An interest rate model seeks to identify the elements or *factors* that are believed to explain the dynamics of interest rates. These factors are random or *stochastic* in nature, so we cannot predict the path of any particular factor. An interest rate model must, therefore, specify a statistical process that describes the stochastic property of these factors in order to arrive at a reasonably accurate representation of the behavior of interest rates. What is important to understand is that the interest rate models commonly used are based on how short-term interest rates can evolve (i.e., change) over time. Consequently, these interest rate models are referred to as one-factor models because only one interest rate is being modeled over time. More complex models consider how more than one interest rate changes over time (e.g., the short rate and the long rate) and are referred to as two-factor models.

Our task at hand is to describe the binomial interest rate tree framework. The valuation model we are attempting to build is the binomial lattice model. It is so named because the short interest rate can take on one of two possible values consistent with the volatility assumption and an interest rate model. As we will soon discover, the two possible interest rates next period will be consistent with the following three conditions: (1) an interest rate model that governs the random process of interest rates, (2) the assumed level of interest rate volatility, and (3) the current benchmark yield curve. We take the prices of the benchmark bonds as given such that when these bonds are valued in our model we recover the market values for each benchmark bond. In this way, we tie the model to the current yield curve that reflects the underlying economic reality.

3.1 The Binomial Interest Rate Tree

The first step for demonstrating the binomial valuation method is to present the benchmark par curve by using bonds of a particular country or currency. For simplicity in our illustration, we will use US dollars. The same principles hold with equal force regardless of the country or currency. The benchmark par curve is presented in Exhibit 2. For simplicity, we assume that all bonds have annual coupon payments. Benchmark bonds are conveniently priced at par so the yields to maturity and the coupon rates on the bonds are the same. From these par rates, we use the bootstrapping methodology to uncover the underlying spot rates shown in Exhibit 3. Because the par curve is upward sloping, it comes as no surprise that after Year 1 the spot rates are higher than the par rates. In Exhibit 4 we present the one-year implied forward rates derived from the spot curve using no arbitrage. Because the par, spot, and forward curves reflect the same information about interest rates, if one of the three curves is known, it is possible to generate the other two curves. The three curves are only identical if the yield curve is flat.

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Exhibit 2 Benchmark Par Curve

Maturity (Years)	Par Rate	Bond Price
1	1.00%	100
2	1.20%	100
3	1.25%	100
4	1.40%	100
5	1.80%	100

Exhibit 3 Underlying One-Year Spot Rates of Par Rates

Maturity (Years)	One-Year Spot Rate
1	1.0000%
2	1.2012%
3	1.2515%
4	1.4045%
5	1.8194%

Exhibit 4 One-Year Implied Forward Rates

Maturity (Years)	Forward Rate
Current one-year rate	1.0000%
One-year rate, one year forward	1.4028%
One-year rate, two years forward	1.3521%
One-year rate, three years forward	1.8647%
One-year rate, four years forward	3.4965%

Recall from our earlier discussion that if we value the benchmark bonds using rates derived from these curves, we will recover the market price of par for all five bonds in Exhibit 2. Specifically, par rates represent the single interest applied to all the cash flows that will produce the market prices. Discounting each cash flow separately with the set of spot rates will also give the same answer. Finally, forward rates are the discount rates of a single cash flow over a single period. If we discount each cash flow with the appropriate discount rate for each period, the computed values will match the observed prices.

When we approach the valuation of bonds with cash flows that are interest rate dependent, we must explicitly allow interest rates to change. We accomplish this task by introducing interest rate volatility and generating an interest rate tree (see Section 3.2 for a discussion of interest rate volatility). An interest rate tree is simply a visual representation of the possible values of interest rates based on an interest rate model and an assumption about interest rate volatility.

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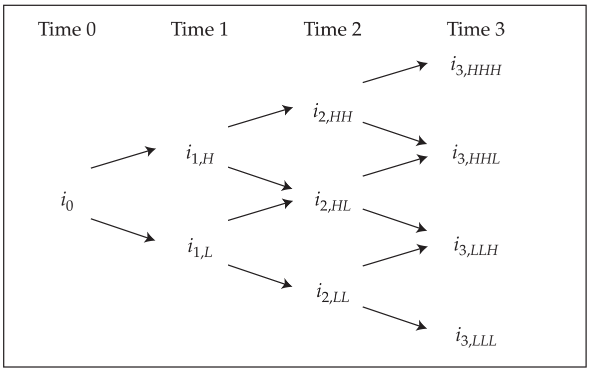
A binomial interest rate tree is presented in Exhibit 5. Our goal is to learn how to populate this structure with interest rates. Notice the *i*'s, which represent different potential values one-year interest rates may take over time. As we move from left to right on the tree, the number of possible interest rates increases. The first is the current time (in years), or formally Time 0. The interest rate displayed at Time 0 is the discount rate that converts Time 1 payments to Time 0 present values. At the bottom of the graph, time is the unit of measurement. Notice that there is one year between possible interest rates. This is called the "time step" and, in our illustration, it matches the frequency of the annual cash flows. The *i*'s in Exhibit 5 are called nodes. The first node is called the root of the tree and is simply the current one-year rate at Time 0.

Exhibit 5 Binomial Interest Rate Tree



We now turn to the question of how to obtain the two possible values for the one-year interest rate one year from today. Two assumptions are required: an interest rate model and a volatility of interest rates. Recall an interest rate model puts structure on the randomness. We are going to use the lognormal random walk, and the resulting tree structure is often referred to as a lognormal tree. A lognormal model of interest rates insures two appealing properties: (1) non-negativity of interest rates and (2) higher volatility at higher interest rates. At each node, there are two possible rates one year forward at Time 1. We will assume for the time being that each has an equal probability of occurring. The two possible rates we will calculate are going to be higher and lower than the one-year forward rate at Time 1 one year from now.

We denote i_L to be the rate lower than the implied forward rate and i_H to be the higher forward rate. The lognormal random walk posits the following relationship between $i_{1,L}$ and $i_{1,H}$:

$$i_{1,H} = i_{1,L}e^{2\sigma}$$

where σ is the standard deviation and *e* is Euler's number, the base of natural logarithms, which is a constant 2.7183.[4] The random possibilities each period are (nearly) centered on the forward rates calculated from the benchmark curve. The intuition of this relationship is deceptively quick and simple. Think of the one-year forward implied interest rate from the yield curve as the average of possible values for the one-year rate at Time 1. The lower of the two rates, i_L, is one standard deviation below the mean (one-year implied forward rate) and i_H is one standard deviation above the mean. Thus, the higher and lower values (i_L and i_H) are multiples of each other and the multiplier is $e^{2\sigma}$. Note that as the standard deviation (i.e., volatility)

4 The number *e* is transcendental and continues infinitely without repeating.

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increases, the multiplier increases and the two rates will grow farther apart but will still be (nearly) centered on the implied forward rate derived from the spot curve. We will demonstrate this soon.

We use the following notation to describe the tree at Time 1. Let

σ = assumed volatility of the one-year rate,
$i_{1,L}$ = the lower one-year forward rate one year from now at Time 1, and
$i_{1,H}$ = the higher one-year forward rate one year from now at Time 1.

For example, suppose that $i_{1,L}$ is 1.194% and σ is 15% per year, then $i_{1,H}$ = 1.194%($e^{2\times0.15}$) = 1.612%.

At Time 2, there are three possible values for the one-year rate, which we will denote as follows:

$i_{2,LL}$ = one-year forward rate at Time 2 assuming the lower rate at Time 1 and the lower rate at Time 2
$i_{2,HH}$ = one-year forward rate at Time 2 assuming the higher rate at Time 1 and the higher rate at Time 2
$i_{2,HL}$ = one-year forward rate at Time 2 assuming the higher rate at Time 1 and the lower rate at Time 2, or equivalently, the lower rate at Time 1 and the higher rate at Time 2

The middle rate will be close to the implied one-year forward rate two years from now derived from the spot curve, whereas the other two rates are two standard deviations above and below this value. (Recall that the multiplier for adjacent rates on the tree differs by a multiple of e raised to the 2σ.) This type of tree is called a recombining tree because there are two paths to get to the middle rate. This feature of the model results in faster computation because the number of possible outcomes each period grows linearly rather than exponentially.

The relationship between $i_{2,LL}$ and the other two one-year rates is as follows:

$i_{2,HH} = i_{2,LL}(e^{4\sigma})$ and $i_{2,HL} = i_{2,LL}(e^{2\sigma})$

In a given period, adjacent possible outcomes in the tree are two standard deviations apart. So, for example, if $i_{2,LL}$ is 0.980%, and assuming once again that σ is 15%, we calculate

$i_{2,HH}$ = 0.980%($e^{4\times0.15}$) = 1.786%

and

$i_{2,HL}$ = 0.980%($e^{2\times0.15}$) = 1.323%.

There are four possible values for the one-year forward rate at Time 3. These are represented as follows: $i_{3,HHH}$, $i_{3,HHL}$, $i_{3,LLH}$ and $i_{3,LLL}$. Once again all the forward rates in the tree are multiples of the lowest possible rates each year. The lowest possible forward rate at Time 3 is $i_{3,LLL}$ and is related to the other three as given below:

$i_{3,HHH} = (e^{6\sigma})i_{3,LLL}$
$i_{3,HHL} = (e^{4\sigma})i_{3,LLL}$
$i_{3,LLH} = (e^{2\sigma})i_{3,LLL}$

Exhibit 6 shows the notation for a four-year binomial interest rate tree. We can simplify the notation by centering the one-year rates on the tree on implied forward rates on the benchmark yield curve and letting i_t be the one-year rate t years from now be the centering rates. The subscripts indicate the rates at the end of the year, so in the second year, it is the rate at the end of Time 2 to the end of Time 3. Exhibit 6 uses this uniform notation. Note that adjacent forward rates in the tree are two standard deviations (σs) apart.

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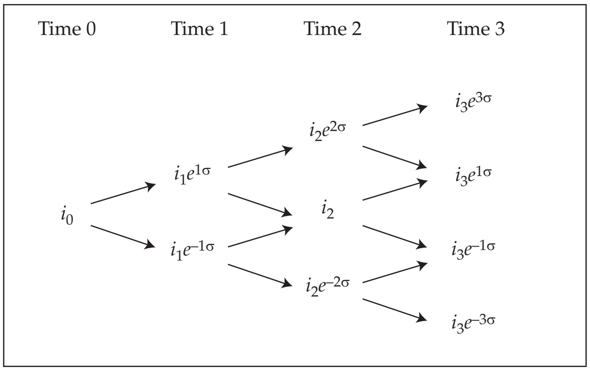
Exhibit 6 Four-Year Binomial Tree



Before we attempt to build an interest rate tree, two additional tools are needed. These tools are introduced in the next two sections.

3.2 What Is Volatility and How Is It Estimated?

Recall that variance is a measure of dispersion of a probability distribution. The standard deviation is the square root of the variance and it is a statistical measure of volatility in the same units as the mean. With a simple lognormal distribution, the changes in interest rates are proportional to the level of the one-period interest rates each period. Volatility is measured relative to the current level of rates. It can be shown that for a lognormal distribution the standard deviation of the one-year rate is equal to $i_0\sigma$.[5] For example, if σ is 10% and the one-year rate (i_0) is 2%, then the standard deviation of the one-year rate is 2% × 10% = 0.2% or 20 bps. As a result, interest rate moves are larger when interest rates are high and are smaller when interest rates are low. One of the benefits of a lognormal distribution is that if interest rates get too close to zero, the absolute change in interest rates becomes smaller and smaller. Negative interest rates are not possible.

There are two methods commonly used to estimate interest rate volatility. The first method is by estimating historical interest rate volatility; volatility is calculated by using data from the recent past with the assumption that what has happened recently is indicative of the future. A second method to estimate interest rate volatility is based on observed market prices of interest rate derivatives (e.g., swaptions, caps, floors). This approach is called implied volatility.

3.3 Determining the Value of a Bond at a Node

To find the value of the bond at a particular node, we use the backward induction valuation methodology. Barring default, we know that at maturity the bonds will be valued at par. So, we start at maturity, fill in those values, and work back from right to left to find the bond's value at the desired node. Suppose we want to determine the bond's value at the lowest node at Time 1. To find this value, we must first calculate the bond's value at the two nodes to the right of the node we selected. The bond's value at the two nodes immediately to the right must be available.

5 Given that $e^{2\sigma} \approx 1 + 2\sigma$, the standard deviation of the one-year rate is $\frac{re^{2\sigma} - r}{2} \approx \frac{r + 2\sigma r - r}{2} = \sigma r$.

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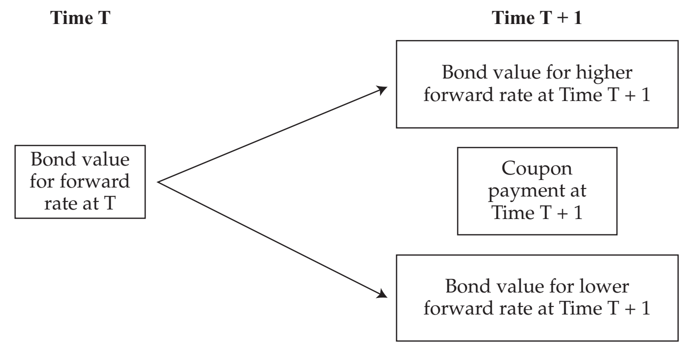
A bond's value at any node will depend on the future coupon payment, C, and the expected future value for the bond. This expected value is the average of the value for the forward rate being higher, to be denoted below by *VH*, and the value for the forward rate being lower, *VL*. It is a simple average because in the lognormal model the probabilities are equal for the rate going up or down. This is illustrated in Exhibit 7. Notice that the coupon payment due at the end of the period, at Time T + 1, is placed directly to the right of the node for Time T. The arrows point to the two possible future bond values, one for the forward rate going up at Time T + 1 and the other for the rate going down.

Exhibit 7 Finding a Bond's Value at Any Node



The next step is to determine the present value of the coupon payment and the expected future bond value. The relevant discount rate is the one-year forward rate prevailing at the beginning of the time period, *i*, at Time T. The bond's value at any node is determined by the following expression:

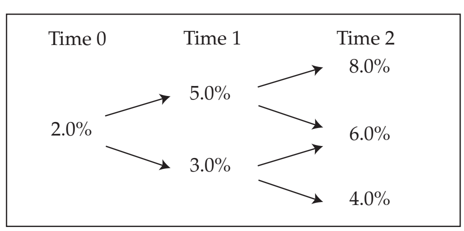
$$\text{Bond value at a node} = \frac{C + (0.5 \times VH + 0.5 \times VL)}{1 + i}$$

EXAMPLE 3

Pricing a Bond Using a Binomial Tree

Using the interest rate tree in Exhibit 8, find the correct price for a three-year, annual-pay bond with a coupon rate of 5%.

Exhibit 8 Three-Year Binomial Interest Rate Tree



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Solution:

Exhibit 9 shows the binomial tree to value the three-year, 5% bond. We start with Time 3. The cash flow is 105, the redemption of par value (100) plus the final coupon payment (5), regardless of the level of the forward rate at Time 2. Using backward induction, we next calculate the present value of the bond as of Time 2 for the three possible forward rates:

105/1.08 = 97.2222

105/1.06= 99.0566

105/1.04 = 100.9615

Working back to Time 1 requires the use of the general expression above for the value at any node. If the forward rate is 5.0% at Time 1, the bond value is 98.2280:

$$\frac{5 + (0.5 \times 97.2222 + 0.5 \times 99.0566)}{1.05} = 98.2280$$

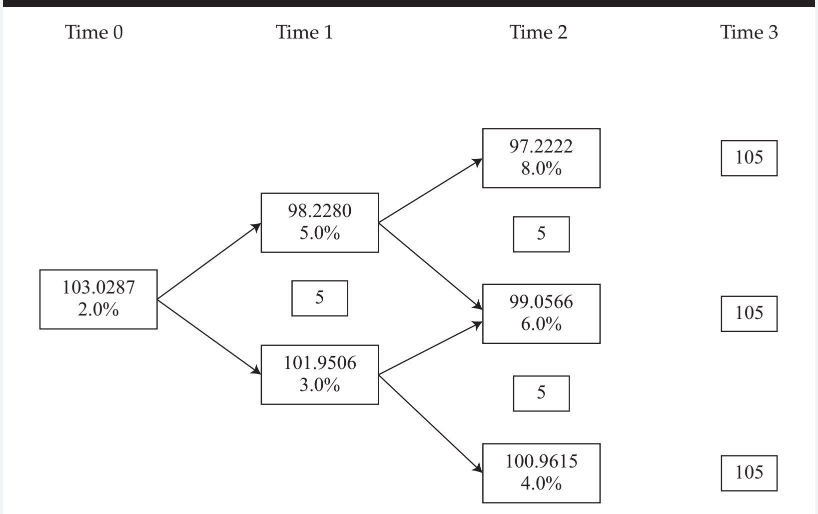
If the forward rate instead is 3.0%, the bond value is 101.9506.

$$\frac{5 + (0.5 \times 99.0566 + 0.5 \times 100.9615)}{1.03} = 101.9506$$

Finally, the value of bond at Time 0 is 103.0287:

$$\frac{5 + (0.5 \times 98.2280 + 0.5 \times 101.9506)}{1.02} = 103.0287$$

Exhibit 9 Three-Year Binomial Tree



3.4 Constructing the Binomial Interest Rate Tree

The construction of a binomial interest rate tree requires multiple steps, but keep in mind what we are trying to accomplish. We are making an assumption about the process that generates interest rates and volatility. The first step is to describe the process of calibrating a binomial interest rate tree to match a specific term structure. We do this to ensure that the model is arbitrage free. We fit the interest rate tree to

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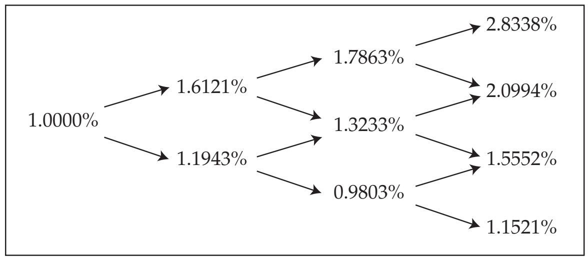
the current yield curve by choosing interest rates so that the model produces the benchmark bond values reported in Section 3.1. By doing this, we tie the model to the underlying economic reality.

Recall from Exhibits 2, 3, and 4 the benchmark bond price information and the relevant par, spot, and forward curves. We will assume that volatility, σ, is 15% and construct a four-year tree using the two-year bond that carries a coupon rate of 1.20%. A complete four-year binomial interest rate tree is presented in Exhibit 10. We will demonstrate how these rates are determined. The current one-year rate is 1%, i_0.

Exhibit 10 Four-Year Binomial Interest Rate Tree



Finding the rates in the tree is an iterative process, and the interest rates are found numerically. There are two possible rates at Time 1—the higher rate and the lower rate. We observe these rates one year from today. These two rates must be consistent with the volatility assumption, the interest rate model, and the observed market value of the benchmark bond. Assume that the interest rate volatility is 15%. From our discussion earlier, we know that at Time 1 the lower one-year rate is lower than the implied one-year forward rate and the higher rate is a multiple of the lower rate. We iterate to a solution with constraints in mind. Once we select these rates, how will we know the rates are correct? The answer is when we discount the cash flows using the tree and produce a value that matches the price of the two-year benchmark bond. If the model does not produce the correct price with this result, we need to select another forward rate and repeat the process. The process of calibrating a binomial interest rate tree to match a specific term structure is illustrated in the following paragraphs.

The procedure starts with the selection of a trial rate for one of the Time 1 forward rates, for instance, $i_{1,L..}$ This rate should be lower than the implied forward rate of 1.4028%. Suppose that we select 1.2500%. The other forward rate will be 1.6873% [= 1.2500% × ($e^{2\times0.15}$)]. Exhibit 11 shows that the Time 0 value for the 1.20%, two-year bond is 99.9363. The redemption of principal and the final interest payment are placed across from the two nodes for the forward rates. At Time 1, the interest payment due is placed across from the initial rate for Time 0. These are the calculations:

$$101.20/1.016873 = 99.5208$$

$$101.20/1.012500 = 99.9506$$

$$\frac{1.20 + (0.5 \times 99.5208 + 0.5 \times 99.9506)}{1.01} = 99.9363$$

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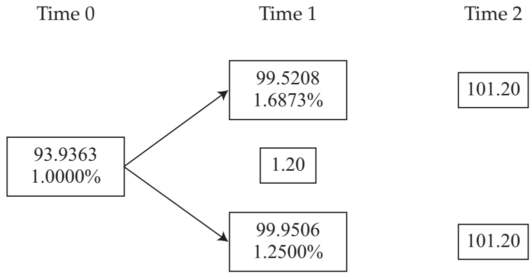
Exhibit 11 Calibrating the Two-Year Binomial Tree



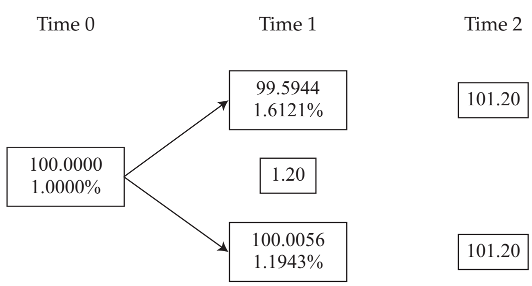
These two trial rates are clearly too high. They need to be lowered somewhat to raise the bond value to attain a Time 0 price for the bond of 100.0000. We could proceed with further trial-and-error search or use an analytic tool, such as Solver in Excel, to carry out this calculation. Essentially, we need to set the cell for the Time 0 bond price to a value of 100.0000 by changing the cell containing the initial lower forward rate for Time 1.

This procedure eventually obtains a value for $i_{1,L}$ of 1.1943%. This is the lower one-year rate. The higher one-year rate is 1.6122% [= 1.1943% × ($e^{2\times0.15}$)]. Notice that the average of these two forward rates is 1.4032% [= 1.6122% + 1.1943%)/2], slightly above the implied forward rate. The binomial tree spreads out around the forward rate curve. The average is slightly higher than the implied forward rate because of the assumption of log-normality.

Recall from the information on the benchmark bonds, that the two-year bond will pay its maturity value of 100 in Time 2 and an annual coupon payment of 1.20. The bond's value at Time 2 is 101.20. The present value of the coupon payment plus the bond's maturity value if the higher one-year rate is realized, *VH*, is 99.5944 (= 101.20/1.016122). Alternatively, the present value of the coupon payment plus the bond's maturity value if the lower one-year rate is realized, *VL*, is 100.0056 (= 101.20/1.011943). These two calculations determine the bond's value one year forward. Effectively, the forward rates move the bond's value from Time 2 to Time 1. Exhibit 12 demonstrates that the arbitrage-free forward rates for Time 1 are 1.6122% and 1.1943%. The value for the bond at Time 0 is 100.0000, confirming the calibration:

$$\frac{1.20 + \left(0.5 \times 99.5944 + 0.5 \times 100.0056\right)}{1.010000} = 100.0000$$

Exhibit 12 Building the Two-Year Binomial Tree



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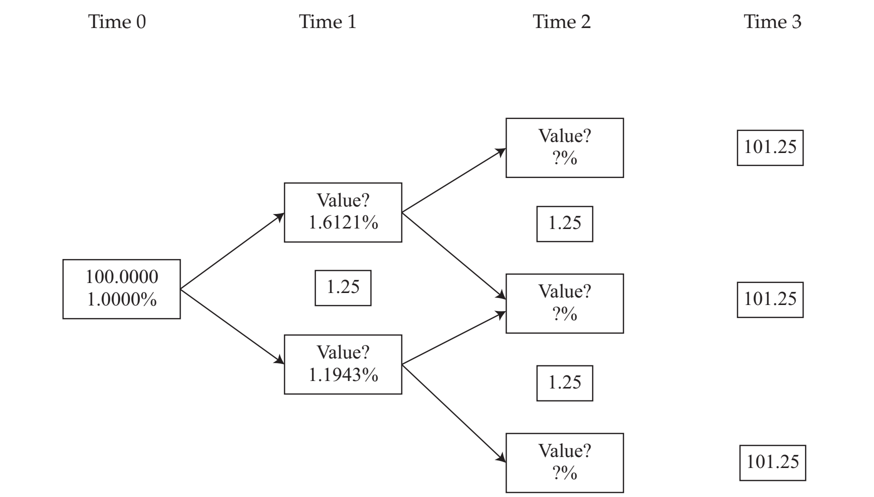
To build out the tree one more year, we repeat the same process, this time using a three-year benchmark bond with a coupon rate of 1.25%. Now, we are looking for three forward rates that are consistent with (1) the interest rate model assumed, (2) the assumed volatility of 15%, (3) a current one-year rate of 1.0%, and (4) the two possible forward rates one year from now (at Time 1) of 1.1943% (the lower rate) and 1.6121% (the higher rate).

At Time 3, we receive the final coupon payment and maturity value of 101.25. In Exhibit 13, we see the known coupon payments of 1.25 for Times 1 and 2. Also entered are the Time 1 forward rates and the target price of par value for the three-year bond. The unknown items to determine are the Time 1 and Time 2 bond values (Value?) and the Time 2 forward rates (?%).

Exhibit 13 Finding the Time 2 Forward Rates



We need to select a trial value for the middle rate, $i_{2,HL}$. A good choice is the implied forward rate of 1.3521%. The trial value for the upper rate, $i_{2,HH}$, would need to be 1.3521% × ($e^{2\times0.15}$) and the lower rate, $i_{2,LL}$, 1.3521%/($e^{2\times0.15}$). The middle rate is then changed, changing the others as well, until the value for the 1.25% three-year bond is 100.0000. It turns out that the three forward rates are 1.7863%, 1.3233%, and 0.9803%. To demonstrate that these are the correct values, we simply work backward from the cash flows at Time 3 of the tree in Exhibit 13. The same procedure is used to obtain the values at the other nodes. The completed tree is shown in Exhibit 14.

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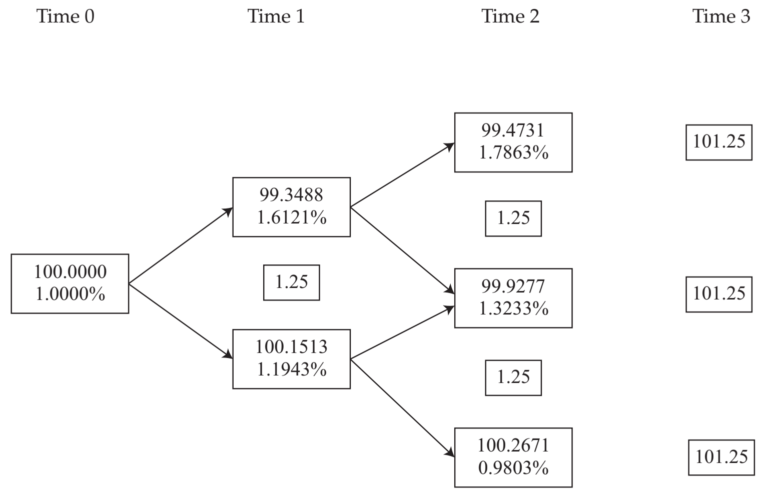
Exhibit 14 Completed Binomial Tree with Calculated Forward Rates



Let us focus on the impact of volatility on the possible forward rates in the tree. If we were to use a higher estimate of volatility, say 20%, the possible forward rates should spread farther out around the forward curve. If we were to use a lower estimate of volatility, say 0.01%, the rates should collapse to the implied forward rates from the current yield curve. Exhibits 15 and 16 depict the interest rate trees for the volatilities of 20% and 0.01%, respectively, and confirm the expected outcome. Notice that in Exhibit 16 for 0.01% volatility, the Time 1 forward rates are very close to the implied forward rate of 1.4028% shown in Exhibit 4. Likewise, the Time 2 and Time 3 rates are a small range around the forward rates of 1.3521% and 1.8647%, respectively. In fact, if σ = 0, the binomial tree is simply the implied forward curve.

Exhibit 15 Completed Tree with σ = 20%

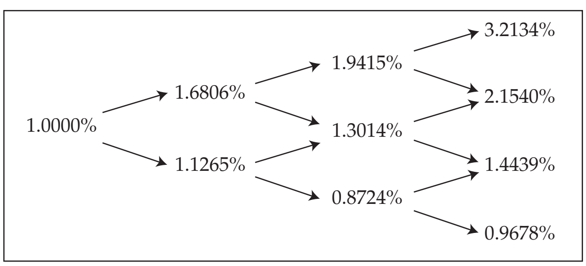


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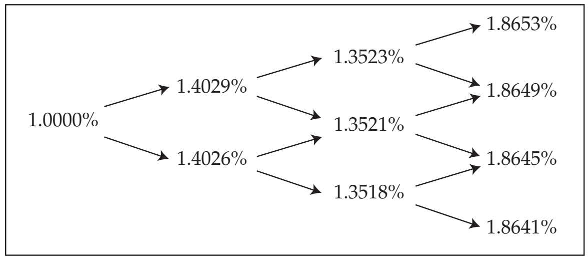
Exhibit 16 Completed Tree with σ = 0.01%



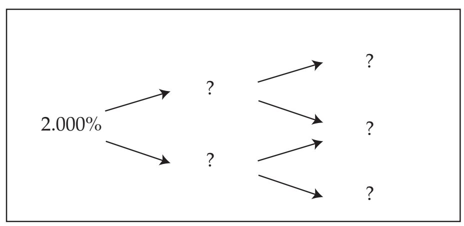
EXAMPLE 4

Calibrating a Binomial Tree to Match a Specific Term Structure

As in Example 2, the one-year par rate is 2.000%, the two-year par rate is 3.000%, and the three-year par rate is 4.000%. Consequently, the spot rates are S_0 = 2.000%, S_1 = 3.015%, and S_2 = 4.055%. The forward rates are F_0 = 2.000%, F_1 = 4.040%, and F_2 = 6.166%. Interest volatility is 15% for all years.

Calibrate the binomial tree in Exhibit 17.

Exhibit 17 Binomial Tree to Calibrate



Solution:

Time 0

The par, spot, and forward rates are all the same for the first period in a binomial tree. Consequently, $Y_0 = S_0 = F_0$ = 2.000%.

Time 1

We need to use trial-and-error search (or Solver in Excel) to find the two forward rates that produce a value of 100.000 for the 3%, two-year bond. The lower trial rate needs to be lower than the implied forward rate of 4.040%, for instance, 3.500%. The higher trial rate would be 3.500% × ($e^{2\times0.15}$) = 4.725%. These lead to a Time 0 value for the bond of 99.936. Therefore, the next stage in the procedure lowers the trial rates. Finally, the calibrated forward rates are 4.646% and 3.442%. Exhibit 18 shows that these are the correct rates because the value of the bond at Time 0 is 100.000. These are the calculations:

103/1.04646 = 98.427

103/1.03442 = 99.573

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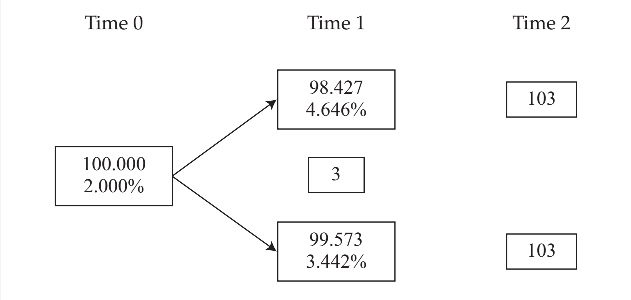
$$\frac{3 + (0.5 \times 98.427 + 0.5 \times 99.573)}{1.02} = 100.0000$$

Exhibit 18 Calibration of Time 1 Forward Rates



Time 2

The initial trial rate for the middle node for Time 2 is the implied forward rate of 6.166%. The rate for the upper node is 8.323% [= 6.166% × ($e^{2\times0.15}$)] and the rate for the lower node is 4.568% [= 6.166%/($e^{2\times0.15}$)]. Exhibit 19 shows that these rates for Time 2, and the already calibrated rates for Time 1, lead to a value of 99.898 for the 4% three-year bond as of Time 0. These are not the arbitrage-free rates—the Time 2 rates need to be lowered slightly to get the price up to 100.000.

Exhibit 19 Calibration of Time 2 Forward Rates

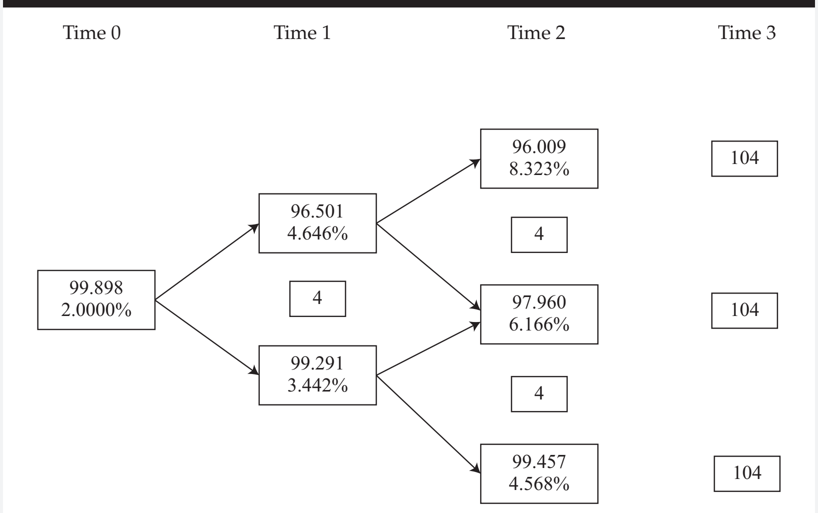


Exhibit 20 displays the completed binomial tree. The calibrated forward rates for Time 2 are 8.167%, 6.050%, and 4.482%. These are the calculations:

104/1.08167 = 96.148

104/1.06050 = 98.067

104/1.04482 = 99.538

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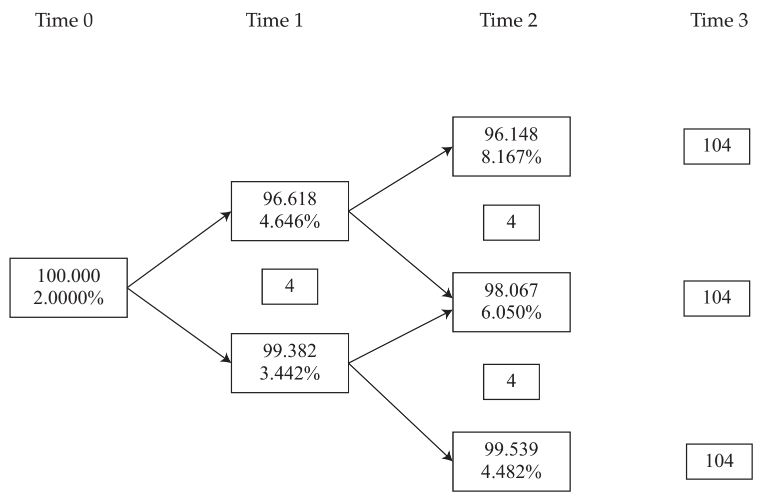
$$\frac{4 + (0.5 \times 96.148 + 0.5 \times 98.067)}{1.04646} = 96.618$$

$$\frac{4 + (0.5 \times 98.067 + 0.5 \times 99.539)}{1.03442} = 99.382$$

$$\frac{4 + (0.5 \times 96.618 + 0.5 \times 99.382)}{1.02000} = 100.000$$

Exhibit 20 Completed Binomial Tree



Now that our tree gives the correct prices for the underlying par bonds maturing in one, two, and three years, we say that our tree is calibrated to be arbitrage free. It will price option-free bonds correctly, including prices for the zero-coupon bonds used to find the spot rates and, to the extent that we have chosen an appropriate interest rate process and interest rate volatility, it will provide insights into the value of bonds with embedded options and their risk parameters.

3.5 Valuing an Option-Free Bond with the Tree

Our next task is twofold. First, we calculate the arbitrage-free value of an option-free, fixed-rate coupon bond. Second, we compare the pricing using the zero-coupon yield curve with the pricing using an arbitrage-free binomial lattice. Because these two valuation methods are arbitrage-free, these two values must be the same.

Now, consider an option-free bond with four years remaining to maturity and a coupon rate of 2%. Note that this is not a benchmark bond and it carries a higher coupon than the four-year benchmark bond, which is priced at par. The value of this bond can be calculated by discounting the cash flow at the spot rates in Exhibit 3 as shown in the following equation:

$$\frac{2}{(1.01)^1} + \frac{2}{(1.012012)^2} + \frac{2}{(1.012515)^3} + \frac{102}{(1.014044)^4} = 102.3254$$

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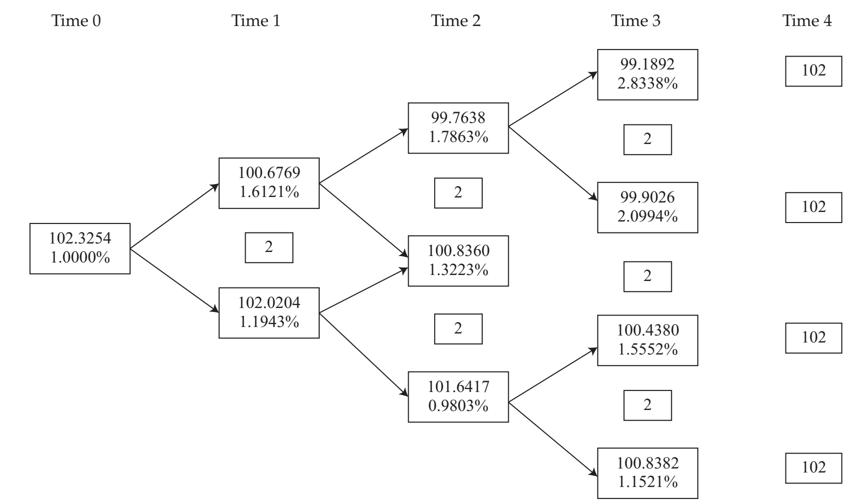
The binomial interest rate tree should produce the same value as discounting the cash flows with the spot rates. An option-free bond that is valued by using the binomial interest rate tree should have the same value as discounting by the spot rates, which is true because the binomial interest rate tree is arbitrage-free.

Let us give the tree a test run and use the 2% option-free bond with four years remaining to maturity. Also assume that the issuer's benchmark yield curve is the one given in Exhibit 2, hence the appropriate binomial interest rate tree is the one in Exhibit 10. Exhibit 21 shows the various values in the discounting process and obtains a bond value of 102.3254. The tree produces the same value for the bond as the spot rates and is therefore consistent with our standard valuation model.

Exhibit 21 Sample Valuation for an Option-Free Bond using a Binomial Tree



EXAMPLE 5

Confirming the Arbitrage-Free Value of a Bond

Using the par curve from Example 2 and Example 4, the yield to maturity for a one-year annual-pay bond is 2%, for a two-year annual-pay bond is 3%, and for a three-year annual-pay bond is 4%. Because this is the same curve as that used in Example 4, we can use the calibrated tree from that example to price a bond. Let us use a three-year annual-pay bond with a 5% coupon, just as we did in Example 2. We know that if the calibrated tree was built correctly and we perform calculations to value the bond with the tree shown in Exhibit 22, its price should be 102.8105.

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Exhibit 22

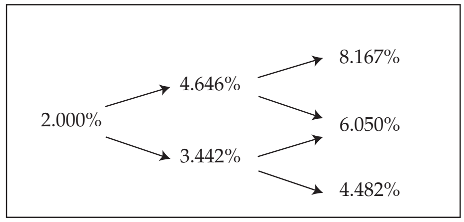
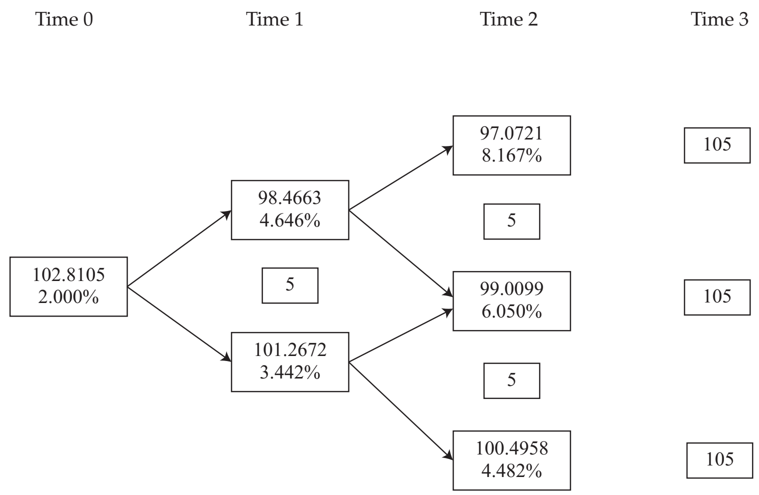


Exhibit 23 Valuing a 5%, Three-Year Bond



Because the tree was calibrated to the same par curve (and spot curve) that was used to price this option-free bond using spot rates only, the tree gives the same price as the spot rate pricing (the small difference is due to rounding).

3.6 Pathwise Valuation

An alternative approach to backward induction in a binomial tree is called pathwise valuation. The binomial interest rate tree specifies all potential rate paths in the model, whereas an interest rate path is the route an interest rate takes from the current time to the security's maturity. Pathwise valuation calculates the present value of a bond for each possible interest rate path and takes the average of these values across paths. We will use the pathwise valuation approach to produce the same value as the backward induction method for an option-free bond. Pathwise valuation involves the following steps: (1) specify a list of all potential paths through the tree, (2) determine the present value of a bond along each potential path, and (3) calculate the average across all possible paths.

Determining all potential paths is just like the following experiment. Suppose you are tossing a fair coin and are keeping track of the number of ways heads and tails can be combined. We will use a device called Pascal's Triangle, displayed in Exhibit 24. Pascal's Triangle can be built as follows: Start with the number 1 at the top of the triangle. The numbers in the boxes below are the sum of the two numbers above it except

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that the edges on each side are all 1. The shaded numbers show that 3 is the sum of 2 and 1. Now toss the coin while keeping track of the possible outcomes. The possible groupings are listed in Exhibit 25 where H stands for heads and T stands for tails.

Exhibit 24

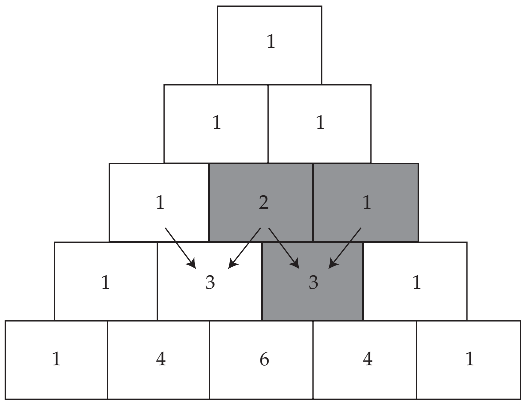

Exhibit 25

Number of Tosses	Outcomes	Pascal's Triangle
1	H T	1, 1
2	HH HT TH TT	1,2,1
3	HHH HHT HTH THH HTT THT TTH TTT	1, 3, 3, 1

This experiment mirrors exactly the number of interest rate paths in our binomial interest rate tree. The total number of paths for each period/year can be easily determined by using Pascal's Triangle. Let us work through an example for a three-year zero-coupon bond. From Pascal's Triangle, there are four possible paths to arrive at Year 3: HH, HT, TH, TT. Using the same binomial tree from Section 3.4, we specify the four paths as well as the possible forward rates along those paths. In Exhibit 26, the last column on the right shows the present value for each path. For example, 100/[(1.01000) × (1.016121) × (1.017863)] = 95.7291. In the bottom right corner is the average present value across all paths.

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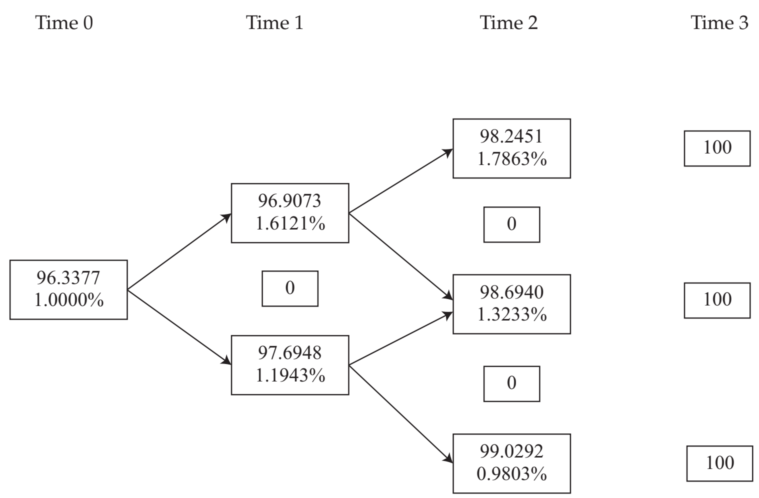
Exhibit 26 Four Interest Rate Paths for a Three-Year Zero-Coupon Bond

Path	Forward Rate Year 1	Forward Rate Year 2	Forward Rate Year 3	Present Value
1	1.0000%	1.6121%	1.7863%	95.7291
2	1.0000%	1.6121%	1.3233%	96.1665
3	1.0000%	1.1943%	1.3233%	96.5636
4	1.0000%	1.1943%	0.9803%	96.8916
				96.3377

Now, we can use the binomial tree to confirm our calculations for the three-year zero-coupon bond. The analysis is presented in Exhibit 27. The interest rate tree does indeed produce the same value.

Exhibit 27 Binomial Tree to Confirm Bond's Value



EXAMPLE 6

Pathwise Valuation Based on a Binomial Interest Rate Tree

Using the par curve from Example 2, Example 4, and Example 5, the yield to maturity for a one-year annual-pay bond is 2%, for a two-year annual-pay bond is 3%, and for a three-year annual-pay bond is 4%. We know that if we generate the paths in the tree correctly and discount the cash flows directly, the three-year, annual-pay, 5% coupon bond should still be priced at 102.8105, as calculated in Example 5.

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There are four paths through the three-year tree.[6] We discount the cash flows along each of the four paths and take their average, as shown in Exhibits 28, 29, and 30.

Exhibit 28 Cash Flows

Path	Time 0	Time 1	Time 2	Time 3
1	0	5	5	105
2	0	5	5	105
3	0	5	5	105
4	0	5	5	105

Exhibit 29 Discount Rates

Path	Time 0	Time 1	Time 2	Time 3
1	2.000%	4.646%	8.167%	
2	2.000%	4.646%	6.050%	
3	2.000%	3.442%	6.050%	
4	2.000%	3.442%	4.482%	

Exhibit 30 Present Values

Path	Time 0
1	100.5298
2	102.3452
3	103.4794
4	104.8877
Average	**102.8105**

The present values are calculated by discounting the cash flows in Exhibit 28 by the forward rates in Exhibit 29. For example, the present value for the bond along path 1 is 100.5298:

$$\frac{5}{1.02} + \frac{5}{(1.02)(1.04646)} + \frac{105}{(1.02)(1.04646)(1.08167)} = 100.5298$$

6 An "N-period" recombining tree has N + 1 terminal nodes at time N. There are 2^N paths through the tree to those (N + 1) nodes. So a 3-period tree has 4 terminal nodes and 8 paths to those nodes. However, for instruments whose terminal payment/value is the same for all paths, e.g., fixed coupon bonds, we really only need interest rates at times 0, 1, …, (N – 1). That is, an (N – 1)-period tree and hence 2^{N-1} paths. So to value an annual-pay, 3-year bond there are really only 4 paths, not 8. Note, however, that for instruments with terminal payments that depend on the interest rate at maturity (time N) we need all 2^N paths. As an example, this would be the case for an "arrears" swap.

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The present value along path 3 is 103.4794:

$$\frac{5}{1.02}+\frac{5}{(1.02)(1.03442)}+\frac{105}{(1.02)(1.03442)(1.06050)}=103.4794$$

The average for the bond prices using pathwise valuation is 102.8105, which matches the result obtained using backward induction in Exhibit 23.

4 MONTE CARLO METHOD

The Monte Carlo method is an alternative method for simulating a sufficiently large number of potential interest rate paths in an effort to discover how a value of a security is affected. This method involves randomly selecting paths in an effort to approximate the results of a complete pathwise valuation. Monte Carlo methods are often used when a security's cash flows are path dependent. Cash flows are path dependent when the cash flow to be received in a particular period depends on the path followed to reach its current level as well as the current level itself. For example, the valuation of mortgage-backed securities depends to a great extent on the level of prepayments, which are interest rate path dependent. Interest rate paths are generated based on some probability distribution, an assumption about volatility, and the model is fit to the current benchmark term structure of interest rates. The benchmark term structure is represented by the current spot rate curve such that the average present value across all scenario interest rate paths for each benchmark bond equals its actual market value. By using this approach, the model is rendered arbitrage free, which is equivalent to calibrating the interest rate tree as discussed in Section 3.

Suppose we intend to value a 30-year bond with the Monte Carlo method. For simplicity, assume the bond has monthly coupon payments (e.g., mortgage-backed securities). The following steps are taken: (1) simulate numerous (say, 500) paths of one-month interest rates under some volatility assumption and probability distribution, (2) generate spot rates from the simulated future one-month interest rates, (3) determine the cash flow along each interest rate path, (4) calculate the present value for each path, and (5) calculate the average present value across all interest rate paths.

Using the procedure just described, the model will produce benchmark bond values equal to the market prices only by chance. We want to ensure this is the case, otherwise the model will neither fit the current spot curve nor be arbitrage free. A constant is added to all interest rates on all paths such that the average present value for each benchmark bond equals its market value. The constant added to all short interest rates is called a drift term. When this technique is used, the model is said to be drift adjusted.

A question that arises concerns how many paths are appropriate for the Monte Carlo method. Increasing the number of paths increases the accuracy of the estimate in a statistical sense. It does not mean the model is closer to the true fundamental value of the security. The Monte Carlo method is only as good as the valuation model used and the accuracy of the inputs.

One other element that yield curve modelers often include in their Monte Carlo estimation is mean reversion. Mean reversion starts with the common-sense notion that history suggests that interest rates almost never get "too high" or "too low." What is meant by "too high" and "too low" is left to the discretion of the modeler. We implement mean reversion by implementing upper and lower bounds on the random process generating future interest rates. Mean reversion has the effect of moving the interest rate toward the implied forward rates from the yield curve.

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EXAMPLE 7

The Application of Monte Carlo Simulation to Bond Pricing

Replace the interest rate paths from Example 6 with randomly generated paths that have been calibrated to the same initial par and spot curves, as shown in Exhibit 31.

Exhibit 31 Discount Rates

Path	Time 0	Time 1	Time 2
1	2.000%	2.500%	4.548%
2	2.000%	3.600%	6.116%
3	2.000%	4.600%	7.766%
4	2.000%	5.500%	3.466%
5	2.000%	3.100%	8.233%
6	2.000%	4.500%	6.116%
7	2.000%	3.800%	5.866%
8	2.000%	4.000%	8.233%

Exhibit 32 Present Values

Path	Time 0
1	105.7459
2	103.2708
3	100.9104
4	103.8543
5	101.9075
6	102.4236
7	103.3020
8	101.0680
Average	**102.8103**

Because we continue to get 102.8103, as shown in Exhibit 32, as the price for our three-year, annual-pay, 5% coupon bond, we know that the Monte Carlo simulation has been calibrated correctly. The paths are now different enough such that path dependent securities, such as mortgage-backed securities, can be analyzed in ways that provide insights not possible in binomial trees.

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SUMMARY

This reading presents the principles and tools for arbitrage valuation of fixed-income securities. Much of the discussion centers on the binomial interest rate tree, which can be used extensively to value both option-free bonds and bonds with embedded options. The following are the main points made in the reading:

- A fundamental principle of valuation is that the value of any financial asset is equal to the present value of its expected future cash flows.
- A fixed-income security is a portfolio of zero-coupon bonds.
- Each zero-coupon bond has its own discount rate that depends on the shape of the yield curve and when the cash flow is delivered in time.
- In well-functioning markets, prices adjust until there are no opportunities for arbitrage.
- The law of one price states that two goods that are perfect substitutes must sell for the same current price in the absence of transaction costs.
- An arbitrage opportunity is a transaction that involves no cash outlay yet results in a riskless profit.
- Using the arbitrage-free approach, viewing a security as a package of zero-coupon bonds means that two bonds with the same maturity and different coupon rates are viewed as different packages of zero-coupon bonds and valued accordingly.
- For bonds that are option free, an arbitrage-free value is simply the present value of expected future values using the benchmark spot rates.
- A binomial interest rate tree permits the short interest rate to take on one of two possible values consistent with the volatility assumption and an interest rate model.
- An interest rate tree is a visual representation of the possible values of interest rates (forward rates) based on an interest rate model and an assumption about interest rate volatility.
- The possible interest rates for any following period are consistent with the following three assumptions: (1) an interest rate model that governs the random process of interest rates, (2) the assumed level of interest rate volatility, and (3) the current benchmark yield curve.
- From the lognormal distribution, adjacent interest rates on the tree are multiples of e raised to the 2σ power.
- One of the benefits of a lognormal distribution is that if interest rates get too close to zero, then the absolute change in interest rates becomes smaller and smaller.
- We use the backward induction valuation methodology that involves starting at maturity, filling in those values, and working back from right to left to find the bond's value at the desired node.
- The interest rate tree is fit to the current yield curve by choosing interest rates that result in the benchmark bond value. By doing this, the bond value is arbitrage free.
- An option-free bond that is valued by using the binomial interest rate tree should have the same value as discounting by the spot rates.

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- Pathwise valuation calculates the present value of a bond for each possible interest rate path and takes the average of these values across paths.
- The Monte Carlo method is an alternative method for simulating a sufficiently large number of potential interest rate paths in an effort to discover how the value of a security is affected and involves randomly selecting paths in an effort to approximate the results of a complete pathwise valuation.

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PRACTICE PROBLEMS

The following information relates to Questions 1–6

Katrina Black, portfolio manager at Coral Bond Management, Ltd., is conducting a training session with Alex Sun, a junior analyst in the fixed income department. Black wants to explain to Sun the arbitrage-free valuation framework used by the firm. Black presents Sun with Exhibit 1, showing a fictitious bond being traded on three exchanges, and asks Sun to identify the arbitrage opportunity of the bond. Sun agrees to ignore transaction costs in his analysis.

Exhibit 1 Three-Year, €100 par, 3.00% Coupon, Annual Pay Option-Free Bond

	Eurex	NYSE Euronext	Frankfurt
Price	€103.7956	€103.7815	€103.7565

Black shows Sun some exhibits that were part of a recent presentation. Exhibit 3 presents most of the data of a binomial lognormal interest rate tree fit to the yield curve shown in Exhibit 2. Exhibit 4 presents most of the data of the implied values for a four-year, option-free, annual pay bond with a 2.5% coupon based on the information in Exhibit 3.

Exhibit 2 Yield to Maturity Par Rates for One-, Two-, and Three-Year Annual Pay Option-Free Bonds

One-year	Two-year	Three-year
1.25%	1.50%	1.70%

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Exhibit 3 Binomial Interest Rate Tree Fit to the Yield Curve (Volatility = 10%)

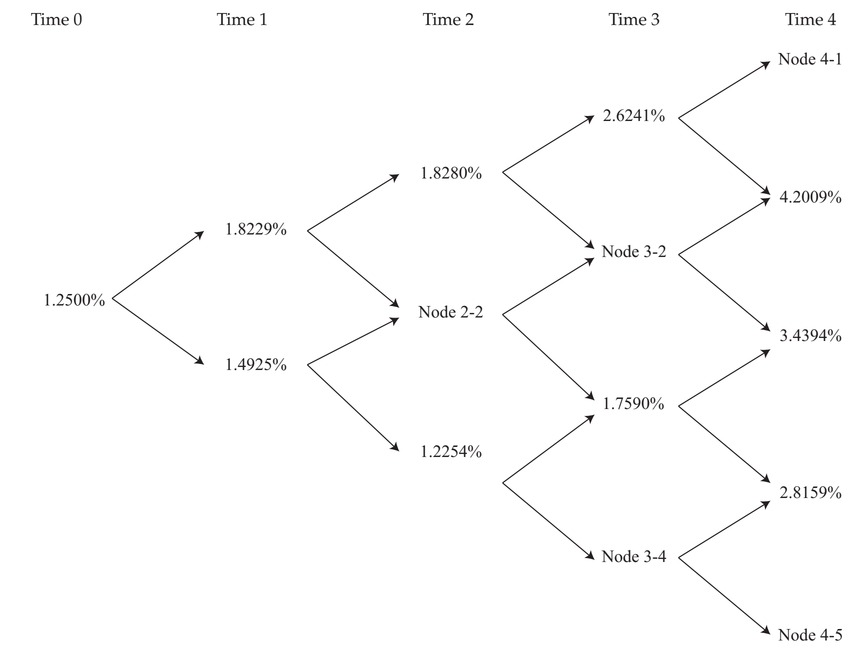
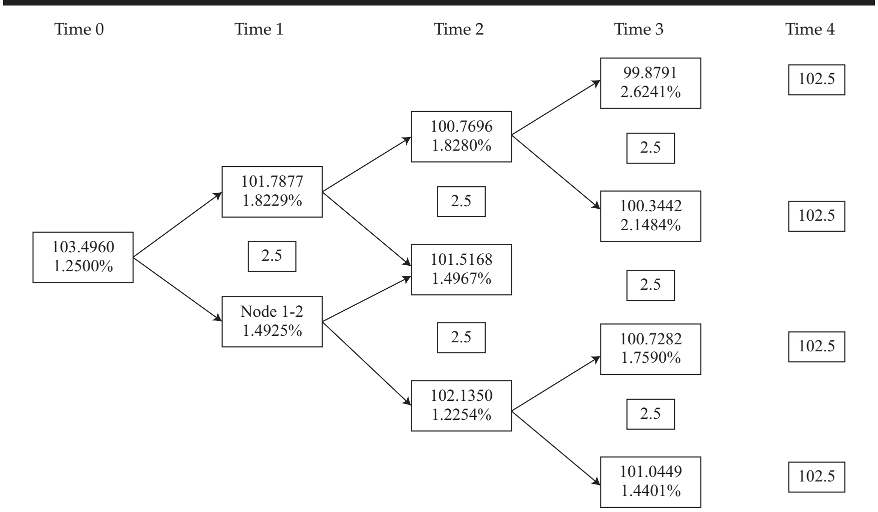


Exhibit 4 Implied Values (in Euros) for a 2.5%, Four-Year, Option-Free, Annual Pay Bond Based on Exhibit 3



Black asks about the missing data in Exhibits 3 and 4 and directs Sun to complete the following tasks related to those exhibits:

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Task 1	Test that the binomial interest tree has been properly calibrated to be arbitrage-free.
Task 2	Develop a spreadsheet model to calculate pathwise valuations. To test the accuracy of the spreadsheet, use the data in Exhibit 3 and calculate the value of the bond if it takes a path of lowest rates in Year 1 and Year 2 and the second lowest rate in Year 3.
Task 3	Identify a type of bond where the Monte Carlo calibration method should be used in place of the binomial interest rate method.
Task 4	Update Exhibit 3 to reflect the current volatility, which is now 15%.

1 Based on Exhibit 1, the *best* action that an investor should take to profit from the arbitrage opportunity is to:
   - A buy on Frankfurt, sell on Eurex.
   - B buy on NYSE Euronext, sell on Eurex.
   - C buy on Frankfurt, sell on NYSE Euronext.

2 Based on Exhibits 1 and 2, the exchange that reflects the arbitrage-free price of the bond is:
   - A Eurex.
   - B Frankfurt.
   - C NYSE Euronext.

3 Which of the following statements about the missing data in Exhibit 3 is correct?
   - A Node 3–2 can be derived from Node 2–2.
   - B Node 4–1 should be equal to Node 4–5 multiplied by $e^{0.4}$.
   - C Node 2–2 approximates the implied one-year forward rate two years from now.

4 Based on the information in Exhibits 3 and 4, the bond price in euros at Node 1–2 in Exhibit 4 is *closest* to:
   - A 102.7917.
   - B 104.8640.
   - C 105.2917.

5 A benefit of performing Task 1 is that it:
   - A enables the model to price bonds with embedded options.
   - B identifies benchmark bonds that have been mispriced by the market.
   - C allows investors to realize arbitrage profits through stripping and reconstitution.

6 If the assumed volatility is changed as Black requested in Task 4, the forward rates shown in Exhibit 3 will *most likely*:
   - A spread out.
   - B remain unchanged.
   - C converge to the spot rates.

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The following information relates to Questions 7–10

Betty Tatton is a fixed income analyst with the hedge fund Sailboat Asset Management (SAM). SAM invests in a variety of global fixed-income strategies, including fixed-income arbitrage. Tatton is responsible for pricing individual investments and analyzing market data to assess the opportunity for arbitrage. She uses two methods to value bonds:

Method 1 Discount each year's cash flow separately using the appropriate interest rate curve.

Method 2 Build and use a binomial interest rate tree.

Tatton compiles pricing data for a list of annual pay bonds (Exhibit 1). Each of the bonds will mature in two years, and Tatton considers the bonds as being risk-free; both the one-year and two-year benchmark spot rates are 2%. Tatton calculates the arbitrage-free prices and identifies an arbitrage opportunity to recommend to her team.

Exhibit 1 Market Data for Selected Bonds

Asset	Coupon	Market Price
Bond A	1%	98.0584
Bond B	3%	100.9641
Bond C	5%	105.8247

Next, Tatton uses the benchmark yield curve provided in Exhibit 2 to consider arbitrage opportunities of both option-free corporate bonds and corporate bonds with embedded options. The benchmark bonds in Exhibit 2 pay coupons annually, and the bonds are priced at par.

Exhibit 2 Benchmark Par Curve

Maturity (years)	Yield to Maturity (YTM)
1	3.0%
2	4.0%
3	5.0%

Tatton then identifies three mispriced three-year annual-pay bonds and compiles data on the bonds (see Exhibit 3).

Exhibit 3 Market Data of Annual-Pay Corporate Bonds

Company	Coupon	Market Price	Yield	Embedded Option?
Hutto-Barkley Inc.	3%	94.9984	5.6%	No
Luna y Estrellas Intl.	0%	88.8996	4.0%	Yes
Peaton Scorpio Motors	0%	83.9619	6.0%	No

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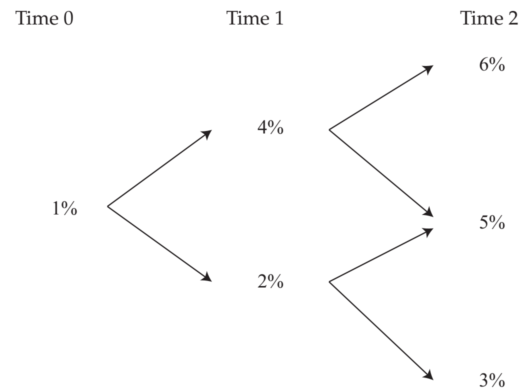
Lastly, Tatton identifies two mispriced Swiss bonds, Bond X, a three-year bond, and Bond Y, a five-year bond. Both are annual-pay bonds with a coupon rate of 6%. To calculate the bonds' values, Tatton devises the first three years of the interest rate lognormal tree presented in Exhibit 4 using historical interest rate volatility data. Tatton considers how this data would change if implied volatility, which is higher than historical volatility, were used instead.

Exhibit 4 Interest Rate Tree; Forward Rates Based on Swiss Market



7 Based on Exhibit 1, which of the following bonds *most likely* includes an arbitrage opportunity?

A Bond A

B Bond B

C Bond C

8 Based on Exhibits 2 and 3 and using Method 1, the amount (in absolute terms) by which the Hutto-Barkley corporate bond is mispriced is *closest* to:

A 0.3368 per 100 of par value.

B 0.4682 per 100 of par value.

C 0.5156 per 100 of par value.

9 Method 1 would *most likely* **not** be an appropriate valuation technique for the bond issued by:

A Hutto-Barkley Inc.

B Luna y Estrellas Intl.

C Peaton Scorpio Motors.

10 Based on Exhibit 4 and using Method 2, the correct price for Bond X is *closest* to:

A 97.2998.

B 109.0085.

C 115.0085.

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The following information relates to Questions 11–18

Meredith Alvarez is a junior fixed-income analyst with Canzim Asset Management. Her supervisor, Stephanie Hartson, asks Alvarez to review the asset price and payoff data shown in Exhibit 1 to determine whether an arbitrage opportunity exists.

Exhibit 1 Price and Payoffs for Two Risk-Free Assets

Asset	Price Today	Payoff in One Year
Asset A	\$500	\$525
Asset B	\$1,000	\$1,100

Hartson also shows Alvarez data for a bond that trades in three different markets in the same currency. These data appear in Exhibit 2.

Exhibit 2 2% Coupon, Five-Year Maturity, Annual Pay Bond

	New York	Hong Kong	Mumbai
Yield to Maturity	1.9%	2.3%	2.0%

Hartson asks Alvarez to value two bonds (Bond C and Bond D) using the binomial tree in Exhibit 3. Exhibit 4 presents selected data for both bonds.

Exhibit 3 Binomial Interest Rate Tree with Volatility = 25%

Time 0	Time 1	Time 2
		2.7183%
	2.8853%	
1.500%		1.6487%
	1.7500%	
		1.0000%

Exhibit 4 Selected Data on Annual Pay Bonds

Bond	Maturity	Coupon Rate
Bond C	2 years	2.5%
Bond D	3 years	3.0%

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Hartson tells Alvarez that she and her peers have been debating various viewpoints regarding the conditions underlying binomial interest rate trees. The following statements were made in the course of the debate.

Statement 1 The only requirements needed to create a binomial interest rate tree are current benchmark interest rates and an assumption about interest rate volatility.

Statement 2 Potential interest rate volatility in a binomial interest rate tree can be estimated using historical interest rate volatility or observed market prices from interest rate derivatives.

Statement 3 A bond value derived from a binomial interest rate tree with a relatively high volatility assumption will be different from the value calculated by discounting the bond's cash flows using current spot rates.

Based on data in Exhibit 5, Hartson asks Alvarez to calibrate a binomial interest rate tree starting with the calculation of implied forward rates shown in Exhibit 6.

Exhibit 5 Selected Data for a Binomial Interest Rate Tree

Maturity	Par Rate	Spot Rate
1	2.5000%	2.5000%
2	3.5000%	3.5177%

Exhibit 6 Calibration of Binomial Interest Rate Tree with Volatility = 25%

Time 0	Time 1
	5.8365%
2.500%	
	Lower one-period forward rate

Hartson mentions pathwise valuations as another method to value bonds using a binomial interest rate tree. Using the binomial interest rate tree in Exhibit 3, Alvarez calculates the possible interest rate paths for Bond D shown in Exhibit 7.

Exhibit 7 Interest Rate Paths for Bond D

Path	Time 0	Time 1	Time 2
1	1.500%	2.8853%	2.7183%
2	1.500	2.8853	1.6487
3	1.500	1.7500	1.6487
4	1.500	1.7500	1.0000

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Before leaving for the day, Hartson asks Alvarez about the value of using the Monte Carlo method to simulate a large number of potential interest rate paths to value a bond. Alvarez makes the following statements.

Statement 4 Increasing the number of paths increases the estimate's statistical accuracy.

Statement 5 The bond value derived from a Monte Carlo simulation will be closer to the bond's true fundamental value.

11 Based on Exhibit 1, Alvarez finds that an arbitrage opportunity is:

A not available.

B available based on the dominance principle.

C available based on the value additivity principle.

12 Based on the data in Exhibit 2, the *most* profitable arbitrage opportunity would be to buy the bond in:

A Mumbai and sell it in Hong Kong.

B Hong Kong and sell it in New York.

C New York and sell it in Hong Kong.

13 Based on Exhibits 3 and 4, the value of Bond C at the upper node at Time 1 is *closest* to:

A 97.1957.

B 99.6255.

C 102.1255.

14 Based on Exhibits 3 and 4, the price for Bond D is *closest* to:

A 97.4785.

B 103.3230.

C 106.3230.

15 Which of the various statements regarding binomial interest rate trees is correct?

A Statement 1

B Statement 2

C Statement 3

16 Based on Exhibits 5 and 6, the value of the lower one-period forward rate is *closest to*:

A 3.5122%.

B 3.5400%.

C 4.8037%.

17 Based on Exhibits 4 and 7, the present value of Bond D's cash flows following Path 2 is *closest* to:

A 97.0322.

B 102.8607.

C 105.8607.

18 Which of the statements regarding Monte Carlo simulation is correct?

A Only Statement 4 is correct.

B Only Statement 5 is correct.

C Both Statement 4 and Statement 5 are correct.

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SOLUTIONS

1 A is correct. This is the same bond being sold at three different prices so an arbitrage opportunity exists by buying the bond from the exchange where it is priced lowest and immediately selling it on the exchange that has the highest price. Accordingly, an investor would maximize profit from the arbitrage opportunity by buying the bond on the Frankfurt exchange (which has the lowest price of €103.7565) and selling it on the Eurex exchange (which has the highest price of €103.7956) to generate a risk-free profit of €0.0391 (as mentioned, ignoring transaction costs) per €100 par.

B is incorrect because buying on NYSE Euronext and selling on Eurex would result in an €0.0141 profit per €100 par (€103.7956 – €103.7815 = €0.0141), which is not the maximum arbitrage profit available. A greater profit would be realized if the bond were purchased in Frankfurt and sold on Eurex.

C is incorrect because buying on Frankfurt and selling on NYSE Euronext would result in an €0.0250 profit per €100 par (€103.7815 – €103.7565 = €0.0250). A greater profit would be realized if the bond were purchased in Frankfurt and sold on Eurex.

2 C is correct. The bond from Exhibit 1 is selling for its calculated value on the NYSE Euronext exchange. The arbitrage-free value of a bond is the present value of its cash flows discounted by the spot rate for zero coupon bonds maturing on the same date as each cash flow. The value of this bond, 103.7815, is calculated as follows:

	Year 1	Year 2	Year 3	Total PV
Yield to maturity	1.2500%	1.500%	1.700%	
Spot rate[1]	1.2500%	1.5019%	1.7049%	
Cash flow	3.00	3.00	103.00	
Present value of payment[2]	2.9630	2.9119	97.9066	103.7815

	Eurex	NYSE Euronext	Frankfurt
Price	€103.7956	€103.7815	€103.7565
Mispricing (per 100 par value)	0.141	0	–0.025

Notes:

1 Spot rates calculated using bootstrapping; for example: Year 2 spot rate (z_2): 100 = $1.5/1.0125 + 101.5/(1+z_2)^2 = 0.015019$.

2 Present value calculated using the formula PV = $FV/(1+r)^n$, where n = number of years until cash flow, FV=cash flow amount, and r = spot rate.

A is incorrect because the price on the Eurex exchange, €103.7956, was calculated using the yield to maturity rate to discount the cash flows when the spot rates should have been used. C is incorrect because the price on the Frankfurt exchange, €103.7565, uses the Year 3 spot rate to discount all the cash flows.

3 C is correct. Because Node 2–2 is the middle node rate in Year 2, it will be close to the implied one-year forward rate two years from now (as derived from the spot curve). Node 4–1 should be equal to the product of Node 4–5 and $e^{0.8}$.

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Lastly, Node 3–2 cannot be derived from Node 2–2; it can be derived from any other Year 3 node; for example, Node 3–2 can be derived from Node 3–4 (equal to the product of Node 3–4 and $e^{4\sigma}$).

4 A is correct. The value of a bond at a particular node, in this case Node 1–2, can be derived by determining the present value of the coupon payment and expected future bond values to the right of that node on the tree. In this case, those two nodes are the middle node in Year 2, equal to 101.5168, and the lower node in Year 2, equal to 102.1350. The coupon payment is 2.5. The bond value at Node 1–2 is calculated as follows:

$$\text{Value} = \frac{2.5 + (0.5 \times 101.5816 + 0.5 \times 102.1350)}{1.014925}$$

$$= 102.7917$$

5 A is correct. Calibrating a binomial interest rate tree to match a specific term structure is important because we can use the known valuation of a benchmark bond from the spot rate pricing to verify the accuracy of the rates shown in the binomial interest rate tree. Once its accuracy is confirmed, the interest rate tree can then be used to value bonds with embedded options. While discounting with spot rates will produce arbitrage-free valuations for option-free bonds, this spot rate method will not work for bonds with embedded options where expected future cash flows are interest-rate dependent (as rate changes impact the likelihood of options being exercised). The interest rate tree allows for the alternative paths that a bond with embedded options might take.

B is incorrect because calibration does not identify mispriced benchmark bonds. In fact, benchmark bonds are employed to prove the accuracy of the binomial interest rate tree, as they are assumed to be correctly priced by the market.

C is incorrect because the calibration of the binomial interest rate tree is designed to produce an arbitrage-free valuation approach and such an approach does not allow a market participant to realize arbitrage profits though stripping and reconstitution.

6 A is correct. Volatility is one of the two key assumptions required to estimate rates for the binomial interest rate tree. Increasing the volatility from 10% to 15% would cause the possible forward rates to spread out on the tree as it increases the exponent in the relationship multiple between nodes ($e^{x\sigma}$, where x = 2 times the number of nodes above the lowest node in a given year in the interest rate tree). Conversely, using a lower estimate of volatility would cause the forward rates to narrow or converge to the implied forward rates from the prevailing yield curve.

B is incorrect because volatility is a key assumption in the binomial interest rate tree model. Any change in volatility will cause a change in the implied forward rates.

C is incorrect because increasing the volatility from 10% to 15% causes the possible forward rates to spread out on the tree, not converge to the implied forward rates from the current yield curve. Rates will converge to the implied forward rates when lower estimates of volatility are assumed.

7 B is correct. Bond B's arbitrage-free price is calculated as follows:

$$\frac{3}{1.02} + \frac{103}{1.02^2} = 101.9416$$

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which is higher than the bond's market price of 100.9641. Therefore, an arbitrage opportunity exists. Since the bond's value (100.9641) is less than the sum of the values of its discounted cash flows individually (101.9416), a trader would perceive an arbitrage opportunity and could buy the bond while selling claims to the individual cash flows (zeros), capturing the excess value. The arbitrage-free prices of Bond A and Bond C are equal to the market prices of the respective bonds, so there is no arbitrage opportunity for these two bonds:

$$\text{Bond A: } \frac{1}{1.02} + \frac{101}{1.02^2} = 98.0584$$

$$\text{Bond C: } \frac{5}{1.02} + \frac{105}{1.02^2} = 105.8247$$

8 C is correct. The first step in the solution is to find the correct spot rate (zero-coupon rates) for each year's cash flow. The benchmark bonds in Exhibit 2 are conveniently priced at par so the yields to maturity and the coupon rates on the bonds are the same. Because the one-year issue has only one cash flow remaining, the YTM equals the spot rate of 3% (or z_1 = 3%). The spot rates for Year 2 (z_2) and Year 3 (z_3) are calculated as follows:

$$100 = \frac{4}{1.0300} + \frac{104}{(1 + z_2)^2}; z_2 = 4.02\%$$

$$100 = \frac{5}{1.0300} + \frac{5}{(1.0402)^2} + \frac{105}{(1 + z_3)^3}; z_3 = 5.07\%$$

The correct arbitrage-free price for the Hutto-Barkley Inc. bond is:

$$P_0 = \frac{3}{(1.0300)} + \frac{3}{(1.0402)^2} + \frac{103}{(1.0507)^3} = 94.4828$$

Therefore, the bond is mispriced by 94.9984 – 94.4828 = 0.5156 per 100 of par value.

A is incorrect because the correct spot rates are not calculated and instead the Hutto-Barkley Inc. bond is discounted using the respective YTM for each maturity. Therefore, this leads to an incorrect mispricing of 94.6616 – 94.9984 = –0.3368 per 100 of par value.

B is incorrect because the spot rates are derived using the coupon rate for Year 3 (maturity) instead of using each year's respective coupon rate to employ the bootstrap methodology. This leads to an incorrect mispricing of 94.5302 – 94.9984 = –0.4682 per 100 of par value.

9 B is correct. The Luna y Estrellas Intl. bond contains an embedded option. Method 1 will produce an arbitrage-free valuation for option-free bonds; however, for bonds with embedded options, changes in future interest rates impact the likelihood the option will be exercised and so impact future cash flows. Therefore, to develop a framework that values bonds with embedded options, interest rates must be allowed to take on different potential values in the future based on some assumed level of volatility (Method 2).

A and C are incorrect because the Hutto-Barkley Inc. bond and the Peaton Scorpio Motors bond are both option-free bonds and can be valued using either Method 1 or Method 2 to produce an arbitrage-free valuation.

10 B is correct. This is the binomial tree that obtains a bond value of 109.0085.

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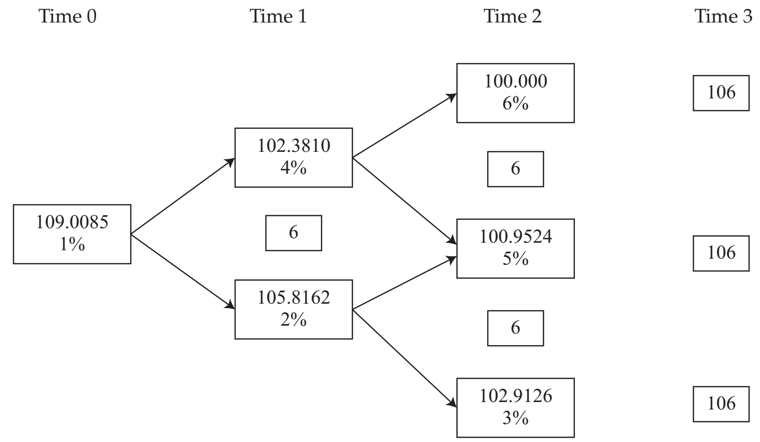
Valuing a 6%, Three-Year Bond



These are the calculations:

$$106/1.06 = 100.0000$$

$$106/1.05 = 100.9524$$

$$106/1.03 = 102.9126$$

$$\frac{6 + (0.5 \times 100.0000 + 0.5 \times 100.9524)}{1.04} = 102.3810$$

$$\frac{6 + (0.5 \times 100.9524 + 0.5 \times 102.9126)}{1.02} = 105.8162$$

$$\frac{6 + (0.5 \times 102.3810 + 0.5 \times 105.8162)}{1.01} = 109.0085$$

A is incorrect because the Time T coupon payment is subtracted from the value in each node calculation for Time T. C is incorrect because it assumes that a coupon is paid at Time 0.

11 B is correct. Based on the dominance principle, an arbitrage opportunity exists. The dominance principle asserts that a financial asset with a risk-free payoff in the future must have a positive price today. Because Asset A and Asset B are both risk-free assets, they should have the same discount rate. Relative to its payoff, Asset A is priced at $500/525, or 0.95238, and Asset B is priced at $1,000/1,100, or 0.90909. Given its higher implied discount rate (10%) and lower corresponding price, Asset B is cheap relative to Asset A, which has a lower implied discount rate (5%) and higher corresponding price.

The arbitrage opportunity based on dominance is to sell two units of Asset A for $1,000 and buy one unit of Asset B. There is no cash outlay today, and in one year, the portfolio delivers a net cash inflow of $50 [= $1,100 – (2 × $525)].

12 B is correct. Of the three markets, the New York bond has the lowest yield to maturity and, correspondingly, the highest bond price. Similarly, the Hong Kong bond has the highest yield to maturity and the lowest bond price of the three markets. Therefore, the most profitable arbitrage trade would be to buy the bond in Hong Kong and sell it in New York.

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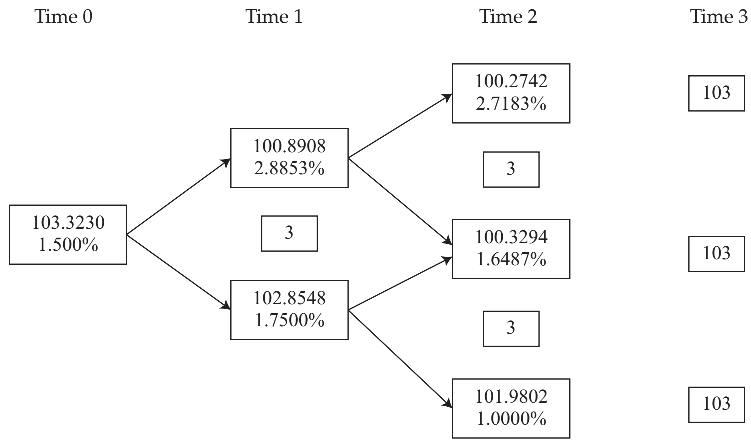
13 B is correct. The bond value at the upper node at Time 1 is closest to 99.6255. The cash flow at Time 2 is 102.5, the redemption of par value (100) plus the final coupon payment (2.5). Using backward induction, we calculate the present value of the bond at the upper node of Time 1 as 102.5/1.028853 = 99.6255.

14 B is correct. The price of Bond D is closest to 103.3230 and can be calculated using backward induction.



$$\text{Bond value at a node} = \frac{C + (0.5 \times VH + 0.5 \times VL)}{1 + i}.$$

Calculations:

The cash flow at Time 3 is 103, the redemption of par value (100) plus the final coupon payment (3).

Time 2 node values:

Upper node: 103/1.027183 = 100.2742

Middle node: 103/1.016487 = 101.3294

Lower node: 103/1.010000 = 101.9802

Working back to Time 1 requires the use of the general expression above.

Time 1 node values:

$$\text{Upper node: } \frac{3 + (0.5 \times 100.2742 + 0.5 \times 101.3294)}{1.028853} = 100.8908$$

$$\text{Lower node: } \frac{3 + (0.5 \times 101.3294 + 0.5 \times 101.9802)}{1.0175} = 102.8548$$

Time 0 node value:

$$\frac{3 + (0.5 \times 100.8908 + 0.5 \times 102.8548)}{1.015} = 103.3230$$

Therefore, the price of the bond is 103.3230.

15 B is correct. Two methods are commonly used to estimate potential interest rate volatility in a binomial interest rate tree. The first method bases estimates on historical interest rate volatility. The second method uses observed market prices of interest rate derivatives.

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Statement 1 is incorrect because there are three requirements to create a binomial interest rate tree, not two. The third requirement is an assumption regarding the interest rate model. Statement 3 is incorrect because the valuation of a bond using spot rates and the valuation of a bond from an interest rate tree will be the same regardless of the volatility assumption used in the model.

16 B is correct. The value of the lower one-period forward rate is closest to 3.5400%, calculated as $0.058365 \times e^{-0.50} = 0.035400$.

17 B is correct. The present value of Bond D's cash flows following Path 2 is 102.8607 and can be calculated as follows:

$$\frac{3}{1.015} + \frac{3}{(1.015)(1.028853)} + \frac{103}{(1.015)(1.028853)(1.016487)} = 102.8607$$

18 A is correct. Increasing the number of paths using the Monte Carlo method does increase the estimate's statistical accuracy. It does not, however, provide a value that is closer to the bond's true fundamental value.

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FIXED INCOME

STUDY SESSION 13

Fixed Income (2)

This study session continues use of the binomial valuation method to value bonds with embedded options. Sensitivity to interest rates and interest rate volatility are key considerations. Option-adjusted spreads are introduced for the evaluation of risky bonds. Credit analysis concepts, tools, and applications are then discussed along with the term structure of credit spreads. The study session concludes with credit default swaps and their use in managing credit exposure.

READING ASSIGNMENTS

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READING

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Valuation and Analysis of Bonds with Embedded Options

by Leslie Abreo, MFE, Ioannis Georgiou, CFA, and Andrew Kalotay, PhD

Leslie Abreo, MFE, is at Andrew Kalotay Associates, Inc. (USA). Ioannis Georgiou, CFA, is at Finovex.com (Cyprus). Andrew Kalotay, PhD, is at Andrew Kalotay Associates, Inc. (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** describe fixed-income securities with embedded options;
☐	**b.** explain the relationships between the values of a callable or putable bond, the underlying option-free (straight) bond, and the embedded option;
☐	**c.** describe how the arbitrage-free framework can be used to value a bond with embedded options;
☐	**d.** explain how interest rate volatility affects the value of a callable or putable bond;
☐	**e.** explain how changes in the level and shape of the yield curve affect the value of a callable or putable bond;
☐	**f.** calculate the value of a callable or putable bond from an interest rate tree;
☐	**g.** explain the calculation and use of option-adjusted spreads;
☐	**h.** explain how interest rate volatility affects option-adjusted spreads;
☐	**i.** calculate and interpret effective duration of a callable or putable bond;
☐	**j.** compare effective durations of callable, putable, and straight bonds;
☐	**k.** describe the use of one-sided durations and key rate durations to evaluate the interest rate sensitivity of bonds with embedded options;
☐	**l.** compare effective convexities of callable, putable, and straight bonds;

(continued)

The presentation of the binomial trees in this reading was revised to conform with other readings in 2018 by Donald J. Smith, PhD, Boston University (USA).

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LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	**m.** calculate the value of a capped or floored floating-rate bond;
☐	**n.** describe defining features of a convertible bond;
☐	**o.** calculate and interpret the components of a convertible bond's value;
☐	**p.** describe how a convertible bond is valued in an arbitrage-free framework;
☐	**q.** compare the risk–return characteristics of a convertible bond with the risk–return characteristics of a straight bond and of the underlying common stock.

1 INTRODUCTION

The valuation of a fixed-rate option-free bond generally requires determining its future cash flows and discounting them at the appropriate rates. Valuation becomes more complicated when a bond has one or more embedded options because the values of embedded options are typically contingent on interest rates.

Understanding how to value and analyze bonds with embedded options is important for practitioners. Issuers of bonds often manage interest rate exposure with embedded options such as call provisions. Investors in callable bonds must appreciate the risk of being called. The perception of this risk is collectively represented by the premium, in terms of increased coupon or yield, that the market demands for callable bonds relative to otherwise identical option-free bonds. Issuers and investors must also understand how other types of embedded options, such as put provisions, conversion options, caps, and floors, affect bond values and the sensitivity of these bonds to interest rate movements.

We begin this reading with a brief overview in Section 2 of various types of embedded options. We then discuss bonds that include a call or put provision. Taking a building-block approach, we show in Section 3 how the arbitrage-free valuation framework discussed in a previous reading can be applied to the valuation of callable and putable bonds, first in the absence of interest rate volatility and then when interest rates fluctuate. We also discuss how option-adjusted spreads are used to value risky callable and putable bonds. Section 4 covers interest rate sensitivity. It highlights the need to use effective duration, including one-sided durations and key rate durations, as well as effective convexity to assess the effect of interest rate movements on the value of callable and putable bonds.

We then turn to bonds that include other familiar types of embedded options. Section 5 focuses on the valuation of capped and floored floating-rate bonds (floaters). Convertible bonds are discussed in Section 6. The valuation of convertible bonds, which are typically callable and may also be putable, is complex because it depends not only on interest rate movements but also on future price movements of the issuer's underlying common stock.

Section 7 briefly highlights the importance of analytic software in bond valuation and analysis. Section 8 summarizes the reading.

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OVERVIEW OF EMBEDDED OPTIONS

2

The term "embedded bond options" or **embedded options** refers to contingency provisions found in the bond's indenture or offering circular. These options represent rights that enable their holders to take advantage of interest rate movements. They can be exercised by the issuer or the bondholder, or they may be exercised automatically depending on the course of interest rates. For example, a call option allows the issuer to benefit from lower interest rates by retiring the bond issue early and refinancing at a lower cost. In contrast, a put option allows the bondholder to benefit from higher interest rates by putting back the bonds to the issuer and reinvesting the proceeds of the retired bond at a higher yield. These options are not independent of the bond and thus cannot be traded separately—hence the adjective "embedded." In this section, we provide a review of familiar embedded options.

Corresponding to every embedded option, or combination of embedded options, is an underlying bond with a specified issuer, issue date, maturity date, principal amount and repayment structure, coupon rate and payment structure, and currency denomination. In this reading, this underlying option-free bond is also referred to as the **straight bond**. The coupon of an underlying bond can be fixed or floating. Fixed-coupon bonds may have a single rate for the life of the bond, or the rate may step up or step down according to a coupon schedule. The coupons of floaters are reset periodically according to a formula based on a reference rate plus a credit spread—for example, six-month Libor + 100 basis points (bps). Except when we discuss capped and floored floaters, this reading focuses on fixed-coupon, single-rate bonds, also referred to as fixed-rate bonds.

2.1 Simple Embedded Options

Call and put options are standard examples of embedded options. In fact, the vast majority of bonds with embedded options are callable, putable, or both. The call provision is by far the most prevalent type of embedded option.

2.1.1 *Call Options*

A **callable bond** is a bond that includes an embedded call option. The call option is an issuer option—that is, the right to exercise the option is at the discretion of the bond's issuer. The call provision allows the issuer to redeem the bond issue prior to maturity. Early redemption usually happens when the issuer has the opportunity to replace a high-coupon bond with another bond that has more favorable terms, typically when interest rates have fallen or when the issuer's credit quality has improved.

Until the 1990s, most long-term corporate bonds in the United States were callable after either five or 10 years. The initial call price (exercise price) was typically at a premium above par, the premium depended on the coupon, and the call price gradually declined to par a few years prior to maturity. Today, most investment-grade corporate bonds are essentially non-refundable. They may have a "make-whole call," so named because the call price is such that the bondholders are more than "made whole" (compensated) in exchange for surrendering their bonds. The call price is calculated at a narrow spread to a benchmark security, usually an on-the-run sovereign bond such as Treasuries in the United States or gilts in the United Kingdom. Thus, economical refunding is virtually out of the question, and investors need have no fear of receiving less than their bonds are worth.

Most callable bonds include a **lockout period** during which the issuer cannot call the bond. For example, a 10-year callable bond may have a lockout period of three years, meaning that the first potential call date is three years after the bond's issue date. Lockout periods may be as short as one month or extend to several years. For

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example, high-yield corporate bonds are often callable a few years after issuance. Holders of such bonds are usually less concerned about early redemption than about possible default. Of course, this perspective can change over the life of the bond—for example, if the issuer's credit quality improves.

Callable bonds include different types of call features. The issuer of a European-style callable bond can only exercise the call option on a single date at the end of the lockout period. An American-style callable bond is continuously callable from the end of the lockout period until the maturity date. A Bermudan-style call option can be exercised only on a predetermined schedule of dates after the end of the lockout period. These dates are specified in the bond's indenture or offering circular.

With a few exceptions, bonds issued by government-sponsored enterprises in the United States (e.g., Fannie Mae, Freddie Mac, Federal Home Loan Banks, and Federal Farm Credit Banks) are callable. These bonds tend to have relatively short maturities (5–10 years) and very short lockout periods (three months to one year). The call price is almost always at 100% of par, and the call option is often Bermudan style.

Tax-exempt municipal bonds (often called "munis"), a type of non-sovereign (local) government bond issued in the United States, are almost always callable at 100% of par any time after the end of the 10th year. They may also be eligible for advance refunding—a highly specialized topic that is not discussed here.

Although the bonds of US government-sponsored enterprises and municipal issuers account for most of the callable bonds issued and traded globally, bonds that include call provisions are also found in other countries in Asia Pacific, Europe, Canada, and Central and South America. The vast majority of callable bonds are denominated in US dollars or euros because of investors' demand for securities issued in these currencies. Australia, the United Kingdom, Japan, and Norway are examples of countries where there is a market for callable bonds denominated in local currency.

2.1.2 *Put Options and Extension Options*

A **putable bond** is a bond that includes an embedded put option. The put option is an investor option—that is, the right to exercise the option is at the discretion of the bondholder. The put provision allows the bondholders to put back the bonds to the issuer prior to maturity, usually at par. This usually happens when interest rates have risen and higher-yielding bonds are available.

Similar to callable bonds, most putable bonds include lockout periods. They can be European or, rarely, Bermudan style, but there are no American-style putable bonds.

Another type of embedded option that resembles a put option is an extension option: At maturity, the holder of an **extendible bond** has the right to keep the bond for a number of years after maturity, possibly with a different coupon. In this case, the terms of the bond's indenture or offering circular are modified, but the bond remains outstanding. Examples of extendible bonds can be found among Canadian issuers such as Royal Bank of Canada, which, as of July 2013, has a 1.125% semi-annual coupon bond outstanding that matures on 22 July 2016 but is extendible to 21 July 2017. We will discuss the resemblance between a putable and an extendible bond in Section 3.5.2.

2.2 Complex Embedded Options

Although callable and putable bonds are the most common types of bonds with embedded options, there are bonds with other types of options or combinations of options.

For instance, a bond can be both callable and putable. For example, as of July 2013, DIC Asset AG, a German corporate issuer, had a 5.875% annual coupon bond outstanding that matured on 16 May 2016. This bond can be either called by the issuer or put by the bondholders.

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Convertible bonds are another type of bond with an embedded option. The conversion option allows bondholders to convert their bonds into the issuer's common stock. Convertible bonds are usually also callable by the issuer; the call provision enables the issuer to take advantage of lower interest rates or to force conversion. We will discuss convertible bonds thoroughly in Section 6.

Another layer of complexity is added when the option is contingent on some particular event. An example is the estate put or survivor's option that may be available to retail investors. For example, as of July 2013, GE Capital, a US corporate issuer, has a 5% semi-annual coupon callable bond outstanding that matures on 15 March 2018. In the event of its holder's death, this bond can be put at par by his or her heirs. Because the estate put comes into play only in the event of the bondholder's death, the value of a bond with an estate put is contingent on the life expectancy of its holder, which is uncertain.

BONDS WITH ESTATE PUTS

Colloquially known as "death-put" bonds, bonds with an estate put or survivor's option can be redeemed at par by the heirs of a deceased bondholder. The bonds should be put only if they sell at a discount—that is, if the prevailing price is below par. Otherwise, they should be sold in the market at a premium.

There is usually a ceiling on the principal amount of the bond the issuer is required to accept in a given year, such as 1% of the original principal amount. Estates giving notice of a put that would result in exceeding this ceiling go into a queue in chronological order.

The value of the estate put depends on the bondholder's life expectancy. The shorter the life expectancy, the greater the value of the estate put. A complicating factor is that most bonds with an estate put are also callable, usually at par and within five years of the issue date. If the issuer calls the bond early, the estate put is extinguished. Needless to say, valuing a callable bond with an estate put requires specialized tools. The key concept to keep in mind is that the value of such a bond depends not only on interest rate movements, like any bond with an embedded option, but also on the investor's life expectancy.

Bonds may contain several interrelated issuer options without any investor option. A prime example is a **sinking fund bond** (sinker), which requires the issuer to set aside funds over time to retire the bond issue, thus reducing credit risk. Such a bond may be callable and may also include options unique to sinking fund bonds, such as an acceleration provision and a delivery option.

SINKING FUND BONDS

The underlying bond has an amortizing structure—for example, a 30-year maturity with level annual principal repayments beginning at the end of the 11th year. In this case, each payment is 5% of the original principal amount. A typical sinking fund bond may include the following options:

- A standard *call option* above par, with declining premiums, starting at the end of Year 10. Thus, the entire bond issue could be called from Year 10 onward.
- An *acceleration provision*, such as a "triple up." Such a provision allows the issuer to repurchase at par three times the mandatory amount, or in this case 15% of the original principal amount, on any scheduled sinking fund date. Assume that the issuer wants to retire the bonds at the end of Year 11. Instead of calling the entire

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outstanding amount at a premium, it would be more cost effective to "sink" 15% at par and call the rest at a premium. Thus, the acceleration provision provides an additional benefit to the issuer if interest rates decline.

- A *delivery option*, which allows the issuer to satisfy a sinking fund payment by delivering bonds to the bond's trustee in lieu of cash.[1] If the bonds are currently trading below par, say at 90% of par, it is more cost effective for the issuer to buy back bonds from investors to meet the sinking fund requirements than to pay par. The delivery option benefits the issuer if interest rates rise. Of course, the benefit can be materialized only if there is a liquid market for the bonds. Investors can take defensive action by accumulating the bonds and refusing to sell them at a discount.

From the issuer's perspective, the combination of the call option and the delivery option is effectively a "long straddle."[2] As a consequence, a sinking fund bond benefits the issuer not only if interest rates decline but also if they rise. Determining the combined value of the underlying bond and the three options is quite challenging.

EXAMPLE 1

Types of Embedded Options

1 Investors in putable bonds *most likely* seek to take advantage of:
   - **A** interest rate movements.
   - **B** changes in the issuer's credit rating.
   - **C** movements in the price of the issuer's common stock.

2 The decision to exercise the option embedded in an extendible bond is made by:
   - **A** the issuer.
   - **B** the bondholder.
   - **C** either the issuer or the bondholder.

3 The conversion option in a convertible bond is a right held by:
   - **A** the issuer.
   - **B** the bondholders.
   - **C** jointly by the issuer and the bondholders.

Solution to 1:

A is correct. A putable bond offers the bondholder the ability to take advantage of a rise in interest rates by putting back the bond to the issuer and reinvesting the proceeds of the retired bond in a higher-yielding bond.

1 A bond's trustee is typically a financial institution with trust powers. It is appointed by the issuer, but it acts in a fiduciary capacity with the bondholders. In public offerings, it is the trustee that determines, usually by lot, which bonds are to be retired.

2 A long straddle is an option strategy involving the purchase of a put option and a call option on the same underlying with the same exercise price and expiration date. At expiration, if the underlying price is above the exercise price, the put option is worthless but the call option is in the money. In contrast, if the underlying price is below the exercise price, the call option is worthless but the put option is in the money. Thus, a long straddle benefits the investor when the underlying price moves up or down. The greater the move up or down (i.e., the greater the volatility), the greater the benefit for the investor.

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Solution to 2:

B is correct. An extendible bond includes an extension option that gives the bondholder the right to keep the bond for a number of years after maturity, possibly with a different coupon.

Solution to 3:

B is correct. A conversion option is a call option that gives the bondholders the right to convert their bonds into the issuer's common stock.

The presence of embedded options affects a bond's value. To quantify this effect, financial theory and financial technology come into play. The following section presents basic valuation and analysis concepts for bonds with embedded options.

3 VALUATION AND ANALYSIS OF CALLABLE AND PUTABLE BONDS

Under the arbitrage-free framework, the value of a bond with embedded options is equal to the sum of the arbitrage-free values of its parts. We first identify the relationships between the values of a callable or putable bond, the underlying option-free (straight) bond, and the call or put option, and then discuss how to value callable and putable bonds under different risk and interest rate volatility scenarios.

3.1 Relationships between the Values of a Callable or Putable Bond, Straight Bond, and Embedded Option

The value of a bond with embedded options is equal to the sum of the arbitrage-free value of the straight bond and the arbitrage-free values of the embedded options.

For a callable bond, the decision to exercise the call option is made by the issuer. Thus, the investor is long the bond but short the call option. From the investor's perspective, therefore, the value of the call option *decreases* the value of the callable bond relative to the value of the straight bond.

Value of callable bond = Value of straight bond – Value of issuer call option

The value of the straight bond can be obtained by discounting the bond's future cash flows at the appropriate rates, as described in Section 3.2. The hard part is valuing the call option because its value is contingent on future interest rates—specifically, the issuer's decision to call the bond depends on its ability to refinance at a lower cost. In practice, the value of the call option is often calculated as the difference between the value of the straight bond and the value of the callable bond:

Value of issuer call option
= Value of straight bond – Value of callable bond (1)

For a putable bond, the decision to exercise the put option is made by the investor. Thus, the investor has a long position in both the bond and the put option. As a consequence, the value of the put option *increases* the value of the putable bond relative to the value of the straight bond.

Value of putable bond = Value of straight bond + Value of investor put option

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It follows that

Value of investor put option
= Value of putable bond − Value of straight bond **(2)**

Although most investment professionals do not need to be experts in bond valuation, they should have a solid understanding of the basic analytical approach, presented in the following sections.

3.2 Valuation of Default-Free and Option-Free Bonds: A Refresher

An asset's value is the present value of the cash flows the asset is expected to generate in the future. In the case of a default-free and option-free bond, the future cash flows are, by definition, certain. Thus, the question is, at which rates should these cash flows be discounted? The answer is that each cash flow should be discounted at the spot rate corresponding to the cash flow's payment date. Although spot rates might not be directly observable, they can be inferred from readily available information, usually from the market prices of actively traded on-the-run sovereign bonds of various maturities. These prices can be transformed into spot rates, par rates (i.e., coupon rates of hypothetical bonds of various maturities selling at par), or forward rates. Recall from Level I that spot rates, par rates, and forward rates are equivalent ways of conveying the same information; knowing any one of them is sufficient to determine the others.

Suppose we want to value a three-year 4.25% annual coupon bond. Exhibit 1 provides the equivalent forms of a yield curve with maturities of one, two, and three years.

Exhibit 1 Equivalent Forms of a Yield Curve

Maturity (year)	Par Rate (%)	Spot Rate (%)	One-Year Forward Rate (%)	
1	2.500	2.500	0 years from now	2.500
2	3.000	3.008	1 year from now	3.518
3	3.500	3.524	2 years from now	4.564

We start with the par rates provided in the second column of Exhibit 1. Because we are assuming annual coupons and annual compounding, the one-year spot rate is simply the one-year par rate. The hypothetical one-year par bond implied by the given par rate has a single cash flow of 102.500 (principal plus coupon) in Year 1.[3] In order to have a present value of par, this future cash flow must be divided by 1.025. Thus, the one-year spot rate or discount rate is 2.500%.

A two-year 3.000% par bond has two cash flows: 3 in Year 1 and 103 in Year 2. By definition, the sum of the two discounted cash flows must equal 100. We know that the discount rate appropriate for the first cash flow is the one-year spot rate (2.500%). We now solve the following equation to determine the two-year spot rate (S_2):

$$\frac{3}{(1.025)} + \frac{103}{(1 + S_2)^2} = 100$$

3 In this reading, all cash flows and values are expressed as a percentage of par.

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We can follow a similar approach to determine the three-year spot rate (S_3):

$$\frac{3.500}{(1.02500)} + \frac{3.500}{(1.03008)^2} + \frac{103.500}{(1 + S_3)^3} = 100$$

The one-year forward rates are determined by using indifference equations. Assume an investor has a two-year horizon. She could invest for two years either at the two-year spot rate, or at the one-year spot rate for one year and then reinvest the proceeds at the one-year forward rate one year from now ($F_{1,1}$). The result of investing using either of the two approaches should be the same. Otherwise, there would be an arbitrage opportunity. Thus,

$$(1 + 0.03008)^2 = (1 + 0.02500) \times (1 + F_{1,1})$$

Similarly, the one-year forward rate two years from now ($F_{2,1}$) can be calculated using the following equation:

$$(1 + 0.03524)^3 = (1 + 0.03008)^2 \times (1 + F_{2,1})$$

The three-year 4.25% annual coupon bond can now be valued using the spot rates:[4]

$$\frac{4.25}{(1.02500)} + \frac{4.25}{(1.03008)^2} + \frac{104.25}{(1.03524)^3} = 102.114$$

An equivalent way to value this bond is to discount its cash flows one year at a time using the one-year forward rates:

$$\frac{4.25}{(1.02500)} + \frac{4.25}{(1.02500)(1.03518)} + \frac{104.25}{(1.02500)(1.03518)(1.04564)} = 102.114$$

3.3 Valuation of Default-Free Callable and Putable Bonds in the Absence of Interest Rate Volatility

When valuing bonds with embedded options, the approach relying on one-period forward rates provides a better framework than that relying on the spot rates because we need to know the value of the bond at different points in time in the future to determine whether the embedded option will be exercised at those points in time.

3.3.1 *Valuation of a Callable Bond at Zero Volatility*

Let us apply this framework to the valuation of a Bermudan-style three-year 4.25% annual coupon bond that is callable at par one year and two years from now. The decision to exercise the call option is made by the issuer. Because the issuer borrowed money, it will exercise the call option when the value of the bond's future cash flows is higher than the call price (exercise price). Exhibit 2 shows how to calculate the value of this callable bond using the one-year forward rates calculated in Exhibit 1.

4 The examples in this reading were created in Microsoft Excel. Numbers may differ from the results obtained using a calculator because of rounding.

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Exhibit 2 Valuation of a Default-Free Three-Year 4.25% Annual Coupon Bond Callable at Par One Year and Two Years from Now at Zero Volatility

	Today	Year 1	Year 2	Year 3
Cash Flow		4.250	4.250	104.250
Discount Rate		2.500%	3.518%	4.564%
Value of the Callable Bond	$\frac{100 + 4.250}{1.02500} = 101.707$	$\frac{99.700 + 4.250}{1.03518} = \cancel{100.417}$ Called at 100	$\frac{104.250}{1.04564} = 99.700$ Not called	

We start by discounting the bond's cash flow at maturity (104.250) to Year 2 using the one-year forward rate two years from now (4.564%). The present value at Year 2 of the bond's future cash flows is 99.700. This value is lower than the call price of 100, so a rational borrower will not call the bond at that point in time. Next, we add the cash flow in Year 2 (4.250) to the present value of the bond's future cash flows at Year 2 (99.700) and discount the sum to Year 1 using the one-year forward rate one year from now (3.518%). The present value at Year 1 of the bond's future cash flows is 100.417. Here, a rational borrower will call the bond at 100 because leaving it outstanding would be more expensive than redeeming it. Last, we add the cash flow in Year 1 (4.250) to the present value of the bond's future cash flows at Year 1 (100.000), and we discount the sum to today at 2.500%. The result (101.707) is the value of the callable bond.

We can apply Equation 1 to calculate the value of the call option embedded in this callable bond. The value of the straight bond is the value of the default-free and option-free three-year 4.25% annual coupon bond calculated in Section 3.2 (102.114). Thus,

Value of issuer call option = 102.114 – 101.707 = 0.407

Recall from the earlier discussion about the relationships between the value of a callable bond, straight bond, and call option that the investor is long the bond and short the call option. Thus, the value of the call option decreases the value of the callable bond relative to that of an otherwise identical option-free bond.

3.3.2 *Valuation of a Putable Bond at Zero Volatility*

We now apply this framework to the valuation of a Bermudan-type three-year 4.25% annual coupon bond that is putable at par one year and two years from now. The decision to exercise the put option is made by the investor. Because the investor lent money, he will exercise the put option when the value of the bond's future cash flows is lower than the put price (exercise price). Exhibit 3 shows how to calculate the value of the three-year 4.25% annual coupon bond putable at par one year and two years from today.

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Exhibit 3 Valuation of a Default-Free Three-Year 4.25% Annual Coupon Bond Putable at Par One Year and Two Years from Now at Zero Volatility

	Today	Year 1	Year 2	Year 3
Cash Flow		4.250	4.250	104.250
Discount Rate		2.500%	3.518%	4.564%
Value of the Putable Bond	$\frac{100.707 + 4.250}{1.02500} = 102.397$	$\frac{100 + 4.250}{1.03518} = 100.707$	$\frac{104.250}{1.04564} = \cancel{99.700}$	
		Not put	Put at 100	

We can apply Equation 2 to calculate the value of the put option:

Value of investor put option = 102.397 − 102.114 = 0.283

Because the investor is long the bond and the put option, the value of the put option increases the value of the putable bond relative to that of an otherwise identical option-free bond.

OPTIMAL EXERCISE OF OPTIONS

The holder of an embedded bond option can extinguish (or possibly modify the terms of) the bond. Assuming that the option is currently exercisable, the obvious question is, does it pay to exercise? Assuming that the answer is affirmative, the follow-up question is whether it is better to exercise the option at present or to wait.

Let us consider the first question: Would it be profitable to exercise the option? The answer is usually straightforward: Compare the value of exercising with the value of not exercising. For example, suppose that a bond is currently putable at 100. If the bond's market price is above 100, putting the bond makes no sense because the cash value from selling the bond would exceed 100. In contrast, if the bond's market price is 100, putting the bond should definitely be considered. Note that the market price of the bond cannot be less than 100 because such a situation creates an arbitrage opportunity: Buy the bond below 100 and immediately put it at 100.

The logic of a call decision by the issuer is similar. If a bond's market price is significantly less than the call price, calling is foolish because the bonds could be simply repurchased in the market at a lower price. Alternatively, if the price is very close to the call price, calling may make sense.

Assume that we have determined that exercising the option would be profitable. If the option under consideration is European style, it is obvious that it should in fact be exercised: There is no justification for not doing so. But if it is an American-style or Bermudan-style option, the challenge is to determine whether it is better to act now or to wait for a better opportunity in the future. The problem is that although circumstances may become more favorable, they may also get worse. So, option holders must consider the odds and decide to act or wait, depending on their risk preference.

The approach presented in this reading for valuing bonds with embedded options assumes that the option holders, be they issuers or investors, are risk neutral. They exercise if, and only if, the benefit from exercise exceeds the expected benefit from waiting. In reality, option holders may be risk averse and may exercise early even if the option is worth more alive than dead.

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EXAMPLE 2

Valuation of Default-Free Callable and Putable Bonds

George Cahill, a portfolio manager, has identified three five-year annual coupon bonds issued by a sovereign government. The three bonds have identical characteristics, except that Bond A is an option-free bond, Bond B is callable at par in two years and three years from today, and Bond C is callable and putable at par two years and three years from today.

1 Relative to the value of Bond A, the value of Bond B is:
   A lower.
   B the same.
   C higher.
2 Relative to the value of Bond B, the value of Bond C is:
   A lower.
   B the same.
   C higher.
3 Under a steeply upward-sloping yield curve scenario, Bond C will *most likely*:
   A be called by the issuer.
   B be put by the bondholders.
   C mature without exercise of any of the embedded options.

Solution to 1:

A is correct. Bond B is a callable bond, and Bond A is the underlying option-free (straight) bond. The call option embedded in Bond B is an issuer option that decreases the bond's value for the investor. If interest rates decline, bond prices usually increase, but the price appreciation of Bond B will be capped relative to the price appreciation of Bond A because the issuer will call the bond to refinance at a lower cost.

Solution to 2:

C is correct. Relative to Bond B, Bond C includes a put option. A put option is an investor option that increases the bond's value for the investor. Thus, the value of Bond C is higher than that of Bond B.

Solution to 3:

B is correct. As interest rates rise, bond prices decrease. Thus, the bondholders will have an incentive to exercise the put option so that they can reinvest the proceeds of the retired bond at a higher yield.

Exhibits 2 and 3 show how callable and putable bonds are valued in the absence of interest rate volatility. In real life, however, interest rates do fluctuate. Thus, the option holder must consider possible evolutions of the yield curve over time.

3.4 Effect of Interest Rate Volatility on the Value of Callable and Putable Bonds

In this section, we discuss the effects of interest rate volatility as well as the level and shape of the yield curve on the value of embedded options.

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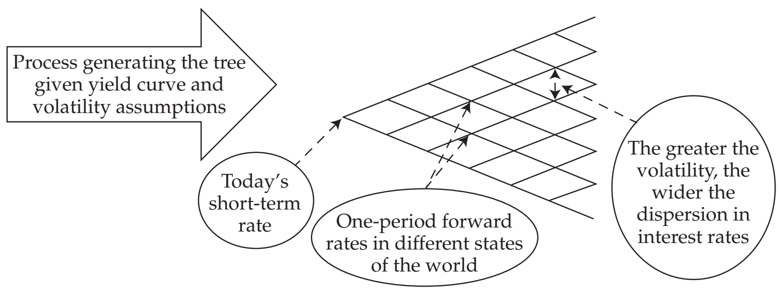
3.4.1 *Interest Rate Volatility*

The value of any embedded option, regardless of the type of option, increases with interest rate volatility. The greater the volatility, the more opportunities exist for the embedded option to be exercised. Thus, it is critical for issuers and investors to understand the effect of interest rate volatility on the value of bonds with embedded options.

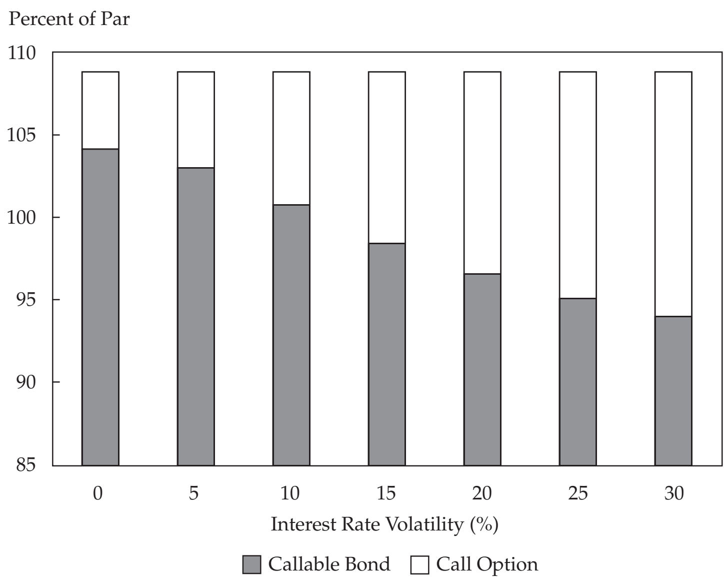
The effect of interest rate volatility is represented in an interest rate tree or lattice, as illustrated in Exhibit 4. From each node on the tree starting from today, interest rates could go up or down. From these two states, interest rates could again go up or down. The dispersion between these up and down states anywhere on the tree is determined by the process generating interest rates based on a given yield curve and interest rate volatility assumptions.

Exhibit 4 Building an Interest Rate Tree



Exhibits 5 and 6 show the effect of interest rate volatility on the value of a callable bond and putable bond, respectively.

Exhibit 5 Value of a 30-Year 4.50% Bond Callable at Par in 10 Years under Different Volatility Scenarios Assuming a 4% Flat Yield Curve



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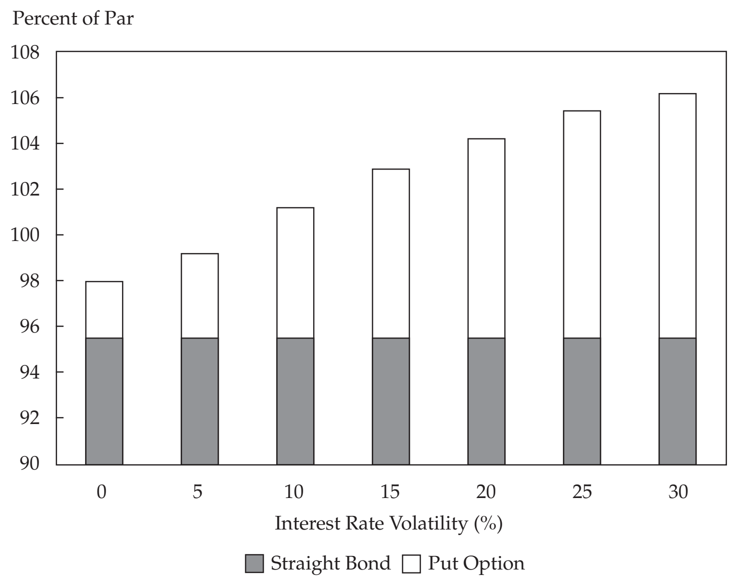
The stacked bars in Exhibit 5 represent the value of the straight bond, which is unaffected by interest rate volatility. The white component is the value of the call option which, when taken away from the value of the straight bond, gives the value of the callable bond—the shaded component. All else being equal, the call option increases in value with interest rate volatility. At zero volatility, the value of the call option is 4.60% of par; at 30% volatility, it is 14.78% of par. Thus, as interest rate volatility increases, the value of the callable bond decreases.

Exhibit 6 Value of a 30-Year 3.75% Bond Putable at Par in 10 Years under Different Volatility Scenarios Assuming a 4% Flat Yield Curve



In Exhibit 6, the shaded component is the value of the straight bond, the white component is the value of the put option, and, thus, the stacked bars represent the value of the putable bond. All else being equal, the put option increases in value with interest rate volatility. At zero volatility, the value of the put option is 2.30% of par; at 30% volatility, it is 10.54% of par. Thus, as interest rate volatility increases, the value of the putable bond increases.

3.4.2 *Level and Shape of the Yield Curve*

The value of a callable or putable bond is also affected by changes in the level and shape of the yield curve.

3.4.2.1 Effect on the Value of a Callable Bond Exhibit 7 shows the value of the same callable bond as in Exhibit 5 under different flat yield curve levels assuming an interest rate volatility of 15%.

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Exhibit 7 Value of a 30-Year 4.50% Bond Callable at Par in 10 Years under Different Flat Yield Curve Levels at 15% Interest Rate Volatility

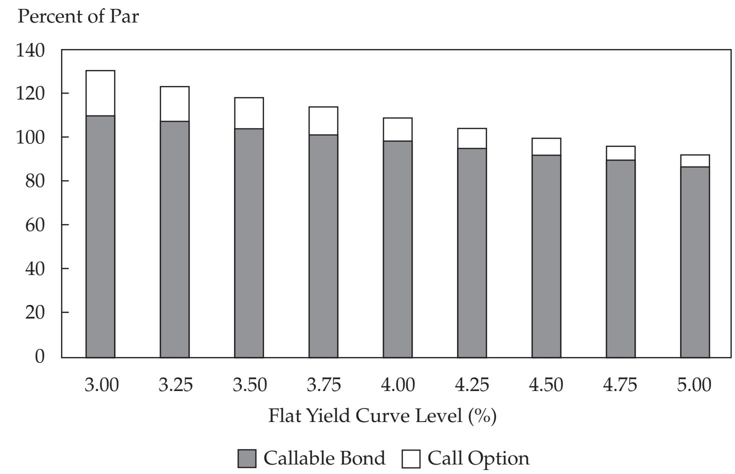
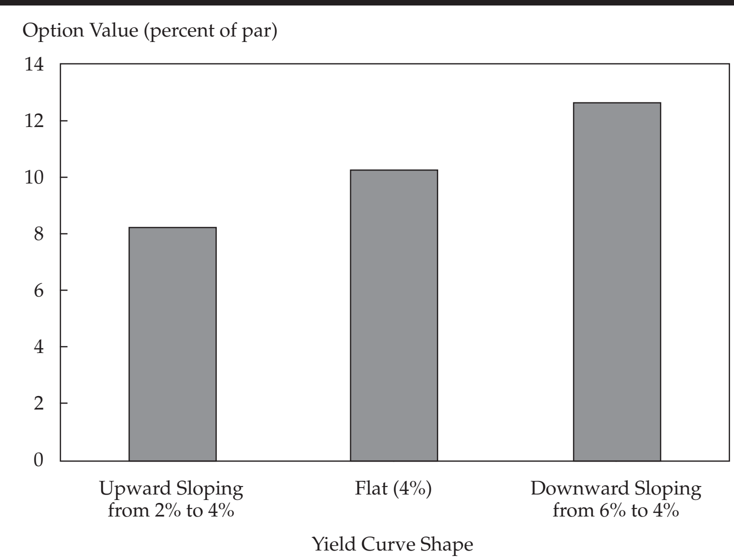


Exhibit 7 shows that as interest rates decline, the value of the straight bond rises, but the rise is partially offset by the increase in the value of the call option. For example, if the yield curve is 5% flat, the value of the straight bond is 92.27% of par and the value of the call option is 5.37% of par, so the value of the callable bond is 86.90% of par. If the yield curve declines to 3% flat, the value of the straight bond rises by 40% to 129.54% of par, but the value of the callable bond only increases by 27% to 110.43% of par. Thus, the value of the callable bond rises less rapidly than the value of the straight bond, limiting the upside potential for the investor.

The value of a call option, and thus the value of a callable bond, is also affected by changes in the shape of the yield curve, as illustrated in Exhibit 8.

Exhibit 8 Value of a Call Option Embedded in a 30-Year 4.50% Bond Callable at Par in 10 Years under Different Yield Curve Shapes at 15% Interest Rate Volatility



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All else being equal, the value of the call option increases as the yield curve flattens. If the yield curve is upward sloping with short-term rates at 2% and long-term rates at 4% (the first bar), the value of the call option represents approximately 8% of par. It rises to approximately 10% of par if the yield curve flattens to 4% (the second bar). The value of the call option increases further if the yield curve actually inverts. Exhibit 8 shows that it exceeds 12% of par if the yield curve is downward sloping with short-term rates at 6% and long-term rates at 4% (the third bar). An inverted yield curve is rare but does happen from time to time.

The intuition to explain the effect of the shape of the yield curve on the value of the call option is as follows. When the yield curve is upward sloping, the one-period forward rates on the interest rate tree are high and opportunities for the issuer to call the bond are fewer. When the yield curve flattens or inverts, many nodes on the tree have lower forward rates, thus increasing the opportunities to call.

Assuming a normal, upward-sloping yield curve at the time of issue, the call option embedded in a callable bond issued at par is out of the money. It would not be called if the arbitrage-free forward rates at zero volatility prevailed. Callable bonds issued at a large premium, as happens frequently in the municipal sector in the United States, are in the money. They will be called if the arbitrage-free forward rates prevail.

3.4.2.2 Effect on the Value of a Putable Bond Exhibits 9 and 10 show how changes in the level and shape of the yield curve affect the value of the putable bond used in Exhibit 6.

Exhibit 9 Value of a 30-Year 3.75% Bond Putable at Par in 10 Years under Different Flat Yield Curve Levels at 15% Interest Rate Volatility

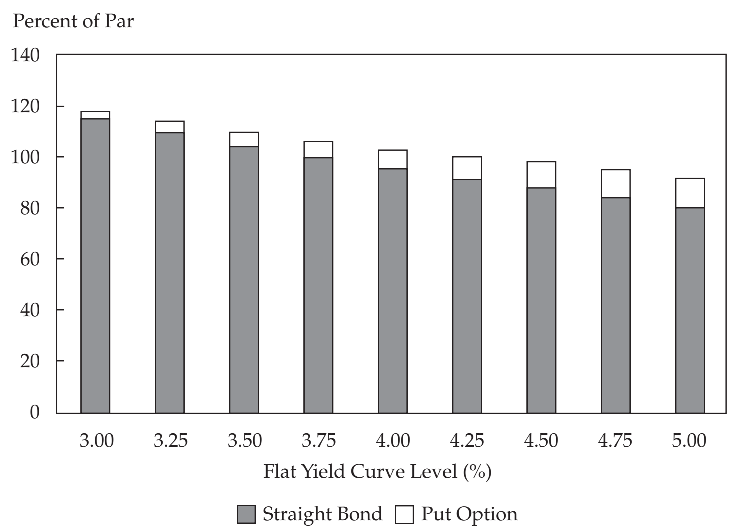

Exhibit 9 illustrates why the put option is considered a hedge against rising interest rates for investors. As interest rates rise, the value of the straight bond declines, but the decline is partially offset by the increase in the value of the put option. For example, if the yield curve moves from 3% flat to 5% flat, the value of the straight bond falls by 30%, but the fall in the value of the putable bond is limited to 22%.

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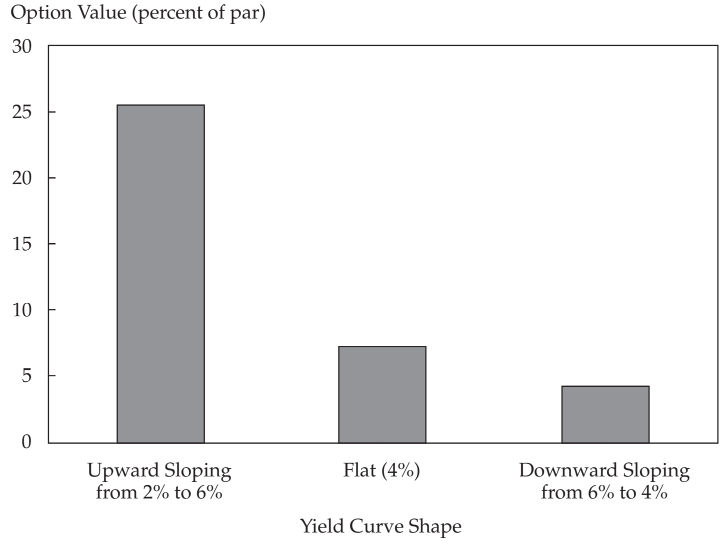
Exhibit 10 Value of the Put Option Embedded in a 30-Year 3.75% Bond Putable at Par in 10 Years under Different Yield Curve Shapes at 15% Interest Rate Volatility



All else being equal, the value of the put option decreases as the yield curve moves from being upward sloping, to flat, to downward sloping. When the yield curve is upward sloping, the one-period forward rates in the interest rate tree are high, which creates more opportunities for the investor to put the bond. As the yield curve flattens or inverts, the number of opportunities declines.

3.5 Valuation of Default-Free Callable and Putable Bonds in the Presence of Interest Rate Volatility

The procedure to value a bond with an embedded option in the presence of interest rate volatility is as follows:

- Generate a tree of interest rates based on the given yield curve and interest rate volatility assumptions.
- At each node of the tree, determine whether the embedded options will be exercised.
- Apply the backward induction valuation methodology to calculate the bond's present value. This methodology involves starting at maturity and working back from right to left to find the bond's present value.

Let us return to the default-free three-year 4.25% annual coupon bonds discussed in Sections 3.3.1 (callable) and 3.3.2 (putable) to illustrate how to apply this valuation procedure. The bonds' characteristics are identical. The yield curve given in Exhibit 1 remains the same with one-year, two-year, and three-year par yields of 2.500%, 3.000%, and 3.500%, respectively. But we now assume an interest rate volatility of 10% instead of 0%. The resulting binomial interest rate tree showing the one-year forward rates zero, one, and two years from now is shown in Exhibit 11. The branching from each node to an up state and a down state is assumed to occur with equal probability.

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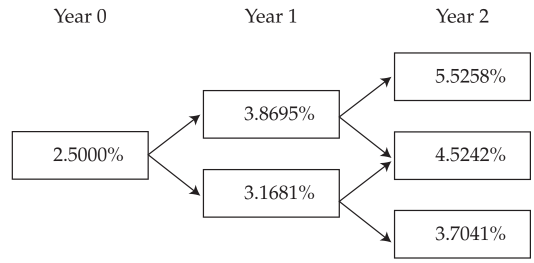
Exhibit 11 Binomial Interest Rate Tree at 10% Interest Rate Volatility



The calibration of a binomial interest rate tree was discussed in a previous reading. As mentioned before, the one-year par rate, the one-year spot rate, and the one-year forward rate zero years from now are identical (2.500%). Because there is no closed-form solution, the one-year forward rates one year from now in the two states are determined iteratively by meeting the following two constraints:

1 The rate in the up state (R_u) is given by

$$R_u = R_d \times e^{2\sigma\sqrt{t}}$$

where R_d is the rate in the down state, σ is the interest rate volatility (10% here), and t is the time in years between "time slices" (a year, so here t = 1).

2 The discounted value of a two-year par bond (bearing a 3.000% coupon rate in this example) equals 100.

In Exhibit 11, at the one-year time slice, R_d is 3.1681% and R_u is 3.8695%. Having established the rates that correctly value the one-year and two-year par bonds implied by the given par yield curve, we freeze these rates and proceed to iterate the rates in the next time slice to determine the one-year forward rates in the three states two years from now. The same constraints as before apply—that is, (1) each rate must be related to its neighbor by the factor $e^{2\sigma\sqrt{t}}$, and (2) the rates must discount a three-year par bond (bearing a 3.500% coupon rate in this example) to a value of 100.

Now that we have determined all the one-year forward rates, we can value the three-year 4.25% annual coupon bonds that are either callable or putable at par one year and two years from now.

3.5.1 *Valuation of a Callable Bond with Interest Rate Volatility*

Exhibit 12 depicts the valuation of a callable bond at 10% volatility.

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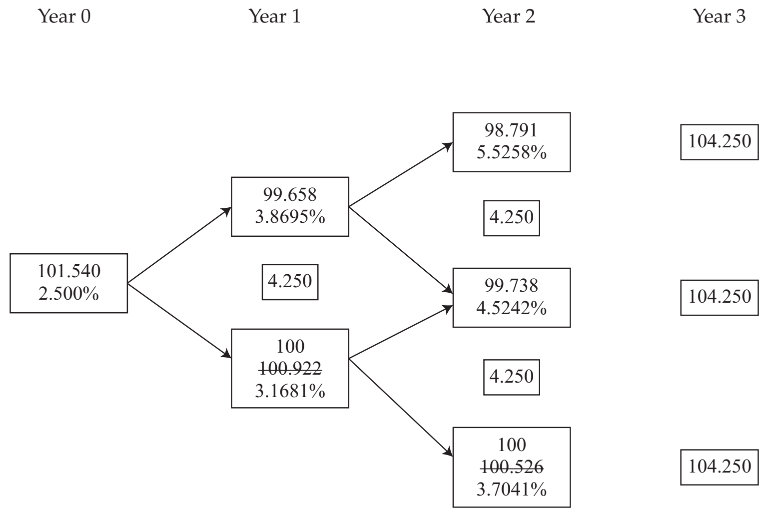
Exhibit 12 Valuation of a Default-Free Three-Year 4.25% Annual Coupon Bond Callable at Par One Year and Two Years from Now at 10% Interest Rate Volatility



The coupon and principal cash flows are placed directly to the right of the interest rate nodes. The calculated bond values at each node are placed above the interest rate. We start by calculating the bond values at Year 2 by discounting the cash flow for Year 3 with the three possible rates.

$$98.791 = \frac{104.250}{1.055258}$$

$$99.738 = \frac{104.250}{1.045242}$$

$$100.526 = \frac{104.250}{1.037041}$$

Because the bond is callable at par in Year 2, we check each scenario to determine whether the present value of the future cash flows is higher than the call price, in which case the issuer calls the bond. Exercise happens only at the bottom of the tree where the rate is 3.7041% and so we reset the value from 100.526 to 100 in that state.

The value in each state of Year 1 is calculated by discounting the values in the two future states emanating from the present state plus the coupon at the appropriate rate in the present state.

$$99.658 = \frac{4.250 + (0.5 \times 98.791 + 0.5 \times 99.738)}{1.038695}$$

The first term in the numerator is the coupon payment and the second term is the expected bond value due at Year 2. In this model the probabilities for moving to the higher and lower node are the same (0.5).

$$100.922 = \frac{4.250 + (0.5 \times 99.738 + 0.5 \times 100)}{1.031681}$$

Notice that the reset value of 100 is used to get the expected bond value. Once again the bond will be callable at the lower node where the interest rate is 3.1681%.

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At Year 0, the value of the callable bond is 101.540.

$$101.540 = \frac{4.250 + (0.5 \times 99.658 + 0.5 \times 100)}{1.025000}$$

The value of the call option, obtained by taking the difference between the value of the straight bond and the value of the callable bond, is now 0.574 (102.114 – 101.540). The fact that the value of the call option is larger at 10% volatility than at 0% volatility (0.407) is consistent with our earlier discussion that option value increases with interest rate volatility.

EXAMPLE 3

Valuation of a Callable Bond Assuming Interest Rate Volatility

Return to the valuation of the Bermudan-style three-year 4.25% annual coupon bond callable at par in one year and two years from now as depicted in Exhibit 12. The one-year, two-year, and three-year par yields are 2.500%, 3.000%, and 3.500%, respectively, and the interest rate volatility is 10%.

1 Assume that nothing changes relative to the initial setting except that the interest rate volatility is now 15% instead of 10%. The new value of the callable bond is:

  A less than 101.540.

  B equal to 101.540.

  C more than 101.540.

2 Assume that nothing changes relative to the initial setting except that the bond is now callable at 102 instead of 100. The new value of the callable bond is *closest to*:

  A 100.000.

  B 102.000.

  C 102.114.

Solution to 1:

A is correct. A higher interest rate volatility increases the value of the call option. Because the value of the call option is subtracted from the value of the straight bond to obtain the value of the callable bond, a higher value for the call option leads to a lower value for the callable bond. Thus, the value of the callable bond at 15% volatility is less than that at 10% volatility—that is, less than 101.540.

Solution to 2:

C is correct. Looking at Exhibit 12, the call price is too high for the call option to be exercised in any scenario. Thus, the value of the call option is zero, and the value of the callable bond is equal to the value of the straight bond—that is, 102.114.

3.5.2 *Valuation of a Putable Bond with Interest Rate Volatility*

The valuation of the three-year 4.25% annual coupon bond putable at par in one year and two years from now at 10% volatility is depicted in Exhibit 13. The procedure for valuing a putable bond is very similar to that described earlier for valuing a callable bond, except that in each state, the bond's value is compared with the put price. The

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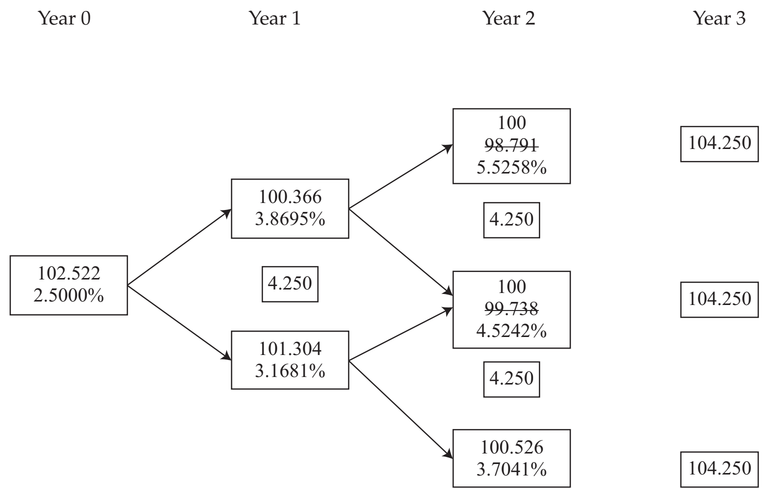
investor puts the bond only when the present value of the bond's future cash flows is lower than the put price. In this case, the value is reset to the put price (100). It happens twice in Year 2, in the states where the interest rates are 5.5258% and 4.5242%. The investor would not exercise the put option in Year 1 because the values for the bond exceed the put price.

Exhibit 13 Valuation of a Default-Free Three-Year 4.25% Annual Coupon Bond Putable at Par One Year and Two Years from Now at 10% Interest Rate Volatility



The value of the putable bond is 102.522. The value of the put option, obtained by taking the difference between the value of the putable bond and the value of the straight bond, is now 0.408 (102.522 – 102.114). As expected, the value of the put option is larger at 10% volatility than at 0% volatility (0.283).

EXAMPLE 4

Valuation of a Putable Bond Assuming Interest Rate Volatility

Return to the valuation of the Bermudan-style three-year 4.25% annual coupon bond putable at par in one year and two years from now, as depicted in Exhibit 13. The one-year, two-year, and three-year par yields are 2.500%, 3.000%, and 3.500%, respectively, and the interest rate volatility is 10%.

1 Assume that nothing changes relative to the initial setting except that the interest rate volatility is now 20% instead of 10%. The new value of the putable bond is:

   A less than 102.522.

   B equal to 102.522.

   C more than 102.522.

2 Assume that nothing changes relative to the initial setting except that the bond is now putable at 95 instead of 100. The new value of the putable bond is *closest to*:

   A 97.522.

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B 102.114.

C 107.522.

Solution to 1:

C is correct. A higher interest rate volatility increases the value of the put option. Because the value of the put option is added to the value of the straight bond to obtain the value of the putable bond, a higher value for the put option leads to a higher value for the putable bond. Thus, the value of the putable bond at 20% volatility is more than that at 10% volatility—that is, more than 102.522.

Solution to 2:

B is correct. Looking at Exhibit 13, the put price is too low for the put option to be exercised in any scenario. Thus, the value of the put option is zero, and the value of the putable bond is equal to the value of the straight bond—that is, 102.114.

PUTABLE VS. EXTENDIBLE BONDS

Putable and extendible bonds are equivalent, except that their underlying option-free bonds are different. Consider a three-year 3.30% bond putable in Year 2. Its value should be exactly the same as that of a two-year 3.30% bond extendible by one year. Otherwise, there would be an arbitrage opportunity. Clearly, the cash flows of the two bonds are identical up to Year 2. The cash flows in Year 3 are dependent on the one-year forward rate two years from now. These cash flows will also be the same for both bonds regardless of the level of interest rates at the end of Year 2.

If the one-year forward rate at the end of Year 2 is higher than 3.30%, the putable bond will be put because the bondholder can reinvest the proceeds of the retired bond at a higher yield, and the extendible bond will not be extended for the same reason. So, both bonds pay 3.30% for two years and are then redeemed. Alternatively, if the one-year forward rate at the end of Year 2 is lower than 3.30%, the putable bond will not be put because the bondholder would not want to reinvest at a lower yield, and the extendible bond will be extended to hold onto the higher interest rate. Thus, both bonds pay 3.30% for three years and are then redeemed.

EXAMPLE 5

Valuation of Bonds with Embedded Options Assuming Interest Rate Volatility

Sidley Brown, a fixed income associate at KMR Capital, is analyzing the effect of interest rate volatility on the values of callable and putable bonds issued by Weather Analytics (WA). WA is owned by the sovereign government, so its bonds are considered default free. Brown is currently looking at three of WA's bonds and has gathered the following information about them:

Characteristic	Bond X	Bond Y	Bond Z
Times to maturity	Three years from today	Three years from today	Three years from today
Coupon	5.2% annual	Not available	4.8% annual

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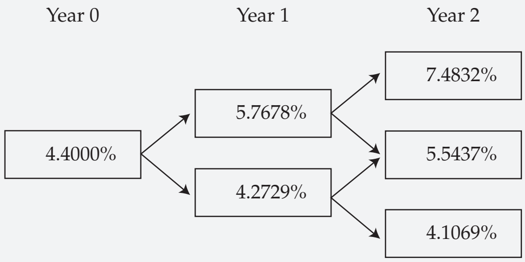
Characteristic	Bond X	Bond Y	Bond Z
Type of bond	Callable at par one year and two years from today	Callable at par one year and two years from today	Putable at par two years from today
Price (as a % of par)	Not available	101.325	Not available

The one-year, two-year, and three-year par rates are 4.400%, 4.700%, and 5.000%, respectively. Based on an estimated interest rate volatility of 15%, Brown has constructed the following binomial interest rate tree:



1 The price of Bond X is *closest to*:

A 96.057% of par.

B 99.954% of par.

C 100.547% of par.

2 The coupon rate of Bond Y is *closest to*:

A 4.200%.

B 5.000%.

C 6.000%.

3 The price of Bond Z is *closest to*:

A 99.638% of par.

B 100.340% of par.

C 100.778% of par.

Brown is now analyzing the effect of interest rate volatility on the price of WA's bonds.

4 Relative to its price at 15% interest rate volatility, the price of Bond X at a lower interest rate volatility will be:

A lower.

B the same.

C higher.

5 Relative to its price at 15% interest rate volatility, the price of Bond Z at a higher interest rate volatility will be:

A lower.

B the same.

C higher.

Solution to 1:

B is correct.

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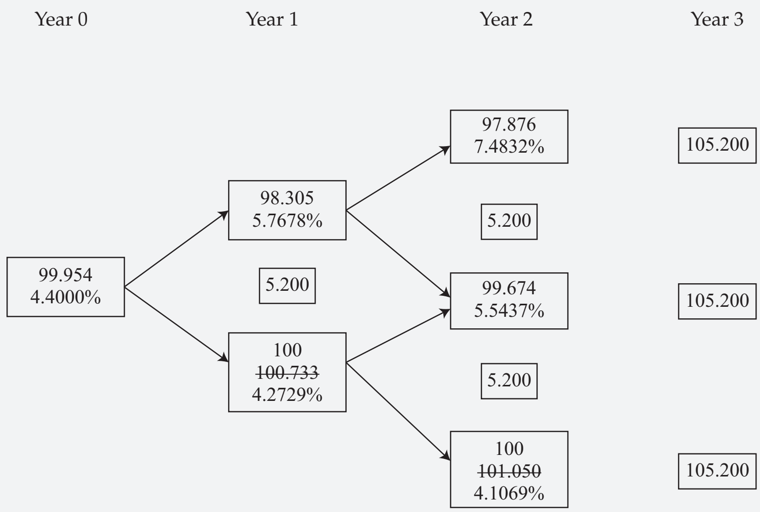
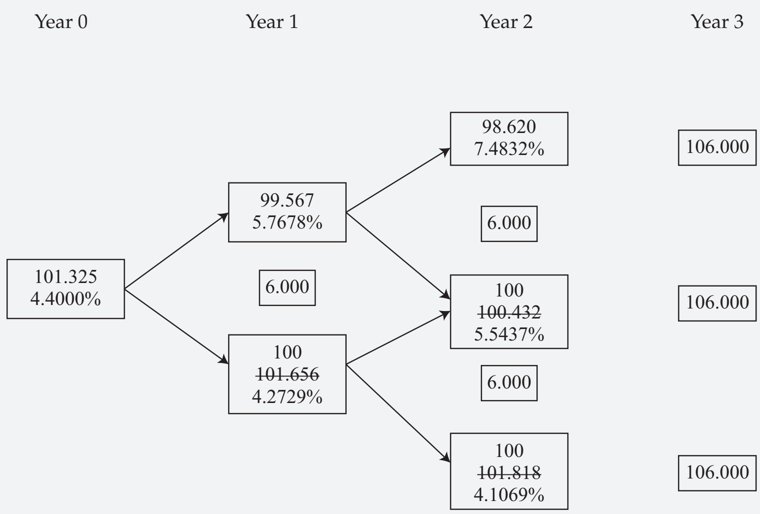


Solution to 2:

C is correct.



Although the correct answer can be found by using the interest rate tree depicted, it is possible to identify it by realizing that the other two answers are clearly incorrect. The three-year 5% straight bond is worth par given that the three-year par rate is 5%. Because the presence of a call option reduces the price of a callable bond, a three-year 5% bond callable at par can only be worth less than par, and certainly less than 101.325 given the yield curve and interest rate volatility assumptions, so B is incorrect. The value of a bond with a coupon rate of 4% is even less, so A is incorrect. Thus, C must be the correct answer.

Solution to 3:

B is correct.

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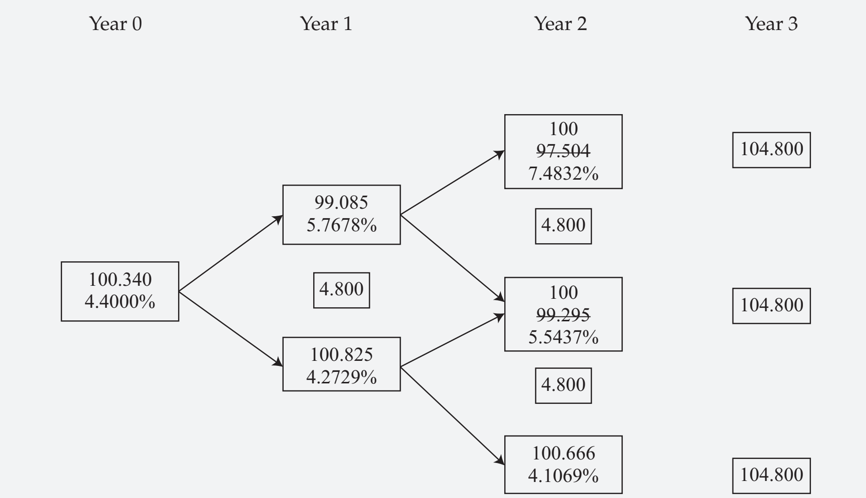

Solution to 4:

C is correct. Bond X is a callable bond. As shown in Equation 1, the value of the call option decreases the value of Bond X relative to the value of the underlying option-free bond. As interest rate volatility decreases, the value of the call option decreases, and thus the value of Bond X increases.

Solution to 5:

C is correct. Bond Z is a putable bond. As shown in Equation 2, the value of the put option increases the value of Bond Z relative to the value of the underlying option-free bond. As interest rate volatility increases, the value of the put option increases, and thus the value of Bond Z increases.

3.6 Valuation of Risky Callable and Putable Bonds

Although the approach described earlier for default-free bonds may apply to securities issued by sovereign governments in their local currency, the fact is that most bonds are subject to default. Accordingly, we have to extend the framework to the valuation of risky bonds.

There are two distinct approaches to valuing bonds that are subject to default risk. The industry-standard approach is to increase the discount rates above the default-free rates to reflect default risk. Higher discount rates imply lower present values, and thus the value of a risky bond will be lower than that of an otherwise identical default-free bond. How to obtain an appropriate yield curve for a risky bond is discussed in Section 3.6.1.

The second approach to valuing risky bonds is by making the default probabilities explicit—that is, by assigning a probability to each time period going forward. For example, the probability of default in Year 1 may be 1%; the probability of default in Year 2, conditional on surviving Year 1, may be 1.25%; and so on. This approach requires specifying the recovery value given default (e.g., 40% of par). Information about default probabilities and recovery values may be accessible from credit default swaps. This important topic is covered in another reading.

3.6.1 *Option-Adjusted Spread*

Depending on available information, there are two standard approaches to construct a suitable yield curve for a risky bond. The more satisfactory but less convenient one is to use an issuer-specific curve, which represents the issuer's borrowing rates over the relevant range of maturities. Unfortunately, most bond professionals do not have

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access to such a level of detail. A more convenient and relatively satisfactory alternative is to uniformly raise the one-year forward rates derived from the default-free benchmark yield curve by a fixed spread, which is estimated from the market prices of suitable bonds of similar credit quality. This fixed spread is known as the zero-volatility spread, or Z-spread.

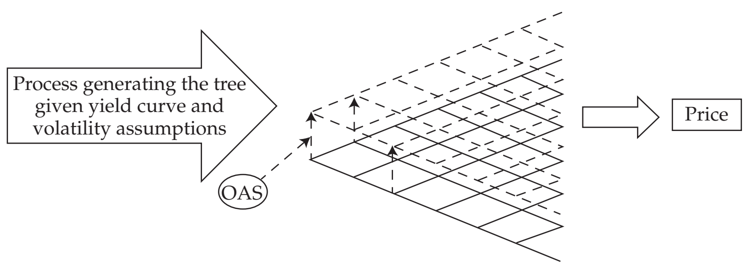
To illustrate, we return to the three-year 4.25% option-free bond introduced in Section 3.2, but we now assume that it is a risky bond and that the appropriate Z-spread is 100 bps. To calculate the arbitrage-free value of this bond, we have to increase each of the one-year forward rates given in Exhibit 1 by the Z-spread of 100 bps:

$$\frac{4.25}{(1.03500)} + \frac{4.25}{(1.03500)(1.04518)} + \frac{104.25}{(1.03500)(1.04518)(1.05564)} = 99.326$$

As expected, the value of this risky bond (99.326) is considerably lower than the value of an otherwise identical but default-free bond (102.114).

The same approach can be applied to the interest rate tree when valuing risky bonds with embedded options. In this case, an **option-adjusted spread** (OAS) is used. As depicted in Exhibit 14, the OAS is the constant spread that, when added to all the one-period forward rates on the interest rate tree, makes the arbitrage-free value of the bond equal to its market price. Note that the Z-spread for an option-free bond is simply its OAS at zero volatility.

Exhibit 14 Interest Rate Tree and OAS



If the bond's price is given, the OAS is determined by trial and error. For example, suppose that the market price of a three-year 4.25% annual coupon bond callable in one year and two years from now, identical to the one valued in Exhibit 12 except that it is risky instead of default-free, is 101.000. To determine the OAS, we try shifting all the one-year forward rates in each state by adding a constant spread. For example, when we add 30 bps to all the one-year forward rates, we obtain a value for the callable bond of 100.973, which is lower than the bond's price. Because of the inverse relationship between a bond's price and its yield, this result means that the discount rates are too high, so we try a slightly lower spread. Adding 28 bps results in a value for the callable bond of 101.010, which is slightly too high. As illustrated in Exhibit 15, the constant spread added uniformly to all the one-period forward rates that justifies the given market price of 101.000 is 28.55 bps; this number is the OAS.

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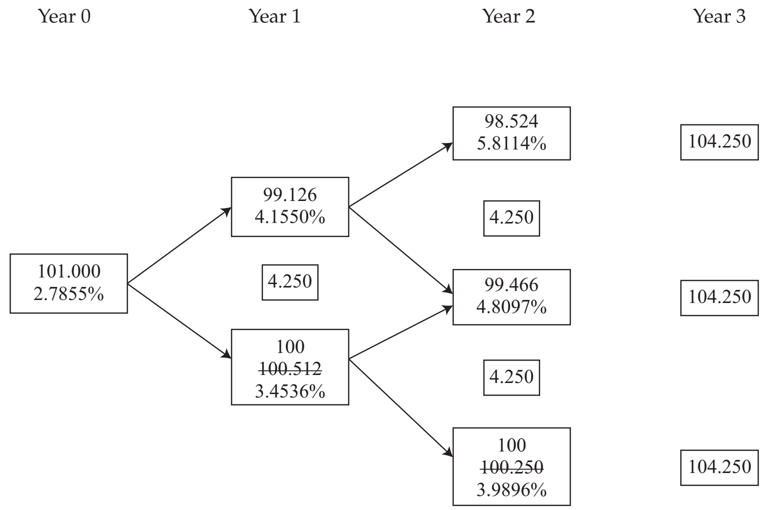
Exhibit 15 OAS of a Risky Three-Year 4.25% Annual Coupon Bond Callable at Par One Year and Two Years from Now at 10% Interest Rate Volatility



As illustrated in Exhibit 15, the value at each node is adjusted based on whether the call option is exercised. Thus, the OAS removes the amount that results from the option risk, which is why this spread is called "option adjusted."

OAS is often used as a measure of value relative to the benchmark. An OAS lower than that for a bond with similar characteristics and credit quality indicates that the bond is likely overpriced (rich) and should be avoided. A larger OAS than that of a bond with similar characteristics and credit quality means that the bond is likely underpriced (cheap). If the OAS is close to that of a bond with similar characteristics and credit quality, the bond looks fairly priced. In our example, the OAS at 10% volatility is 28.55 bps. This number should be compared with the OAS of bonds with similar characteristics and credit quality to make a judgment about the bond's attractiveness.

3.6.2 *Effect of Interest Rate Volatility on Option-Adjusted Spread*

The dispersion of interest rates on the tree is volatility dependent, and so is the OAS. Exhibit 16 shows the effect of volatility on the OAS for a callable bond. The bond is a 5% annual coupon bond with 23 years left to maturity, callable in three years, priced at 95% of par, and valued assuming a flat yield curve of 4%.

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Exhibit 16 Effect of Interest Rate Volatility on the OAS for a Callable Bond

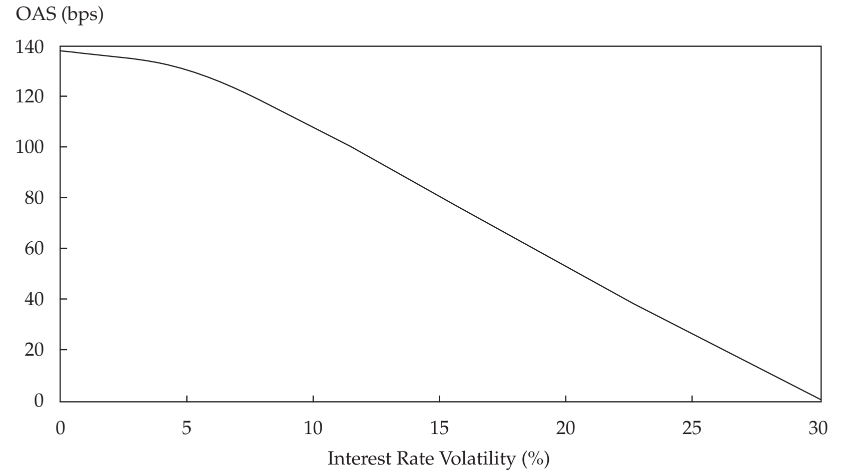
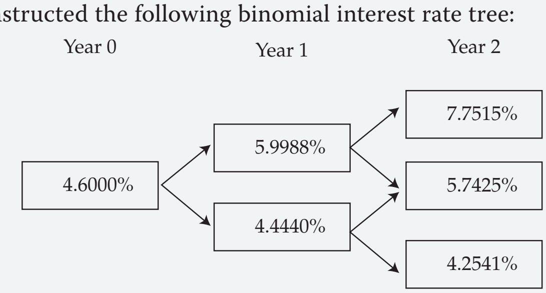


Exhibit 16 shows that as interest rate volatility increases, the OAS for the callable bond decreases. The OAS drops from 138.2 bps at 0% volatility to 1.2 bps at 30% volatility. This exhibit clearly demonstrates the importance of the interest rate volatility assumption. Returning to the example in Exhibit 15, the callable bond may look underpriced at 10% volatility. If an investor assumes a higher volatility, however, the OAS and thus relative cheapness will decrease.

EXAMPLE 6

Option-Adjusted Spread

Robert Jourdan, a portfolio manager, has just valued a 7% annual coupon bond that was issued by a French company and has three years remaining until maturity. The bond is callable at par one year and two years from now. In his valuation, Jourdan used the yield curve based on the on-the-run French government bonds. The one-year, two-year, and three-year par rates are 4.600%, 4.900%, and 5.200%, respectively. Based on an estimated interest rate volatility of 15%, Jourdan constructed the following binomial interest rate tree:



Jourdan valued the callable bond at 102.294% of par. However, Jourdan's colleague points out that because the corporate bond is more risky than French government bonds, the valuation should be performed using an OAS of 200 bps.

1 To update his valuation of the French corporate bond, Jourdan should:

A subtract 200 bps from the bond's annual coupon rate.

B add 200 bps to the rates in the binomial interest rate tree.

C subtract 200 bps from the rates in the binomial interest rate tree.

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2 All else being equal, the value of the callable bond at 15% volatility is *closest to*:

- **A** 99.198% of par.
- **B** 99.247% of par.
- **C** 104.288% of par.

3 Holding the price calculated in the previous question, the OAS for the callable bond at 20% volatility will be:

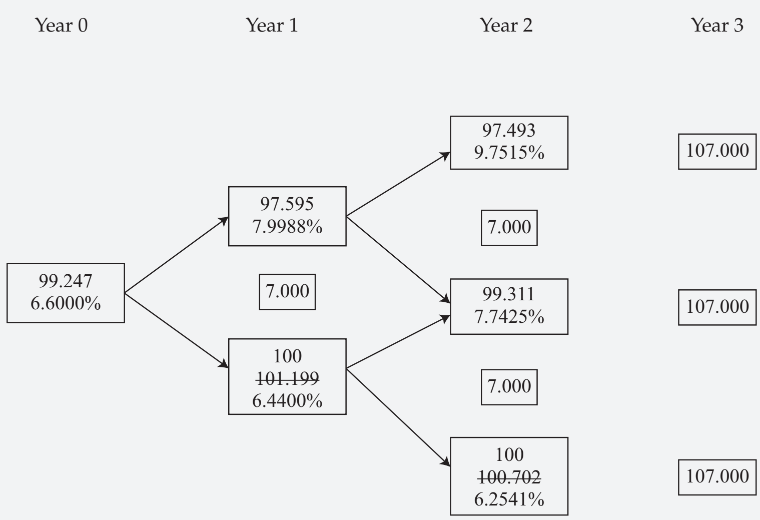
- **A** lower.
- **B** the same.
- **C** higher.

Solution to 1:

B is correct. The OAS is the constant spread that must be *added* to all the one-period forward rates given in the binomial interest rate tree to justify a bond's given market price.

Solution to 2:

B is correct.



Solution to 3:

A is correct. If interest rate volatility increases from 15% to 20%, the OAS for the callable bond will decrease.

SCENARIO ANALYSIS OF BONDS WITH OPTIONS

Another application of valuing bonds with embedded options is scenario analysis over a specified investment horizon. In addition to reinvestment of interest and principal, option valuation comes into play in that callable and putable bonds can be redeemed and their proceeds reinvested during the holding period. Making scenario-dependent, optimal option-exercise decisions involves computationally intensive use of OAS technology because the call or put decision must be evaluated considering the evolution of interest rate scenarios during the holding period.

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Performance over a specified investment horizon entails a trade-off between reinvestment of cash flows and change in the bond's value. Let us take the example of a 4.5% bond with five years left to maturity and assume that the investment horizon is one year. If the bond is option free, higher interest rates increase the reinvestment income but result in lower principal value at the end of the investment horizon. Because the investment horizon is short, reinvestment income is relatively insignificant, and performance will be dominated by the change in the value of the principal. Accordingly, lower interest rates will result in superior performance.

If the bond under consideration is callable, however, it is not at all obvious how the interest rate scenario affects performance. Suppose, for example, that the bond is first callable six months from now and that its current market price is 99.74. Steeply rising interest rates would depress the bond's price, and performance would definitely suffer. But steeply declining interest rates would also be detrimental because the bond would be called and *both interest and principal* would have to be reinvested at lower interest rates. Exhibit 17 shows the return over the one-year investment horizon for the 4.5% bond first callable in six months with five years left to maturity and valued on a 4% flat yield curve.

Exhibit 17 Effect of Interest Rate Changes on a Callable Bond's Total Return

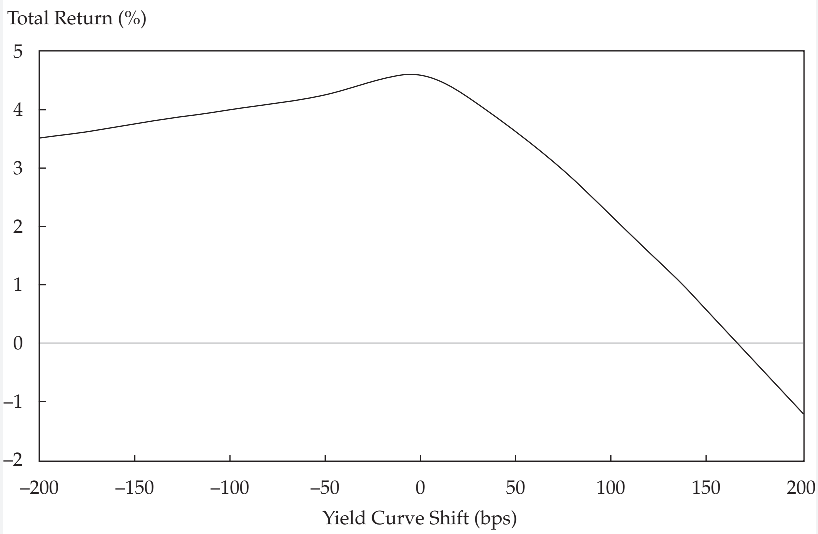

Exhibit 17 clearly shows that lower interest rates do not guarantee higher returns for callable bonds. The point to keep in mind is that the bond may be called long before the end of the investment horizon. Assuming that it is called on the horizon date would overestimate performance. Thus, a realistic prediction of option exercise is essential when performing scenario analysis of bonds with embedded options.

4 INTEREST RATE RISK OF BONDS WITH EMBEDDED OPTIONS

Measuring and managing exposure to interest rate risk are two essential tasks of fixed-income portfolio management. Applications range from hedging a portfolio to asset–liability management of financial institutions. Portfolio managers, whose

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performance is often measured against a benchmark, also need to monitor the interest rate risk of both their portfolio and the benchmark. In this section, we cover two key measures of interest rate risk: duration and convexity.

4.1 Duration

The duration of a bond measures the sensitivity of the bond's full price (including accrued interest) to changes in the bond's yield to maturity (in the case of *yield* duration measures) or to changes in benchmark interest rates (in the case of yield-curve or *curve* duration measures). Yield duration measures, such as modified duration, can be used only for option-free bonds because these measures assume that a bond's expected cash flows do not change when the yield changes. This assumption is in general false for bonds with embedded options because the values of embedded options are typically contingent on interest rates. Thus, for bonds with embedded options, the only appropriate duration measure is the curve duration measure known as effective (or option-adjusted) duration. Because effective duration works for straight bonds as well as for bonds with embedded options, practitioners tend to use it regardless of the type of bond being analyzed.

4.1.1 *Effective Duration*

Effective duration indicates the sensitivity of the bond's price to a 100 bps parallel shift of the benchmark yield curve—in particular, the government par curve—assuming no change in the bond's credit spread.[5] The formula for calculating a bond's effective duration is

$$\text{Effective duration} = \frac{(PV_-) - (PV_+)}{2 \times (\Delta\text{Curve}) \times (PV_0)} \tag{3}$$

where

ΔCurve = the magnitude of the parallel shift in the benchmark yield curve (in decimal);

PV_- = the full price of the bond when the benchmark yield curve is shifted down by ΔCurve;

PV_+ = the full price of the bond when the benchmark yield curve is shifted up by ΔCurve; and

PV_0 = the current full price of the bond (i.e., with no shift).

How is this formula applied in practice? Without a market price, we would need an issuer-specific yield curve to compute PV_0, PV_-, and PV_+. But practitioners usually have access to the bond's current price and thus use the following procedure:

1. Given a price (PV_0), calculate the implied OAS to the benchmark yield curve at an appropriate interest rate volatility.
2. Shift the benchmark yield curve down, generate a new interest rate tree, and then revalue the bond using the OAS calculated in Step 1. This value is PV_-.
3. Shift the benchmark yield curve up by the same magnitude as in Step 2, generate a new interest rate tree, and then revalue the bond using the OAS calculated in Step 1. This value is PV_+.
4. Calculate the bond's effective duration using Equation 3.

5 Although it is possible to explore how arbitrary changes in interest rates affect the bond's price, in practice, the change is usually specified as a parallel shift of the benchmark yield curve.

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Let us illustrate using the same three-year 4.25% bond callable at par one year and two years from now, the same par yield curve (i.e., one-year, two-year, and three-year par yields of 2.500%, 3.000%, and 3.500%, respectively), and the same interest rate volatility (10%) as before. As in Section 3.6, we assume that the bond's current full price is 101.000. We apply the procedure just described:

1 As shown in Exhibit 15, given a price (PV_0) of 101.000, the OAS at 10% volatility is 28.55 bps.
2 We shift the par yield curve down by, say, 30 bps, generate a new interest rate tree, and then revalue the bond at an OAS of 28.55 bps. As shown in Exhibit 18, PV_- is 101.599.
3 We shift the par yield curve up by the same 30 bps, generate a new interest rate tree, and then revalue the bond at an OAS of 28.55 bps. As shown in Exhibit 19, PV_+ is 100.407.
4 Thus,

$$\text{Effective duration} = \frac{101.599 - 100.407}{2 \times 0.0030 \times 101.000} = 1.97$$

An effective duration of 1.97 indicates that a 100-bps increase in interest rate would reduce the value of the three-year 4.25% callable bond by 1.97%.

Exhibit 18 Valuation of a Three-Year 4.25% Annual Coupon Bond Callable at Par One Year and Two Years from Now at 10% Interest Rate Volatility with an OAS of 28.55 bps When Interest Rates Are Shifted Down by 30 bps

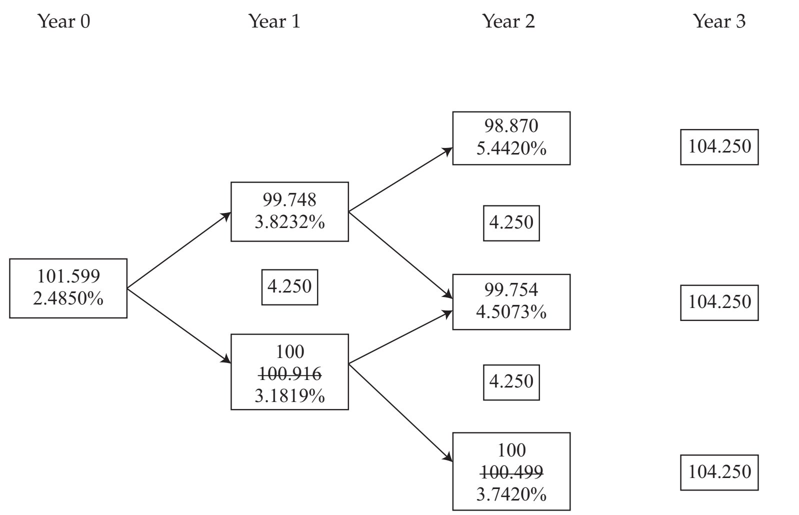

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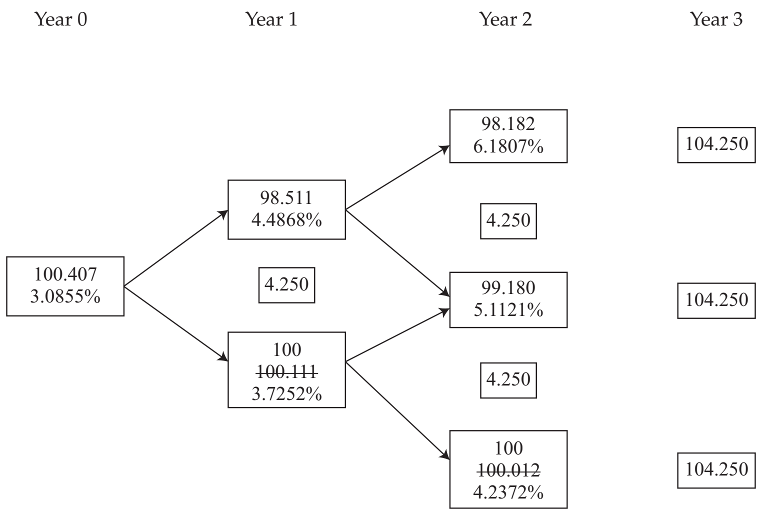
Exhibit 19 Valuation of a Three-Year 4.25% Annual Coupon Bond Callable at Par One Year and Two Years from Now at 10% Interest Rate Volatility with an OAS of 28.55 bps When Interest Rates Shifted Are Shifted Up by 30 bps



The effective duration of a callable bond cannot exceed that of the straight bond. When interest rates are high relative to the bond's coupon, the call option is out of the money, so the bond is unlikely to be called. Thus, the effect of an interest rate change on the price of a callable bond is very similar to that on the price of an otherwise identical option-free bond—the callable and straight bonds have very similar effective durations. In contrast, when interest rates fall, the call option moves into the money. Recall that the call option gives the issuer the right to retire the bond at the call price and thus limits the price appreciation when interest rates decline. As a consequence, the call option reduces the effective duration of the callable bond relative to that of the straight bond.

The effective duration of a putable bond also cannot exceed that of the straight bond. When interest rates are low relative to the bond's coupon, the put option is out of the money, so the bond is unlikely to be put. Thus, the effective duration of the putable bond is in this case very similar to that of an otherwise identical option-free bond. In contrast, when interest rates rise, the put option moves into the money and limits the price depreciation because the investor can put the bond and reinvest the proceeds of the retired bond at a higher yield. Thus, the put option reduces the effective duration of the putable bond relative to that of the straight bond.

When the embedded option (call or put) is deep in the money, the effective duration of the bond with an embedded option resembles that of the straight bond maturing on the first exercise date, reflecting the fact that the bond is highly likely to be called or put on that date.

Exhibit 20 compares the effective durations of option-free, callable, and putable bonds. All bonds are 4% annual coupon bonds with a maturity of 10 years. Both the call option and the put option are European-like and exercisable two months from now. The bonds are valued assuming a 4% flat yield curve and an interest rate volatility of 10%.

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Exhibit 20 Comparison of the Effective Durations of Option-Free, Callable, and Putable Bonds

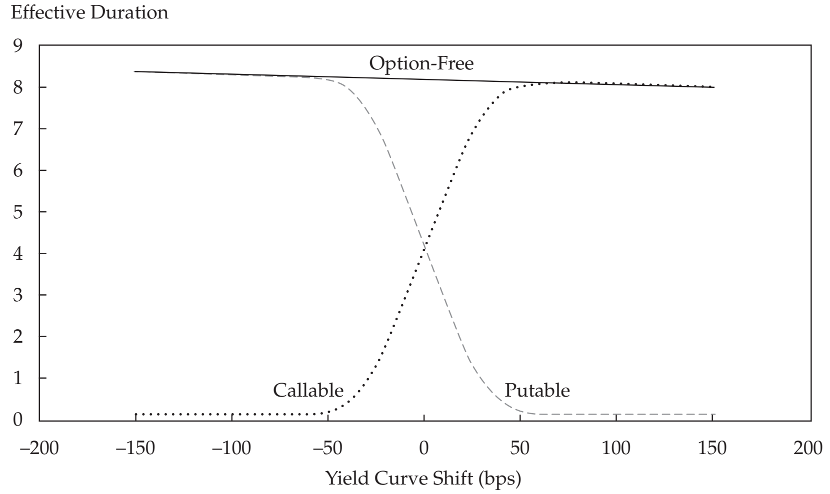

Exhibit 20 shows that the effective duration of an option-free bond changes very little in response to interest rate movements. As expected, when interest rates rise, the put option moves into the money, which limits the price depreciation of the putable bond and shortens its effective duration. In contrast, the effective duration of the callable bond shortens when interest rates fall, which is when the call option moves into the money, limiting the price appreciation of the callable bond.

EFFECTIVE DURATION IN PRACTICE

Effective duration is a concept most practically used in the context of a portfolio. Thus, an understanding of the effective durations of various types of instruments helps manage portfolio duration. In the following table, we show some properties of the effective duration of cash and the common types of bonds:[6]

Type of Bond	Effective Duration
Cash	0
Zero-coupon bond	≈ Maturity
Fixed-rate bond	< Maturity
Callable bond	≤ Duration of straight bond
Putable bond	≤ Duration of straight bond
Floater (Libor flat)	≈ Time (in years) to next reset

In general, a bond's effective duration does not exceed its maturity. There are a few exceptions, however, such as tax-exempt bonds when analyzed on an after-tax basis.

Knowing the effective duration of each type of bond is useful when one needs to change portfolio duration. For example, if a portfolio manager wants to shorten the effective duration of a portfolio of fixed-rate bonds, he or she can add floaters. For the

6 Because the curve shift unit in the denominator of the effective duration formula in Equation 3 is expressed per year, it turns out that the unit of effective duration is in years. In practice, however, effective duration is not viewed as a time measure but as an interest rate risk measure—that is, it reflects the percentage change in price per 100-bps change in interest rates.

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debt manager of a company or other issuing entity, another way of shortening effective duration is to issue callable bonds. The topic of changing portfolio duration is covered thoroughly in Level III.

4.1.2 *One-Sided Durations*

Effective durations are normally calculated by averaging the changes resulting from shifting the benchmark yield curve up and down by the same amount. This calculation works well for option-free bonds but in the presence of embedded options, the results can be misleading. The problem is that when the embedded option is in the money, the price of the bond has limited upside potential if the bond is callable or limited downside potential if the bond is putable. Thus, the price sensitivity of bonds with embedded options is not symmetrical to positive and negative changes in interest rates of the same magnitude.

Consider, for example, a 4.5% bond maturing in five years, which is currently callable at 100. On a 4% flat yield curve at 15% volatility, the value of this callable bond is 99.75. If interest rates declined by 30 bps, the price would rise to 100. In fact, no matter how far interest rates decline, the price of the callable bond cannot exceed 100 because no investor will pay more than the price at which the bond can be immediately called. In contrast, there is no limit to the price decline if interest rates rise. Thus, the average price response to up- and down-shifts of interest rates (effective duration) is not as informative as the price responses to the up-shift (one-sided up-duration) and the down-shift (one-sided down-duration) of interest rates.

Exhibits 21 and 22 illustrate why **one-sided durations**—that is, the effective durations when interest rates go up or down—are better at capturing the interest rate sensitivity of a callable or putable bond than the (two-sided) effective duration, particularly when the embedded option is near the money.

Exhibit 21 Durations for a 4.5% Annual Coupon Bond Maturing in Five Years and Immediately Callable at Par on a 4% Flat Yield Curve at 15% Interest Rate Volatility

	At a 4% Flat Yield Curve	Interest Rate up by 30 bps	Interest Rate down by 30 bps
Value of the Bond	99.75	99.17	100.00
Duration Measure	Effective duration 1.39	One-sided up-duration 1.94	One-sided down-duration 0.84

Exhibit 21 shows that a 30 bps increase in the interest rate has a greater effect on the value of the callable bond than a 30 bps decrease in the interest rate. The fact that the one-sided up-duration is higher than the one-sided down-duration confirms that the callable bond is more sensitive to interest rate rises than to interest rate declines.

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Exhibit 22 Durations for a 4.1% Annual Coupon Bond Maturing in Five Years and Immediately Putable at Par on a 4% Flat Yield Curve at 15% Interest Rate Volatility

	At a 4% Flat Yield Curve	Interest Rate up by 30 bps	Interest Rate down by 30 bps
Value of the Bond	100.45	100.00	101.81
Duration Measure	Effective duration 3.00	One-sided up-duration 1.49	One-sided down-duration 4.51

The one-sided durations in Exhibit 22 indicate that the putable bond is more sensitive to interest rate declines than to interest rate rises.

4.1.3 *Key Rate Durations*

Effective duration is calculated by assuming parallel shifts in the benchmark yield curve. In reality, however, interest rate movements are not as neat. Many portfolio managers and risk managers like to isolate the price responses to changes in the rates of key maturities on the benchmark yield curve. For example, how would the price of a bond be expected to change if only the two-year benchmark rate moved up by 5 bps? The answer is found by using **key rate durations** (also known as partial durations), which reflect the sensitivity of the bond's price to changes in specific maturities on the benchmark yield curve. Thus, key rate durations help portfolio managers and risk managers identify the "shaping risk" for bonds—that is, the bond's sensitivity to changes in the shape of the yield curve (e.g., steepening and flattening).

The valuation procedure and formula applied in the calculation of key rate durations are identical to those used in the calculation of effective duration, but instead of shifting the entire benchmark yield curve, only key points are shifted, one at a time. Thus, the effective duration for each maturity point shift is calculated in isolation.

Exhibits 23, 24, and 25 show the key rate durations for bonds valued at a 4% flat yield curve. Exhibit 23 examines option-free bonds, and Exhibits 24 and 25 extend the analysis to callable and putable bonds, respectively.

Exhibit 23 Key Rate Durations of 10-Year Option-Free Bonds Valued at a 4% Flat Yield Curve

Coupon (%)	Price (% of par)	Key Rate Durations				
		Total	2-Year	3-Year	5-Year	10-Year
0	67.30	9.81	–0.07	–0.34	–0.93	11.15
2	83.65	8.83	–0.03	–0.13	–0.37	9.37
4	100.00	8.18	0.00	0.00	0.00	8.18
6	116.35	7.71	0.02	0.10	0.27	7.32
8	132.70	7.35	0.04	0.17	0.47	6.68
10	149.05	7.07	0.05	0.22	0.62	6.18

As shown in Exhibit 23, for option-free bonds not trading at par (the white rows), shifting any par rate has an effect on the value of the bond, but shifting the maturity-matched (10-year in this example) par rate has the greatest effect. This is simply because the largest cash flow of a fixed-rate bond occurs at maturity with the payment of both the final coupon and the principal.

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For an option-free bond trading at par (the shaded row), the maturity-matched par rate is the only rate that affects the bond's value. It is a definitional consequence of "par" rates. If the 10-year par rate on a curve is 4%, then a 4% 10-year bond valued on that curve at zero OAS will be worth par, regardless of the par rates of the other maturity points on the curve. In other words, shifting any rate other than the 10-year rate on the par yield curve will not change the value of a 10-year bond trading at par. Shifting a par rate up or down at a particular maturity point, however, respectively increases or decreases the *discount rate* at that maturity point. These facts will be useful to remember in the following paragraph.

As illustrated in Exhibit 23, key rate durations can sometimes be negative for maturity points that are shorter than the maturity of the bond being analyzed if the bond is a zero-coupon bond or has a very low coupon. We can explain why this is the case by using the zero-coupon bond (the first row of Exhibit 23). As discussed in the previous paragraph, if we increase the five-year par rate, the value of a 10-year bond trading at par must remain unchanged because the 10-year par rate has not changed. But the five-year zero-coupon rate has increased because of the increase in the five-year par rate. Thus, the value of the five-year coupon of the 10-year bond trading at par will be lower than before the increase. But because the value of the 10-year bond trading at par must remain par, the remaining cash flows, including the cash flow occurring in Year 10, must be discounted at slightly *lower* rates to compensate. This results in a lower 10-year zero-coupon rate, which makes the value of a 10-year zero-coupon bond (whose only cash flow is in Year 10) *rise* in response to an *upward* change in the five-year par rate. Consequently, the five-year key rate duration for a 10-year zero-coupon bond is negative (–0.93).

Unlike for option-free bonds, the key rate durations of bonds with embedded options depend not only on the *time to maturity* but also on the *time to exercise*. Exhibits 24 and 25 illustrate this phenomenon for 30-year callable and putable bonds. Both the call option and the put option are European-like exercisable 10 years from now, and the bonds are valued assuming a 4% flat yield curve and a volatility of 15%.

Exhibit 24 Key Rate Durations of 30-Year Bonds Callable in 10 Years Valued at a 4% Flat Yield Curve with 15% Interest Rate Volatility

Coupon	Price	Key Rate Durations					
(%)	(% of par)	Total	2-Year	3-Year	5-Year	10-Year	30-Year
2	64.99	19.73	–0.02	–0.08	–0.21	–1.97	22.01
4	94.03	13.18	0.00	0.02	0.05	3.57	9.54
6	114.67	9.11	0.02	0.10	0.29	6.00	2.70
8	132.27	7.74	0.04	0.17	0.48	6.40	0.66
10	148.95	7.14	0.05	0.22	0.62	6.06	0.19

The bond with a coupon of 2% (the first row of Exhibit 24) is unlikely to be called, and thus it behaves more like a 30-year option-free bond, whose effective duration depends primarily on movements in the 30-year par rate. Therefore, the rate that has the highest effect on the value of the callable bond is the maturity-matched (30-year) rate. As the bond's coupon increases, however, so does the likelihood of the bond being called. Thus, the bond's total effective duration shortens, and the rate that has the highest effect on the callable bond's value gradually shifts from the 30-year rate to the 10-year rate. At the very high coupon of 10%, because of the virtual certainty of being called, the callable bond behaves like a 10-year option-free bond; the 30-year key rate duration is negligible (0.19) relative to the 10-year key rate duration (6.06).

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Exhibit 25 Key Rate Durations of 30-Year Bonds Putable in 10 Years Valued at a 4% Flat Yield Curve with 15% Interest Rate Volatility

Coupon (%)	Price (% of par)	Key Rate Durations					
		Total	2-Year	3-Year	5-Year	10-Year	30-Year
2	83.89	9.24	–0.03	–0.14	–0.38	8.98	0.81
4	105.97	12.44	0.00	–0.01	–0.05	4.53	7.97
6	136.44	14.75	0.01	0.03	0.08	2.27	12.37
8	169.96	14.90	0.01	0.06	0.16	2.12	12.56
10	204.38	14.65	0.02	0.07	0.21	2.39	11.96

If the 30-year bond putable in 10 years has a high coupon, its price is more sensitive to the 30-year rate because it is unlikely to be put and thus behaves like an otherwise identical option-free bond. The 10% putable bond (the last row of Exhibit 25), for example, is most sensitive to changes in the 30-year rate, as illustrated by a 30-year key rate duration of 11.96. At the other extreme, a low-coupon bond is most sensitive to movements in the 10-year rate. It is almost certain to be put and so behaves like an option-free bond maturing on the put date.

4.2 Effective Convexity

Duration is an approximation of the expected bond price responses to changes in interest rates because actual changes in bond prices are not linear, particularly for bonds with embedded options. Thus, it is useful to measure **effective convexity**—that is, the sensitivity of duration to changes in interest rates—as well. The formula to calculate a bond's effective convexity is

$$\text{Effective convexity} = \frac{(PV_{-}) + (PV_{+}) - \left[2 \times (PV_0)\right]}{(\Delta\text{Curve})^2 \times (PV_0)} \quad (4)$$

where

ΔCurve = the magnitude of the parallel shift in the benchmark yield curve (in decimal);

PV_{-} = the full price of the bond when the benchmark yield curve is shifted down by ΔCurve;

PV_{+} = the full price of the bond when the benchmark yield curve is shifted up by ΔCurve; and

PV_0 = the current full price of the bond (i.e., with no shift).

Let us return to the three-year 4.25% bond callable at par in one year and two years from now. We still use the same par yield curve (i.e., one-year, two-year, and three-year par yields of 2.500%, 3.000%, and 3.500%, respectively) and the same interest rate volatility (10%) as before, but we now assume that the bond's current full price is 100.785 instead of 101.000. Thus, the implied OAS is 40 bps. Given 30 bps shifts in the benchmark yield curve, the resulting PV_{-} and PV_{+} are 101.381 and 100.146, respectively. Using Equation 4, the effective convexity is:

$$\frac{101.381 + 100.146 - 2 \times 100.785}{(0.003)^2 \times 100.785} = -47.41$$

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Exhibit 20 in Section 4.1.1, although displaying effective durations, also illustrates the effective convexities of option-free, callable, and putable bonds. The option-free bond exhibits low positive convexity—that is, the price of an option-free bond rises slightly more when interest rates move down than it declines when interest rates move up by the same amount.

When interest rates are high and the value of the call option is low, the callable and straight bond experience very similar effects from changes in interest rates. They both have positive convexity. However, the effective convexity of the callable bond turns negative when the call option is near the money, as in the example just presented, which indicates that the upside for a callable bond is much smaller than the downside. The reason is because when interest rates decline, the price of the callable bond is capped by the price of the call option if it is near the exercise date.

Conversely, putable bonds always have positive convexity. When the option is near the money, the upside for a putable bond is much larger than the downside because the price of a putable bond is floored by the price of the put option if it is near the exercise date.

Compared side by side, putable bonds have more upside potential than otherwise identical callable bonds when interest rates decline. Putable bonds also have less downside risk than otherwise identical callable bonds when interest rates rise.

EXAMPLE 7

Interest Rate Sensitivity

Erna Smith, a portfolio manager, has two fixed-rate bonds in her portfolio: a callable bond (Bond X) and a putable bond (Bond Y). She wants to examine the interest rate sensitivity of these two bonds to a parallel shift in the benchmark yield curve. Assuming an interest rate volatility of 10%, her valuation software shows how the prices of these bonds change for 30-bps shifts up or down:

	Bond X	Bond Y
Time to maturity	Three years from today	Three years from today
Coupon	3.75% annual	3.75% annual
Type of bond	Callable at par one year from today	Putable at par one year from today
Current price (% of par)	100.594	101.330
Price (% of par) when shifting the benchmark yield curve down by 30 bps	101.194	101.882
Price (% of par) when shifting the benchmark yield curve up by 30 bps	99.860	100.924

1 The effective duration for Bond X is *closest* to:

A 0.67.

B 2.21.

C 4.42.

2 The effective duration for Bond Y is *closest* to:

A 0.48.

B 0.96.

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C 1.58.

3 When interest rates rise, the effective duration of:

A Bond X shortens.

B Bond Y shortens.

C the underlying option-free (straight) bond corresponding to Bond X lengthens.

4 When the option embedded in Bond Y is in the money, the one-sided durations *most likely* show that the bond is:

A more sensitive to a decrease in interest rates.

B more sensitive to an increase in interest rates.

C equally sensitive to a decrease or to an increase in interest rates.

5 The price of Bond X is affected:

A only by a shift in the one-year par rate.

B only by a shift in the three-year par rate.

C by all par rate shifts but is most sensitive to shifts in the one-year and three-year par rates.

6 The effective convexity of Bond X:

A cannot be negative.

B turns negative when the embedded option is near the money.

C turns negative when the embedded option moves out of the money.

7 Which of the following statements is *most* accurate?

A Bond Y exhibits negative convexity.

B For a given decline in interest rate, Bond X has less upside potential than Bond Y.

C The underlying option-free (straight) bond corresponding to Bond Y exhibits negative convexity.

Solution to 1:

B is correct. The effective duration for Bond X is

$$\text{Effective duration} = \frac{101.194 - 99.860}{2 \times 0.003 \times 100.594} = 2.21$$

A is incorrect because the duration of a bond with a single cash flow one year from now is approximately one year, so 0.67 is too low, even assuming that the bond will be called in one year with certainty. C is incorrect because 4.42 exceeds the maturity of Bond X (three years).

Solution to 2:

C is correct. The effective duration for Bond Y is

$$\text{Effective duration} = \frac{101.882 - 100.924}{2 \times 0.003 \times 101.330} = 1.58$$

Solution to 3:

B is correct. When interest rates rise, a put option moves into the money, and the putable bond is more likely to be put. Thus, it behaves like a shorter-maturity bond, and its effective duration shortens. A is incorrect because when interest rates rise, a call option moves out of the money, so the callable bond is less likely to be called. C is incorrect because the effective duration of an option-free bond changes very little in response to interest rate movements.

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Solution to 4:

A is correct. If interest rates rise, the investor's ability to put the bond at par limits the price depreciation. In contrast, there is no limit to the increase in the bond's price when interest rates decline. Thus, the price of a putable bond whose embedded option is in the money is more sensitive to a decrease in interest rates.

Solution to 5:

C is correct. The main driver of the call decision is the two-year forward rate one year from now. This rate is most significantly affected by changes in the one-year and three-year par rates.

Solution to 6:

B is correct. The effective convexity of a callable bond turns negative when the call option is near the money because the price response of a callable bond to lower interest rates is capped by the call option. That is, in case of a decline in interest rates, the issuer will call the bonds and refund at lower rates, thus limiting the upside potential for the investor.

Solution to 7:

B is correct. As interest rates decline, the value of a call option increases whereas the value of a put option decreases. The call option embedded in Bond X limits its price appreciation, but there is no such cap for Bond Y. Thus, Bond X has less upside potential than Bond Y. A is incorrect because a putable bond always has positive convexity—that is, Bond Y has more upside than downside potential. C is incorrect because an option-free bond exhibits low positive convexity.

5 VALUATION AND ANALYSIS OF CAPPED AND FLOORED FLOATING-RATE BONDS

Options in floating-rate bonds (floaters) are exercised automatically depending on the course of interest rates—that is, if the coupon rate rises or falls below the threshold, the cap or floor automatically applies. Similar to callable and putable bonds, capped and floored floaters can be valued by using the arbitrage-free framework.

5.1 Valuation of a Capped Floater

The cap provision in a floater prevents the coupon rate from increasing above a specified maximum rate. As a consequence, a **capped floater** protects the issuer against rising interest rates and is thus an issuer option. Because the investor is long the bond but short the embedded option, the value of the cap decreases the value of the capped floater relative to the value of the straight bond:

$$\text{Value of capped floater} = \text{Value of straight bond} - \text{Value of embedded cap} \quad (5)$$

To illustrate how to value a capped floater, consider a floating-rate bond that has a three-year maturity. The floater's coupon pays the one-year Libor annually, set in arrears, and is capped at 4.500%. The term "set in arrears" means that the coupon rate is set at the *end* of the coupon period—the payment date and the setting date are one and the same. For simplicity, we assume that the issuer's credit quality closely matches the Libor swap curve (i.e., there is no credit spread) and that the Libor swap curve is

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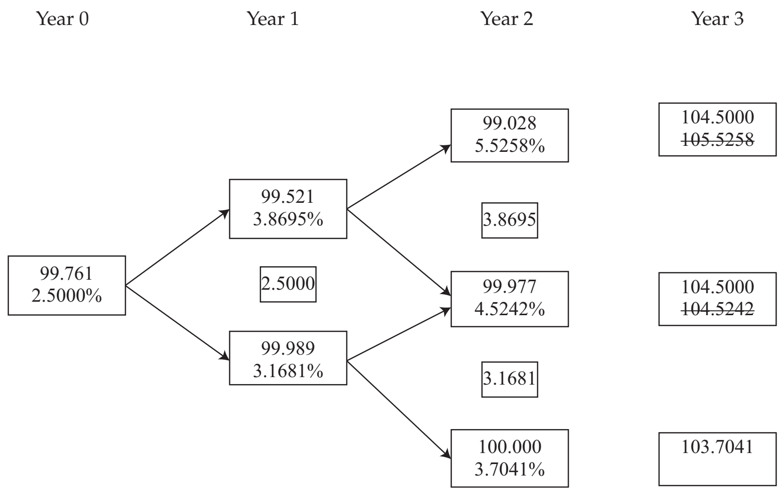
the same as the par yield curve given in Exhibit 1 (i.e., one-year, two-year, and three-year par yields of 2.500%, 3.000%, and 3.500%, respectively). We also assume that the interest rate volatility is 10%.

The valuation of the capped floater is depicted in Exhibit 26.

Exhibit 26 Valuation of a Three-Year Libor Floater Capped at 4.500% at 10% Interest Rate Volatility



Without a cap, the value of this floater would be 100 because in every scenario, the coupon paid would be equal to the discount rate. But because the coupon rate is capped at 4.500%, which is lower than the highest interest rates in the tree, the value of the capped floater will be lower than the value of the straight bond.

For each scenario, we check whether the cap applies, and if it does, the cash flow is adjusted accordingly. For example, at the top of the tree at Year 2, Libor (5.5258%) is higher than the 4.500% cap. Thus, the coupon payment at Year 3 is capped at the 4.500 maximum amount, and the cash flow is adjusted downward from the uncapped amount (105.5258) to the capped amount (104.5000). The coupon is also capped when Libor is 4.5242% at Year 2.

As expected, the value of the capped floater is lower than 100 (99.761). The value of the cap can be calculated by using Equation 5:

Value of embedded cap = 100 – 99.761 = 0.239

RATCHET BONDS: DEBT MANAGEMENT ON AUTOPILOT

Ratchet bonds are floating-rate bonds with both issuer and investor options. As with conventional floaters, the coupon is reset periodically according to a formula based on a reference rate and a credit spread. A capped floater protects the issuer against rising interest rates. Ratchet bonds offer extreme protection: At the time of reset, the coupon can only decline; it can never exceed the existing level. So, over time, the coupon "ratchets down."

The Tennessee Valley Authority (TVA) was the first issuer of ratchet bonds. In 1998, it issued $575 million 6.75% "PARRS" due 1 June 2028. The coupon rate was resettable on 1 June 2003 and annually thereafter. Exhibit 27 shows annual coupon resets since 2003:[7]

7 See Kalotay and Abreo (1999).

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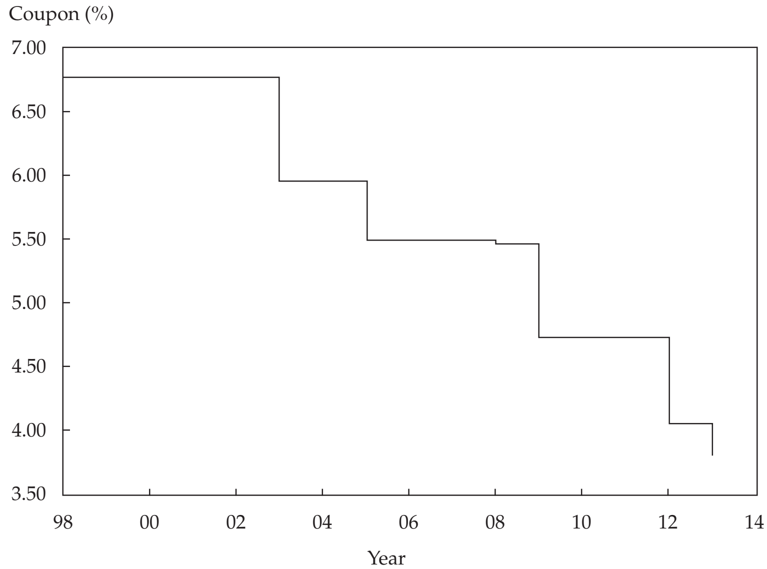
Exhibit 27 TVA Annual Coupon Resets



This ratchet bond has allowed TVA to reduce its borrowing rate by 292 bps without refinancing. You may wonder why anyone would buy such a bond. The answer is that at issuance, the coupon of a ratchet bond is much higher than that of a standard floater. In fact, the initial coupon is set well above the issuer's long-term option-free borrowing rate in order to compensate investors for the potential loss of interest income over time. In this regard, a ratchet bond is similar to a conventional callable bond: When the bond is called, the investor must purchase a replacement in the prevailing lower rate environment. The initial above-market coupon of a callable bond reflects this possibility.

A ratchet bond can be thought of as the lifecycle of a callable bond through several possible calls, in which the bond is replaced by one that is itself callable, to the original maturity. The appeal for the issuer is that these "calls" entail no transaction cost, and the call decision is on autopilot.

Ratchet bonds also contain investor options. Whenever a coupon is reset, the investor has the right to put the bonds back to the issuer at par. The embedded option is called a "contingent put" because the right to put is available to the investor *only* if the coupon is reset. The coupon reset formula of ratchet bonds is designed to assure that the market price at the time of reset is above par, provided that the issuer's credit quality does not deteriorate. Therefore, the contingent put offers investors protection against an adverse credit event. Needless to say, the valuation of a ratchet bond is rather complex.

5.2 Valuation of a Floored Floater

The floor provision in a floater prevents the coupon rate from decreasing below a specified minimum rate. As a consequence, a **floored floater** protects the investor against declining interest rates and is thus an investor option. Because the investor is long both the bond and the embedded option, the value of the floor increases the value of the floored floater relative to the value of the straight bond:

$$\text{Value of floored floater} = \text{Value of straight bond} + \text{Value of embedded floor} \quad (6)$$

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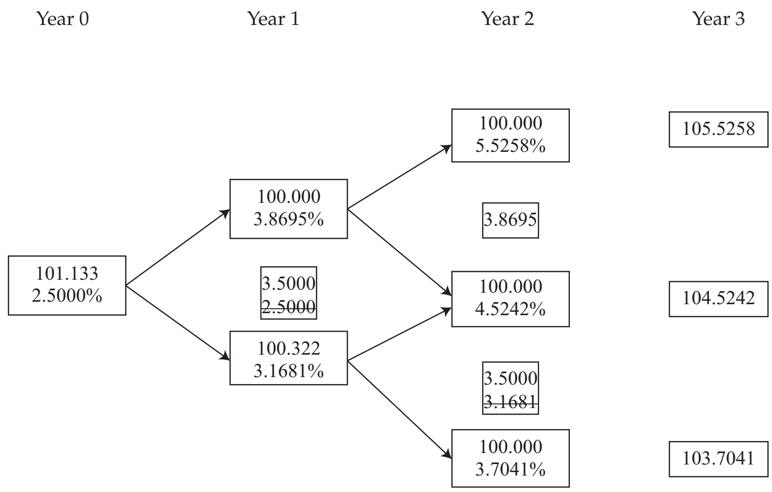
To illustrate how to value a floored floater, we return to the example we used for the capped floater but assume that the embedded option is now a 3.500% floor instead of a 4.500% cap. The other assumptions remain the same. The valuation of the floored floater is depicted in Exhibit 28.

Exhibit 28 Valuation of a Three-Year Libor Floater Floored at 3.500% at 10% Interest Rate Volatility



Recall from the discussion about the capped floater that if there were no cap, the value of the floater would be 100 because the coupon paid would equal the discount rate. The same principle applies here: If there were no floor, the value of this floater would be 100. Because the presence of the floor potentially increases the cash flows, however, the value of the floored floater must be equal to or higher than the value of the straight bond.

Exhibit 28 shows that the floor is binding at Year 0 because Libor (2.5000%) is less than the cap rate (3.5000%) and at Year 1 at the lower node where Libor is 3.1681%. Thus, the corresponding interest payments at Year 1 and 2 are increased to the minimum amount of 3.5000. As a consequence, the value of the floored floater exceeds 100 (101.133). The value of the floor can be calculated by using Equation 6:

Value of embedded floor = 101.133 – 100 = 1.133

EXAMPLE 8

Valuation of Capped and Floored Floaters

1 A three-year floating rate bond pays annual coupons of one-year Libor (set in arrears) and is capped at 5.600%. The Libor swap curve is as given in Exhibit 1 (i.e., the one-year, two-year, and three-year par yields are 2.500%, 3.000%, and 3.500%, respectively), and interest rate volatility is 10%. The value of the capped floater is *closest to*:

A 100.000.

B 105.600.

C 105.921.

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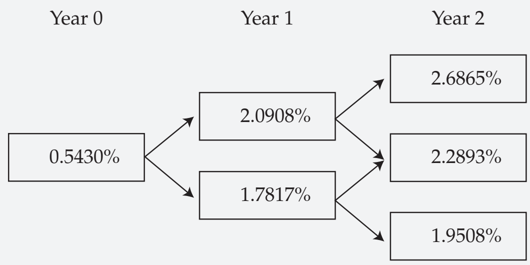
2 A three-year floating-rate bond pays annual coupons of one-year Libor (set in arrears) and is floored at 3.000%. The Libor swap curve is as given in Exhibit 1 (i.e., the one-year, two-year, and three-year par yields are 2.500%, 3.000%, and 3.500%, respectively), and interest rate volatility is 10%. The value of the floored floater is *closest to*:

A 100.000.

B 100.488.

C 103.000.

3 An issuer in the Eurozone wants to sell a three-year floating-rate note at par with an annual coupon based on the 12-month Euribor + 300 bps. Because the 12-month Euribor is currently at an historic low and the issuer wants to protect itself against a sudden increase in interest cost, the issuer's advisers recommend increasing the credit spread to 320 bps and capping the coupon at 5.50%. Assuming an interest rate volatility of 8%, the advisers have constructed the following binomial interest rate tree:



The value of the capped floater is *closest to*:

A 92.929.

B 99.916.

C 109.265.

Solution to 1:

A is correct. As illustrated in Exhibit 26, the cap is higher than any of the rates at which the floater is reset on the interest rate tree. Thus, the value of the bond is the same as if it had no cap—that is, 100.

Solution to 2:

B is correct. One can eliminate C because as illustrated in Exhibit 28, all else being equal, the bond with a higher floor (3.500%) has a value of 101.133. The value of a bond with a floor of 3.000% cannot be higher. Intuitively, B is the likely correct answer because the straight bond is worth 100. However, it is still necessary to calculate the value of the floored floater because if the floor is low enough, it could be worthless.

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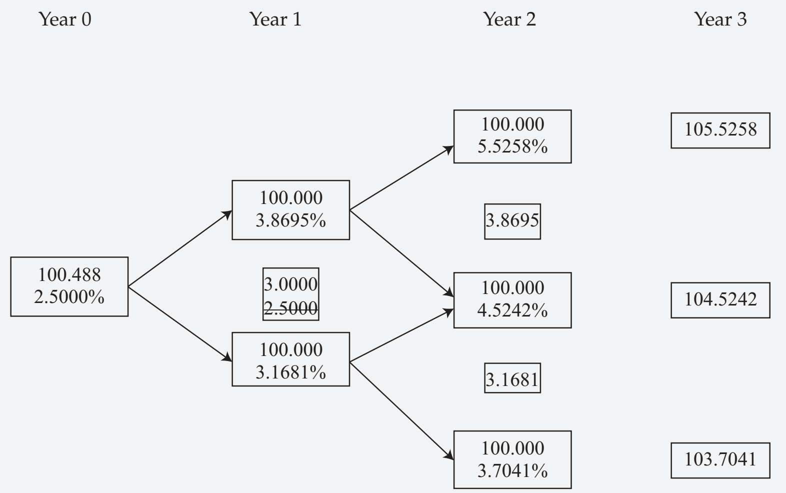
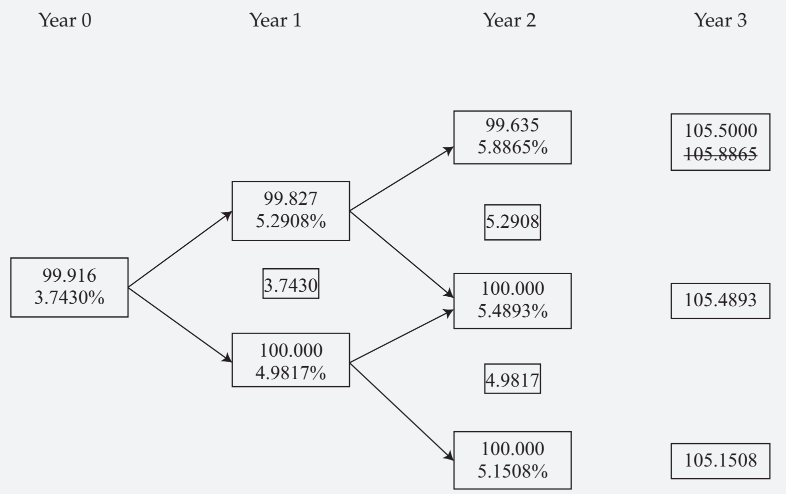


Here, it turns out that the floor adds 0.488 in value to the straight bond. Had the floor been 2.500%, the floored floater and the straight bond would both be worth par.

Solution to 3:

B is correct.



6 VALUATION AND ANALYSIS OF CONVERTIBLE BONDS

So far, we have discussed bonds for which the exercise of the option is at the discretion of the issuer (callable bond), at the discretion of the bondholder (putable bond), or set through a pre-defined contractual arrangement (capped and floored floaters). What distinguishes a convertible bond from the bonds discussed earlier is that exercising the option results in the change of the security from a bond to a common stock. This section describes defining features of convertible bonds and discusses how to analyze and value these bonds.

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6.1 Defining Features of a Convertible Bond

A **convertible bond** is a hybrid security. In its traditional form, it presents the characteristics of an option-free bond and an embedded conversion option. The conversion option is a call option on the issuer's common stock, which gives bondholders the right to convert their debt into equity during a pre-determined period (known as the **conversion period**) at a pre-determined price (known as the **conversion price**).

Convertible bonds have been issued and traded since the 1880s. They offer benefits to both the issuer and the investors. Investors usually accept a lower coupon for convertible bonds than for otherwise identical non-convertible bonds because they can participate in the potential upside through the conversion mechanism—that is, if the share price of the issuer's common stock (underlying share price) exceeds the conversion price, the bondholders can convert their bonds into shares at a cost lower than market value. The issuer benefits from paying a lower coupon. In case of conversion, an added benefit for the issuer is that it no longer has to repay the debt that was converted into equity.

However, what might appear as a win–win situation for both the issuer and the investors is not a "free lunch" because the issuer's existing shareholders face dilution in case of conversion. In addition, if the underlying share price remains below the conversion price and the bond is not converted, the issuer must repay the debt or refinance it, potentially at a higher cost. If conversion is not achieved, the bondholders will have lost interest income relative to an otherwise identical non-convertible bond that would have been issued with a higher coupon and would have thus offered investors an additional spread.

We will use the information provided in Exhibit 29 to describe the features of a convertible bond and then illustrate how to analyze it. This exhibit refers to a callable convertible bond issued by Waste Management Utility PLC (WMU), a company listed on the London Stock Exchange.

Exhibit 29 WMU £100,000,000 4.50% Callable Convertible Bonds Due 3 April 2017

Excerpt from the Bond's Offering Circular

- **Issue Date:** 3 April 2012
- **Status:** Senior unsecured, unsubordinated
- **Interest:** 4.50% of nominal value (par) per annum payable annually in arrears, with the first interest payment date on 3 April 2013 unless prior redeemed or converted
- **Issue Price:** 100% of par denominated into bonds of £100,000 each and integral multiples of £1,000 each thereafter
- **Conversion Period:** 3 May 2012 to 5 March 2017
- **Initial Conversion Price:** £6.00 per share
- **Conversion Ratio:** Each bond of par value of £100,000 is convertible to 16,666.67 ordinary shares
- **Threshold Dividend:** £0.30 per share
- **Change of Control Conversion Price:** £4.00 per share
- **Issuer Call Price:** From the second anniversary of issuance: 110%; from the third anniversary of issuance: 105%; from the fourth anniversary of issuance: 103%

(continued)

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Exhibit 29 (Continued)

Market Information

- **Convertible Bond Price on 4 April 2013:** £127,006
- **Share Price on Issue Date:** £4.58
- **Share Price on 4 April 2013:** £6.23
- **Dividend per Share:** £0.16
- **Share Price Volatility per annum as of 4 April 2013:** 25%

The applicable share price at which the investor can convert the bonds into ordinary (common) shares is called the conversion price. In the WMU example provided in Exhibit 29, the conversion price was set at £6 per share.

The number of shares of common stock that the bondholder receives from converting the bonds into shares is called the **conversion ratio**. In the WMU example, bondholders who have invested the minimum stipulated of £100,000 and convert their bonds into shares will receive 16,666.67 shares each (£100,000/£6) per £100,000 of nominal value. The conversion may be exercised during a particular period or at set intervals during the life of the bond. To accommodate share price volatility and technical settlement requirements, it is not uncommon to see conversion periods similar to the one in Exhibit 29—that is, beginning shortly after the issuance of the convertible bond and ending shortly prior to its maturity.

The conversion price in Exhibit 29 is referred to as the *initial* conversion price because it reflects the conversion price *at issuance*. Corporate actions, such as stock splits, bonus share issuances, and rights or warrants issuances, affect a company's share price and may reduce the benefit of conversion for the convertible bondholders. Thus, the terms of issuance of the convertible bond contain detailed information defining how the conversion price and conversion ratio are adjusted should such a corporate action occur during the life of the bond. For example, suppose that WMU performs a 2:1 stock split to its common shareholders. In this case, the conversion price would be adjusted to £3.00 per share and the conversion ratio would then be adjusted to 33,333.33 shares per £100,000 of nominal value.

As long as the convertible bond is still outstanding and has not been converted, the bondholders receive interest payments (annually in the WMU example). Meanwhile, if the issuer declares and pays dividends, common shareholders receive dividend payments. The terms of issuance may offer no compensation to convertible bondholders for dividends paid out during the life of the bond at one extreme, or they may offer full protection by adjusting the conversion price downward for any dividend payments at the other extreme. Typically, a threshold dividend is defined in the terms of issuance (£0.30 per share in the WMU example). Annual dividend payments below the threshold dividend have no effect on the conversion price. In contrast, the conversion price is adjusted downward for annual dividend payments above the threshold dividend to offer compensation to convertible bondholders.

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Should the issuer be acquired by or merged with another company during the life of the bond, bondholders might no longer be willing to continue lending to the new entity. Change-of-control events are defined in the prospectus or offering circular and, if such an event occurs, convertible bondholders usually have the choice between

- a put option that can be exercised during a specified period following the change-of-control event and that provides full redemption of the nominal value of the bond; or
- an adjusted conversion price that is lower than the initial conversion price. This downward adjustment gives the convertible bondholders the opportunity to convert their bonds into shares earlier and at more advantageous terms, and thus allows them to participate in the announced merger or acquisition as common shareholders.

In addition to a put option in case of a change-of-control event, it is not unusual for a convertible bond to include a put option that convertible bondholders can exercise during specified periods. Put options can be classified as “hard” puts or “soft” puts. In the case of a hard put, the issuer must redeem the convertible bond for cash. In the case of a soft put, the investor has the right to exercise the put but the issuer chooses how the payment will be made. The issuer may redeem the convertible bond for cash, common stock, subordinated notes, or a combination of the three.

It is more frequent for convertible bonds to include a call option that gives the issuer the right to call the bond during a specified period and at specified times. As discussed earlier, the issuer may exercise the call option and redeem the bond early if interest rates are falling or if its credit rating is revised upward, thus enabling the issuance of debt at a lower cost. The issuer may also believe that its share price will increase significantly in the future because of its performance or because of events that will take place in the economy or in its sector. In this case, the issuer may try to maximize the benefit to its existing shareholders relative to convertible bondholders and call the bond. To offer convertible bondholders protection against early repayment, convertible bonds usually have a lockout period. Subsequently, they can be called but at a premium, which decreases as the maturity of the bond approaches. In the WMU example, the convertible bond is not callable until its second anniversary, when it is callable at a premium of 10% above par value. The premium decreases to 5% at its third anniversary and 3% at its fourth anniversary.

If a convertible bond is callable, the issuer has an incentive to call the bond when the underlying share price increases above the conversion price in order to avoid paying further coupons. Such an event is called **forced conversion** because it forces bondholders to convert their bonds into shares. Otherwise, the redemption value that bondholders would receive from the issuer calling the bond would result in a disadvantageous position and a loss compared with conversion. Even if interest rates have not fallen or the issuer’s credit rating has not improved, thus not allowing refinancing at a lower cost, the issuer might still proceed with calling the bond when the underlying share price exceeds the conversion price. Doing so allows the issuer to take advantage of the favorable equity market conditions and force the bondholders to convert their bonds into shares. The forced conversion strengthens the issuer’s capital structure and eliminates the risk that a subsequent correction in equity prices prevents conversion and requires redeeming the convertible bonds at maturity.

6.2 Analysis of a Convertible Bond

There are a number of investment metrics and ratios that help in analyzing and valuing a convertible bond.

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6.2.1 *Conversion Value*

The **conversion value** or parity value of a convertible bond indicates the value of the bond if it is converted at the market price of the shares.

Conversion value = Underlying share price × Conversion ratio

Based on the information provided in Exhibit 29, we can calculate the conversion value for WMU's convertible bonds at the issuance date and on 4 April 2013:

Conversion value at the issuance date = £4.58 × 16,666.67 = £76,333.33

Conversion value on 4 April 2013 = £6.23 × 16,666.67 = £103,833.33

6.2.2 *Minimum Value of a Convertible Bond*

The minimum value of a convertible bond is equal to the greater of

- the conversion value and
- the value of the underlying option-free bond. Theoretically, the value of the straight bond (straight value) can be estimated by using the market value of a non-convertible bond of the issuer with the same characteristics as the convertible bond but without the conversion option. In practice, such a bond rarely exists. Thus, the straight value is found by using the arbitrage-free framework and by discounting the bond's future cash flows at the appropriate rates.

The minimum value of a convertible bond can also be described as a floor value. It is a *moving* floor, however, because the straight value is not fixed; it changes with fluctuations in interest rates and credit spreads. If interest rates rise, the value of the straight bond falls, making the floor fall. Similarly, if the issuer's credit spread increases as a result, for example, of a downgrade of its credit rating from investment grade to non-investment grade, the floor value will fall too.

Using the conversion values calculated in Section 6.2.1, the minimum value of WMU's convertible bonds at the issuance date is

$$\begin{aligned}\text{Minimum value at the issuance date} &= \text{Maximum}(£76{,}333.33;£100{,}000)\\ &= £100{,}000\end{aligned}$$

The straight value at the issuance date is £100,000 because the issue price is set at 100% of par. But after this date, this value will fluctuate. Thus, to calculate the minimum value of WMU's convertible bond on 4 April 2013, it is first necessary to calculate the value of the straight bond that day using the arbitrage-free framework. From Exhibit 29, the coupon is 4.50%, paid annually. Assuming a 2.5% flat yield curve, the straight value on 4 April 2013 is:

$$\frac{£4{,}500}{(1.02500)} + \frac{£4{,}500}{(1.02500)^2} + \frac{£4{,}500}{(1.02500)^3} + \frac{£100{,}000 + £4{,}500}{(1.02500)^4} = £107{,}523.95$$

It follows that the minimum value of WMU's convertible bonds on 4 April 2013 is

$$\begin{aligned}\text{Minimum value on 4 April 2013} &= \text{Maximum}(£103{,}833.33;£107{,}523.95)\\ &= £107{,}523.95\end{aligned}$$

If the value of the convertible bond were lower than the greater of the conversion value and the straight value, an arbitrage opportunity would ensue. Two scenarios help illustrate this concept. Returning to the WMU example, suppose that the convertible bond is selling for £103,833.33 on 4 April 2013—that is, at a price that is lower than the straight value of £107,523.95. In this scenario, the convertible bond is cheap relative to the straight bond; put another way, the convertible bond offers a higher yield than

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an otherwise identical non-convertible bond. Thus, investors will find the convertible bond attractive, buy it, and push its price up until the convertible bond price returns to the straight value and the arbitrage opportunity disappears.

Alternatively, assume that on 4 April 2013, the yield on otherwise identical non-convertible bonds is 5.00% instead of 2.50%. Using the arbitrage-free framework, the straight value is £98,227.02. Suppose that the convertible bond is selling at this straight value—that is, at a price that is lower than its conversion value of £103,833.33. In this case, an arbitrageur can buy the convertible bond for £98,227.02, convert it into 16,666.67 shares, and sell the shares at £6.23 each or £103,833.33 in total. The arbitrageur makes a profit equal to the difference between the conversion value and the straight value—that is, £5,606.31 (£103,833.33 – £98,227.02). As more arbitrageurs follow the same strategy, the convertible bond price will increase until it reaches the conversion value and the arbitrage opportunity disappears.

6.2.3 *Market Conversion Price, Market Conversion Premium per Share, and Market Conversion Premium Ratio*

Many investors do not buy a convertible bond at issuance on the primary market but instead buy such a bond later in its life on the secondary market. The **market conversion premium per share** allows investors to identify the premium or discount payable when buying the convertible bond rather than the underlying common stock.[8]

$$\text{Market conversion premium per share} = \text{Market conversion price} - \text{Underlying share price}$$

where

$$\text{Market conversion price} = \frac{\text{Convertible bond price}}{\text{Conversion ratio}}$$

The market conversion price represents the price that investors effectively pay for the underlying common stock if they buy the convertible bond and then convert it into shares. It can be viewed as a break-even price. Once the underlying share price exceeds the market conversion price, any further rise in the underlying share price is certain to increase the value of the convertible bond by at least the same percentage (we will discuss why this is the case in Section 6.4).

Based on the information provided in Exhibit 29,

$$\text{Market conversion price on 4 April 2013} = \frac{£127{,}006}{16{,}666.67} = £7.62$$

and

$$\text{Market conversion premium per share on 4 April 2013} = £7.62 - £6.23 = £1.39$$

The **market conversion premium ratio** expresses the premium or discount investors have to pay as a percentage of the current market price of the shares:

$$\text{Market conversion premium ratio} = \frac{\text{Market conversion premium per share}}{\text{Underlying share price}}$$

In the WMU example,

$$\text{Market conversion premium ratio on 4 April 2013} = \frac{£1.39}{£6.23} = 22.32\%$$

8 Although discounts are rare, they can theoretically happen given that the convertible bond and the underlying common stock trade in different markets with different types of market participants. For example, highly volatile share prices may result in the market conversion price being lower than the underlying share price.

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Why would investors be willing to pay a premium to buy the convertible bond? Recall that the straight value acts as a floor for the convertible bond price. Thus, as the underlying share price falls, the convertible bond price will not fall below the straight value. Viewed in this context, the market conversion premium per share resembles the price of a call option. Investors who buy a call option limit their downside risk to the price of the call option (premium). Similarly, the premium paid when buying a convertible bond allows investors to limit their downside risk to the straight value. There is a fundamental difference, however, between the buyers of a call option and the buyers of a convertible bond. The former know exactly the amount of the downside risk, whereas the latter know only that the most they can lose is the difference between the convertible bond price and the straight value because the straight value is not fixed.

6.2.4 *Downside Risk with a Convertible Bond*

Many investors use the straight value as a measure of the downside risk of a convertible bond, and calculate the following metric:

$$\text{Premium over straight value} = \frac{\text{Convertible bond price}}{\text{Straight value}} - 1$$

All else being equal, the higher the premium over straight value, the less attractive the convertible bond. In the WMU example,

$$\text{Premium over straight value} = \frac{£127{,}006}{£107{,}523.95} - 1 = 18.11\%$$

Despite its use in practice, the premium over straight value is a flawed measure of downside risk because, as mentioned earlier, the straight value is not fixed but rather fluctuates with changes in interest rates and credit spreads.

6.2.5 *Upside Potential of a Convertible Bond*

The upside potential of a convertible bond depends primarily on the prospects of the underlying common stock. Thus, convertible bond investors should be familiar with the techniques used to value and analyze common stocks. These techniques are covered in other readings.

6.3 Valuation of a Convertible Bond

Historically, the valuation of convertible bonds has been challenging because these securities combine characteristics of bonds, stocks, and options, thus requiring an understanding of what affects the value of fixed income, equity, and derivatives. The complexity of convertible bonds has also increased over time as a result of market innovations as well as additions to the terms and conditions of these securities. For example, convertible bonds have evolved into contingent convertible bonds and convertible contingent convertible bonds, which are even more complex to value and analyze.[9]

The fact that many bond's prospectuses or offering circulars frequently provide for an independent financial valuer to determine the conversion price (and in essence the value of the convertible bond) under different scenarios is evidence of the complexity

9 Contingent convertible bonds, or "CoCos," pay a higher coupon than otherwise identical non-convertible bonds, but they are usually deeply subordinated and may be converted into equity or face principal write-downs if regulatory capital ratios are breached. Convertible contingent convertible bonds, or "CoCoCos," combine a traditional convertible bond and a CoCo. They are convertible at the discretion of the investor, thus offering upside potential if the share price increases, but they are also converted into equity or face principal write-downs in the event of a regulatory capital breach. CoCos and CoCoCos are usually issued by financial institutions, particularly in Europe.

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associated with valuing convertible bonds. Because of this complexity, convertible bonds in many markets come with selling restrictions. They are typically offered in very high denominations and only to professional or institutional investors. Regulators perceive them as securities that are too risky for retail investors to invest in directly.

As with any fixed-income instrument, convertible bond investors should perform a diligent risk–reward analysis of the issuer, including its ability to service the debt and repay the principal, as well as a review of the bond's terms of issuance (e.g., collateral, credit enhancements, covenants, and contingent provisions). In addition, convertible bond investors must analyze the factors that typically affect bond prices, such as interest rate movements. Because most convertible bonds have lighter covenants than otherwise similar non-convertible bonds and are frequently issued as subordinated securities, the valuation and analysis of some convertible bonds can be complex.

The investment characteristics of a convertible bond depend on the underlying share price, so convertible bond investors must also analyze factors that may affect the issuer's common stock, including dividend payments and the issuer's actions (e.g., acquisitions or disposals, rights issues). Even if the issuer is performing well, adverse market conditions might depress share prices and prevent conversion. Thus, convertible bond investors must also identify and analyze the exogenous reasons that might ultimately have a negative effect on convertible bonds.

Academics and practitioners have developed advanced models to value convertible bonds, but the most commonly used model remains the arbitrage-free framework. A traditional convertible bond can be viewed as a straight bond and a call option on the issuer's common stock, so

Value of convertible bond = Value of straight bond
+ Value of call option on the issuer's stock

Many convertible bonds include a call option that gives the issuer the right to call the bond during a specified period and at specified times. The value of such bonds is

Value of callable convertible bond = Value of straight bond + Value of call option on the issuer's stock – Value of issuer call option

Suppose that the callable convertible bond also includes a put option that gives the bondholder the right to require that the issuer repurchases the bond. The value of such a bond is

Value of callable putable convertible bond = Value of straight bond + Value of call option on the issuer's stock – Value of issuer call option + Value of investor put option

No matter how many options are embedded into a bond, the valuation procedure remains the same. It relies on generating a tree of interest rates based on the given yield curve and interest rate volatility assumptions, determining at each node of the tree whether the embedded options will be exercised, and then applying the backward induction valuation methodology to calculate the present value of the bond.

6.4 Comparison of the Risk–Return Characteristics of a Convertible Bond, the Straight Bond, and the Underlying Common Stock

In its simplest form, a convertible bond can be viewed as a straight bond and a call option on the issuer's common stock. When the underlying share price is well below the conversion price, the convertible bond is described as "busted convertible" and exhibits mostly bond risk–return characteristics—that is, the risk–return characteristics of the convertible bond resemble those of the underlying option-free (straight) bond. In this case, the call option is out of the money, so share price movements do not

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significantly affect the price of the call option and, thus, the price of the convertible bond. Consequently, the price movement of the convertible bond closely follows that of the straight bond, and such factors as interest rate movements and credit spreads significantly affect the convertible bond price. The convertible bond exhibits even stronger bond risk–return characteristics when the call option is out of the money and the conversion period is approaching its end because the time value component of the option decreases toward zero, and it is highly likely that the conversion option will expire worthless.

In contrast, when the underlying share price is above the conversion price, a convertible bond exhibits mostly stock risk–return characteristics—that is, the risk–return characteristics of the convertible bond resemble those of the underlying common stock. In this case, the call option is in the money, so the price of the call option and thus the price of the convertible bond is significantly affected by share price movements but mostly unaffected by factors driving the value of an otherwise identical option-free bond, such as interest rate movements. When the call option is in the money, it is more likely to be exercised by the bondholder and the value of the shares resulting from the conversion is higher than the redemption value of the bond. Such convertible bonds trade at prices that follow closely the conversion value of the convertible bond, and their price exhibits similar movements to that of the underlying stock.

In between the bond and the stock extremes, the convertible bond trades like a hybrid instrument. It is important to note the risk–return characteristics of convertible bonds (1) when the underlying share price is below the conversion price and increases toward it and (2) when the underlying share price is above the conversion price but decreases toward it.

In the first case, the call option component increases significantly in value as the underlying share price approaches the conversion price. The return on the convertible bond during such periods increases significantly but at a lower rate than the increase in the underlying share price because the conversion price has not been reached yet. When the share price exceeds the conversion price and goes higher, the change in the convertible bond price converges toward the change in the underlying share price—this is why we noted in Section 6.2.4 that when the underlying share price exceeds the market conversion price, any further rise in the underlying share price is certain to increase the value of the convertible bond by at least the same percentage.

In the second case (that is, when the underlying share price is above the conversion price but decreases toward it), the relative change in the convertible bond price is less than the change in the underlying share price because the convertible bond has a floor. As mentioned earlier, this floor is the minimum value of the convertible bond, which in this case is equal to the value of the underlying option-free bond.

Exhibit 30 illustrates graphically the price behavior of a convertible bond and the underlying common stock.

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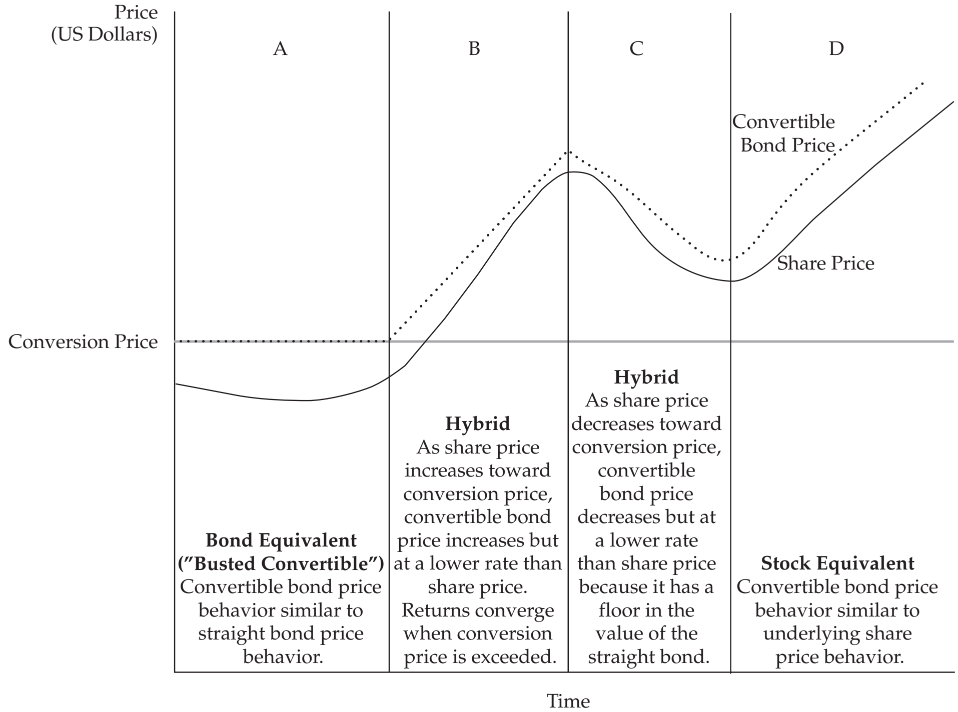
Exhibit 30 Price Behavior of a Convertible Bond and the Underlying Common Stock



Why would an investor not exercise the conversion option when the underlying share price is above the conversion price, as in areas B, C, and D? The call option on the issuer's common stock may be a European-style option that cannot be exercised now but only at the end of a pre-determined period. Even if the call option is an American-style option, making it possible to convert the bond into equity, it may not be optimal for the convertible bondholder to exercise prior to the expiry of the conversion period; as discussed in Section 3.3.2, it is sometimes better to wait than to exercise an option that is in the money. The investor may also prefer to sell the convertible bond instead of exercising the conversion option.

Except for busted convertibles, the most important factor in the valuation of convertible bonds is the underlying share price. However, it is worth mentioning that large movements in interest rates or in credit spreads may significantly affect the value of convertible bonds. For a convertible bond with a fixed coupon, all else being equal, a significant fall in interest rates would result in an increase in its value and price, whereas a significant rise in interest rates would lead in a decrease in its value and price. Similarly, all else being equal, a significant improvement in the issuer's credit quality would result in an increase in the value and price of its convertible bonds, whereas a deterioration of the issuer's credit quality would lead to a decrease in the value and price of its convertible bonds.

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EXAMPLE 9

Valuation of Convertible Bonds

Nick Andrews, a fixed-income investment analyst, has been asked by his supervisor to prepare an analysis of the convertible bond issued by Heavy Element Inc., a chemical industry company, for presentation to the investment committee. Andrews has gathered the following data from the convertible bond's prospectus and market information:

Issuer: Heavy Element Inc.

Issue Date: 15 September 2010

Maturity Date: 15 September 2015

Interest: 3.75% payable annually

Issue Size: $100,000,000

Issue Price: $1,000 at par

Conversion Ratio: 23.26

Convertible Bond Price on 16 September 2012: $1,230

Share Price on 16 September 2012: $52

1 The conversion price is *closest to*:
 - A $19.
 - B $43.
 - C $53.

2 The conversion value on 16 September 2012 is *closest to*:
 - A $24.
 - B $230.
 - C $1,209.

3 The market conversion premium per share on 16 September 2012 is *closest to*:
 - A $0.88.
 - B $2.24.
 - C $9.00.

4 The risk–return characteristics of the convertible bond on 16 September 2012 *most likely* resemble that of:
 - A a busted convertible.
 - B Heavy Element's common stock.
 - C a bond of Heavy Element that is identical to the convertible bond but without the conversion option.

5 As a result of favorable economic conditions, credit spreads for the chemical industry narrow, resulting in lower interest rates for the debt of companies such as Heavy Element. All else being equal, the price of Heavy Element's convertible bond will *most likely*:
 - A decrease significantly.
 - B not change significantly.
 - C increase significantly.

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6 Suppose that on 16 September 2012, the convertible bond is available in the secondary market at a price of $1,050. An arbitrageur can make a risk-free profit by:

A buying the underlying common stock and shorting the convertible bond.

B buying the convertible bond, exercising the conversion option, and selling the shares resulting from the conversion.

C shorting the convertible bond and buying a call option on the underlying common stock exercisable at the conversion price on the conversion date.

7 A few months have passed. Because of chemical spills in lake water at the site of a competing facility, the government has introduced very costly environmental legislation. As a result, share prices of almost all publicly traded chemical companies, including Heavy Element, have decreased sharply. Heavy Element's share price is now $28. Now, the risk–return characteristics of the convertible bond *most likely* resemble that of:

A a bond.

B a hybrid instrument.

C Heavy Element's common stock.

Solution to 1:

B is correct. The conversion price is equal to the par value of the convertible bond divided by the conversion ratio—that is, $1,000/23.26 = $43 per share.

Solution to 2:

C is correct. The conversion value is equal to the underlying share price multiplied by the conversion ratio—that is, $52 × 23.26 = $1,209.

Solution to 3:

A is correct. The market conversion premium per share is equal to the convertible bond price divided by the conversion ratio, minus the underlying share price—that is, ($1,230/23.26) – $52 = $52.88 – $52 = $0.88.

Solution to 4:

B is correct. The underlying share price ($52) is well above the conversion price ($43). Thus, the convertible bond exhibits risk–return characteristics that are similar to those of the underlying common stock. A is incorrect because a busted convertible is a convertible bond for which the underlying common stock trades at a significant discount relative to the conversion price. C is incorrect because it describes a busted convertible.

Solution to 5:

B is correct. The underlying share price ($52) is well above the conversion price ($43). Thus, the convertible bond exhibits mostly stock risk–return characteristics, and its price is mainly driven by the underlying share price. Consequently, the decrease in credit spreads will have little effect on the convertible bond price.

Solution to 6:

B is correct. The convertible bond price ($1,050) is lower than its minimum value ($1,209). Thus, the arbitrageur can buy the convertible bond for $1,050; convert it into 23.26 shares; and sell the shares at $52 each, or $1,209 in total, making a

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profit of $159. A and C are incorrect because in both scenarios, the arbitrageur is short the underpriced asset (convertible bond) and long an overpriced asset, resulting in a loss.

Solution to 7:

A is correct. The underlying share price ($28) is now well below the conversion price ($43), so the convertible bond is a busted convertible and exhibits mostly bond risk–return characteristics. B is incorrect because the underlying share price would have to be close to the conversion price for the risk–return characteristics of the convertible bond to resemble that of a hybrid instrument. C is incorrect because the underlying share price would have to be in excess of the conversion price for the risk–return characteristics of the convertible bond to resemble that of the company's common stock.

7 BOND ANALYTICS

The introduction of OAS analysis in the mid-1980s marked the dawn of modern bond valuation theory. The approach is mathematically elegant, robust, and widely applicable. The typical implementation, however, relies heavily on number crunching. Whether it involves calculating the OAS corresponding to a price, valuing a bond with embedded options, or estimating key rate durations, computers are essential to the process. Needless to say, practitioners must have access to systems that can execute the required calculations correctly and in a timely manner. Most practitioners rely on commercially available systems, but some market participants, in particular financial institutions, may develop analytics in-house.

How can a practitioner tell if such a system is adequate? First, the system should be able to report the correct cash flows, discount rates, and present value of the cash flows. The discount rates can be verified by hand or on a spreadsheet. In practice, it is impossible to examine every calculation, but there are a few relatively simple tests that can be useful, and we present three of these tests below. Also, even if it is difficult to verify that a result is correct, it may be possible to establish that it is wrong.

Check that the put–call parity holds. A simple test for option valuation is to check for put–call parity—that is, the important relationship for European-type options discussed in a separate reading on derivatives. According to put–call parity,

$$\text{Value}(C) - \text{Value}(P) = PV(\text{Forward price of bond on exercise date} - \text{Exercise price})$$

C and P refer to the European-type call option and put option on the same underlying bond and have the same exercise date and the same exercise price, respectively. If the system fails this test, look for an alternative.

Check that the value of the underlying option-free bond does not depend on interest rate volatility. To test the integrity of the interest rate tree calibration, set up and value a callable bond with a very high call price, say 150% of par. This structure should have the same value as that of the straight bond independent of interest rate volatility. The same should be true for a putable bond with a very low put price, say 50% of par.

Check that the volatility term structure slopes downward. As discussed earlier, the specified interest rate volatility is that of the short-term rate. This volatility, in turn, implies the volatilities of longer-term rates. In order for the interest rate process to be stable, the implied volatilities should decline as the term lengthens.

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SUMMARY

This reading covers the valuation and analysis of bonds with embedded options. The following are the main points made in this reading:

- An embedded option represents a right that can be exercised by the issuer, by the bondholder, or automatically depending on the course of interest rates. It is attached to, or embedded in, an underlying option-free bond called a straight bond.
- Simple embedded option structures include call options, put options, and extension options. Callable and putable bonds can be redeemed prior to maturity, at the discretion of the issuer in the former case and of the bondholder in the latter case. An extendible bond gives the bondholder the right to keep the bond for a number of years after maturity. Putable and extendible bonds are equivalent, except that the underlying option-free bonds are different.
- Complex embedded option structures include bonds with other types of options or combinations of options. For example, a convertible bond includes a conversion option that allows the bondholders to convert their bonds into the issuer's common stock. A bond with an estate put can be put by the heirs of a deceased bondholder. Sinking fund bonds make the issuer set aside funds over time to retire the bond issue and are often callable, may have an acceleration provision, and may also contain a delivery option. Valuing and analyzing bonds with complex embedded option structures is challenging.
- According to the arbitrage-free framework, the value of a bond with an embedded option is equal to the arbitrage-free values of its parts—that is, the arbitrage-free value of the straight bond and the arbitrage-free values of each of the embedded options.
- Because the call option is an issuer option, the value of the call option decreases the value of the callable bond relative to an otherwise identical but non-callable bond. In contrast, because the put option is an investor option, the value of the put option increases the value of the putable bond relative to an otherwise identical but non-putable bond.
- In the absence of default and interest rate volatility, the bond's future cash flows are certain. Thus, the value of a callable or putable bond can be calculated by discounting the bond's future cash flows at the appropriate one-period forward rates, taking into consideration the decision to exercise the option. If a bond is callable, the decision to exercise the option is made by the issuer, which will exercise the call option when the value of the bond's future cash flows is higher than the call price. In contrast, if the bond is putable, the decision to exercise the option is made by the bondholder, who will exercise the put option when the value of the bond's future cash flows is lower than the put price.
- In practice, interest rates fluctuate, and interest rate volatility affects the value of embedded options. Thus, when valuing bonds with embedded options, it is important to consider the possible evolution of the yield curve over time.
- Interest rate volatility is modeled using a binomial interest rate tree. The higher the volatility, the lower the value of the callable bond and the higher the value of the putable bond.

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- Valuing a bond with embedded options assuming an interest rate volatility requires three steps: (1) Generate a tree of interest rates based on the given yield curve and volatility assumptions; (2) at each node of the tree, determine whether the embedded options will be exercised; and (3) apply the backward induction valuation methodology to calculate the present value of the bond.
- The most commonly used approach to valuing risky bonds is to add a spread to the one-period forward rates used to discount the bond's future cash flows.
- The option-adjusted spread is the single spread added uniformly to the one-period forward rates on the tree to produce a value or price for a bond. OAS is sensitive to interest rate volatility: The higher the volatility, the lower the OAS for a callable bond.
- For bonds with embedded options, the best measure to assess the sensitivity of the bond's price to a parallel shift of the benchmark yield curve is effective duration. The effective duration of a callable or putable bond cannot exceed that of the straight bond.
- The effective convexity of a straight bond is negligible, but that of bonds with embedded options is not. When the option is near the money, the convexity of a callable bond is negative, indicating that the upside for a callable bond is much smaller than the downside, whereas the convexity of a putable bond is positive, indicating that the upside for a putable bond is much larger than the downside.
- Because the prices of callable and putable bonds respond asymmetrically to upward and downward interest rate changes of the same magnitude, one-sided durations provide a better indication regarding the interest rate sensitivity of bonds with embedded options than (two-sided) effective duration.
- Key rate durations show the effect of shifting only key points, one at a time, rather than the entire yield curve.
- The arbitrage-free framework can be used to value capped and floored floaters. The cap provision in a floater is an issuer option that prevents the coupon rate from increasing above a specified maximum rate. Thus, the value of a capped floater is equal to or less than the value of the straight bond. In contrast, the floor provision in a floater is an investor option that prevents the coupon from decreasing below a specified minimum rate. Thus, the value of a floored floater is equal to or higher than the value of the straight bond.
- The characteristics of a convertible bond include the conversion price, which is the applicable share price at which the bondholders can convert their bonds into common shares, and the conversion ratio, which reflects the number of shares of common stock that the bondholders receive from converting their bonds into shares. The conversion price is adjusted in case of corporate actions, such as stock splits, bonus share issuances, and rights and warrants issuances. Convertible bondholders may receive compensation when the issuer pays dividends to its common shareholders, and they may be given the opportunity to either put their bonds or convert their bonds into shares earlier and at more advantageous terms in the case of a change of control.
- There are a number of investment metrics and ratios that help analyze and value convertible bonds. The conversion value indicates the value of the bond if it is converted at the market price of the shares. The minimum value of a convertible bond sets a floor value for the convertible bond at the greater of the conversion value or the straight value. This floor is moving, however, because the straight value is not fixed. The market conversion premium represents the price investors effectively pay for the underlying shares if they buy the

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convertible bond and then convert it into shares. Scaled by the market price of the shares, it represents the premium payable when buying the convertible bond rather than the underlying common stock.

- Because convertible bonds combine characteristics of bonds, stocks, and options, as well as potentially other features, their valuation and analysis is challenging. Convertible bond investors should consider the factors that affect not only bond prices but also the underlying share price.
- The arbitrage-free framework can be used to value convertible bonds, including callable and putable ones. Each component (straight bond, call option of the stock, and call and/or put option on the bond) can be valued separately.
- The risk–return characteristics of a convertible bond depend on the underlying share price relative to the conversion price. When the underlying share price is well below the conversion price, the convertible bond is "busted" and exhibits mostly bond risk–return characteristics. Thus, it is mainly sensitive to interest rate movements. In contrast, when the underlying share price is well above the conversion price, the convertible bond exhibits mostly stock risk–return characteristics. Thus, its price follows similar movements to the price of the underlying stock. In between these two extremes, the convertible bond trades like a hybrid instrument.

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Kalotay, A., and L. Abreo. 1999. "Ratchet Bonds: Maximum Refunding Efficiency at Minimum Transaction Cost." *Journal of Applied Corporate Finance* 12 (1): 40–47.

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PRACTICE PROBLEMS

The following information relates to Questions 1–10

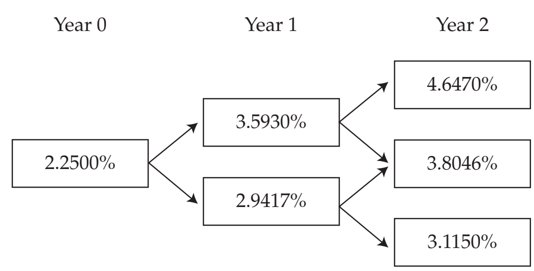
Samuel & Sons is a fixed-income specialty firm that offers advisory services to investment management companies. On 1 October 20X0, Steele Ferguson, a senior analyst at Samuel, is reviewing three fixed-rate bonds issued by a local firm, Pro Star, Inc. The three bonds, whose characteristics are given in Exhibit 1, carry the highest credit rating.

Exhibit 1 Fixed-Rate Bonds Issued by Pro Star, Inc.

Bond	Maturity	Coupon	Type of Bond
Bond #1	1 October 20X3	4.40% annual	Option-free
Bond #2	1 October 20X3	4.40% annual	Callable at par on 1 October 20X1 and on 1 October 20X2
Bond #3	1 October 20X3	4.40% annual	Putable at par on 1 October 20X1 and on 1 October 20X2

The one-year, two-year, and three-year par rates are 2.250%, 2.750%, and 3.100%, respectively. Based on an estimated interest rate volatility of 10%, Ferguson constructs the binomial interest rate tree shown in Exhibit 2.

Exhibit 2 Binomial Interest Rate Tree



On 19 October 20X0, Ferguson analyzes the convertible bond issued by Pro Star given in Exhibit 3. That day, the option-free value of Pro Star's convertible bond is $1,060 and Pro Star's stock price is and $37.50.

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Exhibit 3 Convertible Bond Issued by Pro Star, Inc.

Issue Date:	6 December 20X0
Maturity Date:	6 December 20X4
Coupon Rate:	2%
Issue Price:	$1,000
Conversion Ratio:	31

1 The call feature of Bond #2 is *best* described as:
   A European style.
   B American style.
   C Bermudan style.

2 The bond that would *most likely* protect investors against a significant increase in interest rates is:
   A Bond #1.
   B Bond #2.
   C Bond #3.

3 A fall in interest rates would *most likely* result in:
   A a decrease in the effective duration of Bond #3.
   B Bond #3 having more upside potential than Bond #2.
   C a change in the effective convexity of Bond #3 from positive to negative.

4 The value of Bond #2 is *closest* to:
   A 102.103% of par.
   B 103.121% of par.
   C 103.744% of par.

5 The value of Bond #3 is *closest* to:
   A 102.103% of par.
   B 103.688% of par.
   C 103.744% of par.

6 All else being equal, a rise in interest rates will *most likely* result in the value of the option embedded in Bond #3:
   A decreasing.
   B remaining unchanged.
   C increasing.

7 All else being equal, if Ferguson assumes an interest rate volatility of 15% instead of 10%, the bond that would *most likely* increase in value is:
   A Bond #1.
   B Bond #2.
   C Bond #3.

8 All else being equal, if the shape of the yield curve changes from upward sloping to flattening, the value of the option embedded in Bond #2 will *most likely*:
   A decrease.
   B remain unchanged.
   C increase.

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9 The conversion price of the bond in Exhibit 3 is closest to:
   A $26.67.
   B $32.26.
   C $34.19.

10 If the market price of Pro Star's common stock falls from its level on 19 October 20X0, the price of the convertible bond will *most likely*:
   A fall at the same rate as Pro Star's stock price.
   B fall but at a slightly lower rate than Pro Star's stock price.
   C be unaffected until Pro Star's stock price reaches the conversion price.

The following information relates to Question 11–19

Rayes Investment Advisers specializes in fixed-income portfolio management. Meg Rayes, the owner of the firm, would like to add bonds with embedded options to the firm's bond portfolio. Rayes has asked Mingfang Hsu, one of the firm's analysts, to assist her in selecting and analyzing bonds for possible inclusion in the firm's bond portfolio.

Hsu first selects two corporate bonds that are callable at par and have the same characteristics in terms of maturity, credit quality and call dates. Hsu uses the option adjusted spread (OAS) approach to analyse the bonds, assuming an interest rate volatility of 10%. The results of his analysis are presented in Exhibit 1.

Exhibit 1 Summary Results of Hsu's Analysis Using the OAS Approach

Bond	OAS (in bps)
Bond #1	25.5
Bond #2	30.3

Hsu then selects the four bonds issued by RW, Inc. given in Exhibit 2. These bonds all have a maturity of three years and the same credit rating. Bonds #4 and #5 are identical to Bond #3, an option-free bond, except that they each include an embedded option.

Exhibit 2 Bonds Issued by RW, Inc.

Bond	Coupon	Special Provision
Bond #3	4.00% annual	
Bond #4	4.00% annual	Callable at par at the end of years 1 and 2
Bond #5	4.00% annual	Putable at par at the end of years 1 and 2
Bond #6	One-year Libor annually, set in arrears	

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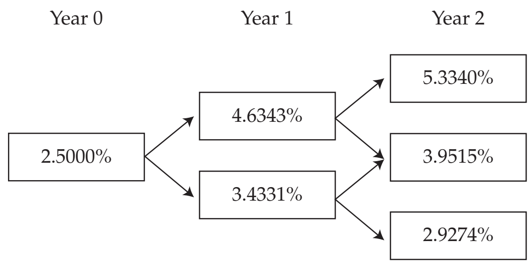
To value and analyze RW's bonds, Hsu uses an estimated interest rate volatility of 15% and constructs the binomial interest rate tree provided in Exhibit 3.

Exhibit 3 Binomial Interest Rate Tree Used to Value RW's Bonds



Rayes asks Hsu to determine the sensitivity of Bond #4's price to a 20 bps parallel shift of the benchmark yield curve. The results of Hsu's calculations are shown in Exhibit 4.

Exhibit 4 Summary Results of Hsu's Analysis about the Sensitivity of Bond #4's Price to a Parallel Shift of the Benchmark Yield Curve

Magnitude of the Parallel Shift in the Benchmark Yield Curve	+20 bps	–20 bps
Full Price of Bond #4 (% of par)	100.478	101.238

Hsu also selects the two floating-rate bonds issued by Varlep, plc given in Exhibit 5. These bonds have a maturity of three years and the same credit rating.

Exhibit 5 Floating-Rate Bonds Issued by Varlep, plc

Bond	Coupon
Bond #7	One-year Libor annually, set in arrears, capped at 5.00%
Bond #8	One-year Libor annually, set in arrears, floored at 3.50%

To value Varlep's bonds, Hsu constructs the binomial interest rate tree provided in Exhibit 6.

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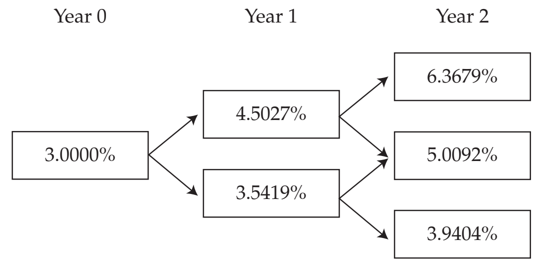
Exhibit 6 Binomial Interest Rate Tree Used to Value Varlep's Bonds



Last, Hsu selects the two bonds issued by Whorton, Inc. given in Exhibit 7. These bonds are close to their maturity date and are identical, except that Bond #9 includes a conversion option. Whorton's common stock is currently trading at $30 per share.

Exhibit 7 Bonds Issued by Whorton, Inc.

Bond	Type of Bond
Bond #9	Convertible bond with a conversion price of $50
Bond #10	Identical to Bond #9 except that it does not include a conversion option

11 Based on Exhibit 1, Rayes would *most likely* conclude that relative to Bond #1, Bond #2 is:

A overpriced.

B fairly priced.

C underpriced.

12 The effective duration of Bond #6 is:

A lower than or equal to 1.

B higher than 1 but lower than 3.

C higher than 3.

13 In Exhibit 2, the bond whose effective duration will lengthen if interest rates rise is:

A Bond #3.

B Bond #4.

C Bond #5.

14 The effective duration of Bond #4 is *closest* to:

A 0.76.

B 1.88.

C 3.77.

15 The value of Bond #7 is *closest* to:

A 99.697% of par.

B 99.936% of par.

C 101.153% of par.

16 The value of Bond #8 is *closest* to:

A 98.116% of par.

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B 100.000% of par.

C 100.485% of par.

17 The value of Bond #9 is equal to the value of Bond #10:

A plus the value of a put option on Whorton's common stock.

B plus the value of a call option on Whorton's common stock.

C minus the value of a call option on Whorton's common stock.

18 The minimum value of Bond #9 is equal to the *greater* of:

A the conversion value of Bond #9 and the current value of Bond #10.

B the current value of Bond #10 and a call option on Whorton's common stock.

C the conversion value of Bond #9 and a call option on Whorton's common stock.

19 The factor that is currently *least likely* to affect the risk-return characteristics of Bond #9 is:

A Interest rate movements.

B Whorton's credit spreads.

C Whorton's common stock price movements.

The following information relates to Question 20–27

John Smith, an investment adviser, meets with Lydia Carter to discuss her pending retirement and potential changes to her investment portfolio. Domestic economic activity has been weakening recently, and Smith's outlook is that equity market values will be lower during the next year. He would like Carter to consider reducing her equity exposure in favor of adding more fixed-income securities to the portfolio.

Government yields have remained low for an extended period, and Smith suggests considering investment-grade corporate bonds to provide additional yield above government debt issues. In light of recent poor employment figures and two consecutive quarters of negative GDP growth, the consensus forecast among economists is that the central bank, at its next meeting this month, will take actions that will lead to lower interest rates.

Smith and Carter review par, spot, and one-year forward rates (Exhibit 1) and four fixed-rate investment-grade bonds issued by Alpha Corporation which are being considered for investment (Exhibit 2).

Exhibit 1 Par, Spot, and One-Year Forward Rates (annual coupon payments)

Maturity (Years)	Par Rate (%)	Spot Rate (%)	One-Year Forward (%)
1	1.0000	1.0000	1.0000
2	1.2000	1.2012	1.4028
3	1.2500	1.2515	1.3522

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Exhibit 2 Selected Fixed-Rate Bonds of Alpha Corporation

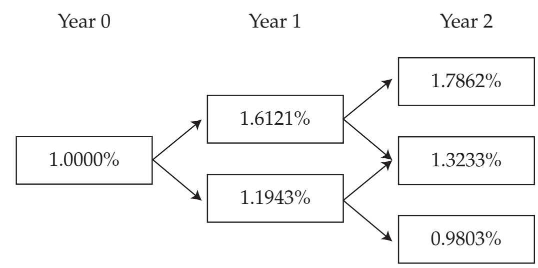
Bond	Annual Coupon	Type of Bond
Bond 1	1.5500%	Straight bond
Bond 2	1.5500%	Convertible bond: currently trading out of the money
Bond 3	1.5500%	Putable bond: putable at par one year and two years from now
Bond 4	1.5500%	Callable bond: callable at par without any lockout periods

Note: All bonds in Exhibit 2 have remaining maturities of exactly three years.

Carter tells Smith that the local news media have been reporting that housing starts, exports, and demand for consumer credit are all relatively strong, even in light of other poor macroeconomic indicators. Smith explains that the divergence in economic data leads him to believe that volatility in interest rates will increase. Smith also states that he recently read a report issued by Brown and Company forecasting that the yield curve could invert within the next six months.

Smith develops a binomial interest rate tree with a 15% interest rate volatility assumption to assess the value of Alpha Corporation's bonds. Exhibit 3 presents the interest rate tree.

Exhibit 3 Binomial Interest Rate Tree for Alpha Corporation 15% Interest Rate Volatility



Carter asks Smith about the possibility of analyzing bonds that have lower credit ratings than the investment-grade Alpha bonds. Smith discusses four other corporate bonds with Carter. Exhibit 4 presents selected data on the four bonds.

Exhibit 4 Selected Information on Fixed-Rate Bonds for Beta, Gamma, Delta, and Rho Corporations

Bond	Issuer	Bond Features	Credit Rating
Bond 5	Beta Corporation	Coupon 1.70% Callable in Year 2 OAS of 45 bps	B
Bond 6	Gamma Corporation	Coupon 1.70% Callable in Year 2 OAS of 65 bps	B

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Exhibit 4 (Continued)

Bond	Issuer	Bond Features	Credit Rating
Bond 7	Delta Corporation	Coupon 1.70% Callable in Year 2 OAS of 85 bps	B
Bond 8	Rho Corporation	Coupon 1.70% Callable in Year 2 OAS of 105 bps	CCC

Notes: All bonds have remaining maturities of three years. OAS stands for option-adjusted spread.

20 Based on Exhibit 2, and assuming that the forecast for interest rates and Smith's outlook for equity returns are validated, which bond's option is *most likely* to be exercised?

A Bond 2

B Bond 3

C Bond 4

21 Based on Exhibit 2, the current price of Bond 1 is *most likely* greater than the current price of:

A Bond 2.

B Bond 3.

C Bond 4.

22 Assuming the forecast for interest rates is proven accurate, which bond in Exhibit 2 will likely experience the smallest price increase?

A Bond 1

B Bond 3

C Bond 4

23 Based on the information in Exhibit 1 and Exhibit 2, the value of the embedded option in Bond 4 is *closest* to:

A nil.

B 0.1906.

C 0.3343.

24 If Smith's interest rate volatility forecast turns out to be true, which bond in Exhibit 2 is likely to experience the greatest price increase?

A Bond 2

B Bond 3

C Bond 4

25 If the Brown and Company forecast comes true, which of the following is *most* likely to occur? The value of the embedded option in:

A Bond 3 decreases.

B Bond 4 decreases.

C both Bond 3 and Bond 4 increases.

26 Based on Exhibit 2 and Exhibit 3, the market price of Bond 4 is *closest* to:

A 100.4578.

B 100.5123.

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C 100.8790.

27 Which of the following conclusions regarding the bonds in Exhibit 4 is correct?

A Bond 5 is relatively cheaper than Bond 6.

B Bond 7 is relatively cheaper than Bond 6.

C Bond 8 is relatively cheaper than Bond 7.

The following information relates to Questions 28-36

Jules Bianchi is a bond analyst for Maneval Investments, Inc. Bianchi gathers data on three corporate bonds, as shown in Exhibit 1.

Exhibit 1 Selected Bond Data

Issuer	Coupon Rate	Price	Bond Description
Ayrault, Inc. (AI)	5.25%	100.200	Callable at par in one year and two years from today
Blum, Inc. (BI)	5.25%	101.300	Option-free
Cresson Enterprises (CE)	5.25%	102.100	Putable at par in one year from today

Note: Each bond has a remaining maturity of three years, annual coupon payments, and a credit rating of BBB.

To assess the interest rate risk of the three bonds, Bianchi constructs two binomial interest rate trees based on a 10% interest rate volatility assumption and a current one-year rate of 1%. Panel A of Exhibit 2 provides an interest rate tree assuming the benchmark yield curve shifts down by 30 bps, and Panel B provides an interest rate tree assuming the benchmark yield curve shifts up by 30 bps. Bianchi determines that the AI bond is currently trading at an option-adjusted spread (OAS) of 13.95 bps relative to the benchmark yield curve.

Exhibit 2 Binomial Interest Rate Trees

Panel A Interest Rates Shift Down by 30 bps

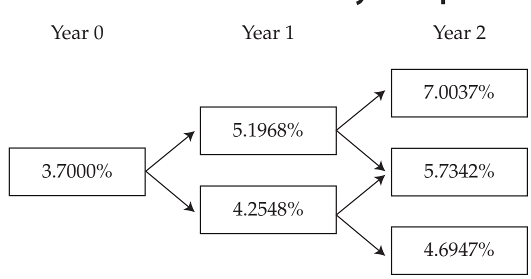

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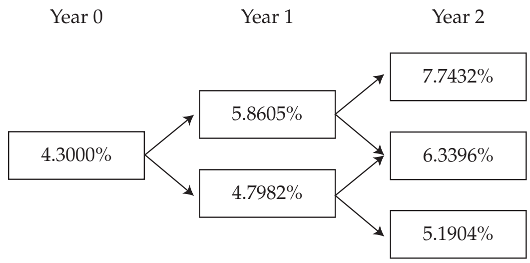
Exhibit 2 (Continued)

Panel B Interest Rates Shift Up by 30 bps



Armand Gillette, a convertible bond analyst, stops by Bianchi's office to discuss two convertible bonds. One is issued by DeLille Enterprises (DE) and the other is issued by Raffarin Incorporated (RI). Selected data for the two bonds are presented in Exhibits 3 and 4.

Exhibit 3 Selected Data for DE Convertible Bond

Issue price	€1,000 at par
Conversion period	13 September 20X5 to 12 September 20X8
Initial conversion price	€10.00 per share
Threshold dividend	€0.50 per share
Change of control conversion price	€8.00 per share
Common stock share price on issue date	€8.70
Share price on 17 September 20X5	€9.10
Convertible bond price on 17 September 20X5	€1,123

Exhibit 4 Selected Data for RI Convertible Bond

Straight bond value	€978
Value of embedded issuer call option	€43
Value of embedded investor put option	€26
Value of embedded call option on issuer's stock	€147
Conversion price	€12.50
Current common stock share price	€11.75

Gillette makes the following comments to Bianchi:

- "The DE bond does not contain any call or put options but the RI bond contains both an embedded call option and put option. I expect that DeLille Enterprises will soon announce a common stock dividend of €0.70 per share."
- "My belief is that, over the next year, Raffarin's share price will appreciate toward the conversion price but not exceed it."

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28 Based on Exhibits 1 and 2, the effective duration for the AI bond is *closest to*:

A 1.98.

B 2.15.

C 2.73.

29 If benchmark yields were to fall, which bond in Exhibit 1 would *most likely* experience a decline in effective duration?

A AI bond

B BI bond

C CE bond

30 Based on Exhibit 1, for the BI bond, one-sided:

A up-duration will be greater than one-sided down-duration.

B down-duration will be greater than one-sided up-duration.

C up-duration and one-sided down-duration will be about equal.

31 Based on Exhibit 1, which key rate duration is the largest for the BI bond?

A One-year key rate duration

B Two-year key rate duration

C Three-year key rate duration

32 Which bond in Exhibit 1 *most likely* has the lowest effective convexity?

A AI bond

B BI bond

C CE bond

33 Based on Exhibit 3, if DeLille Enterprises pays the dividend expected by Gillette, the conversion price of the DE bond will:

A be adjusted downward.

B not be adjusted.

C be adjusted upward.

34 Based on Exhibit 3, the market conversion premium per share for the DE bond on 17 September 20X5 is *closest* to:

A €0.90.

B €2.13.

C €2.53.

35 Based on Exhibit 4, the arbitrage-free value of the RI bond is *closest* to:

A €814.

B €1,056.

C €1,108.

36 Based on Exhibit 4 and Gillette's forecast regarding Raffarin's share price, the return on the RI bond over the next year is *most likely* to be:

A lower than the return on Raffarin's common shares.

B the same as the return on Raffarin's common shares.

C higher than the return on Raffarin's common shares.

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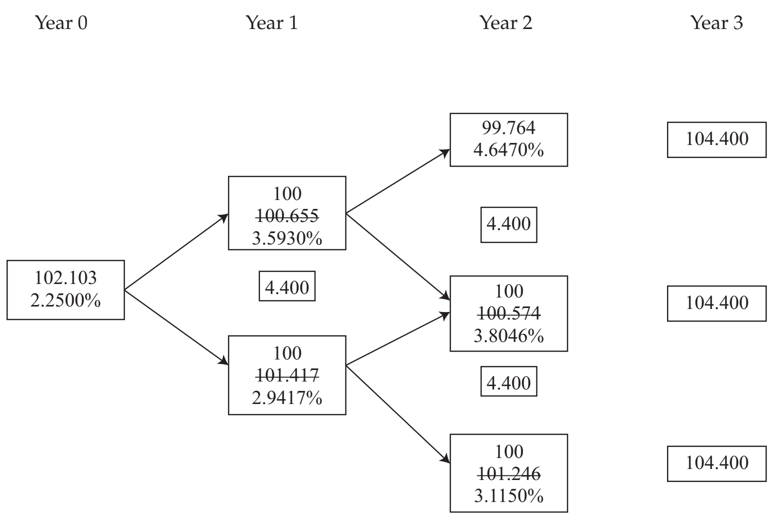
SOLUTIONS

1 C is correct. The call option embedded in Bond #2 can be exercised only at two predetermined dates: 1 October 20X1 and 1 October 20X2. Thus, the call feature is Bermudan style.

2 C is correct. The bond that would most likely protect investors against a significant increase in interest rates is the putable bond, i.e., Bond #3. When interest rates have risen and higher-yield bonds are available, a put option allows the bondholders to put back the bonds to the issuer prior to maturity and to reinvest the proceeds of the retired bonds in higher-yielding bonds.

3 B is correct. A fall in interest rates results in a rise in bond values. For a callable bond such as Bond #2, the upside potential is capped because the issuer is more likely to call the bond. In contrast, the upside potential for a putable bond such as Bond #3 is uncapped. Thus, a fall in interest rates would result in a putable bond having more upside potential than an otherwise identical callable bond. Note that A is incorrect because the effective duration of a putable bond increases, not decreases, with a fall in interest rates—the bond is less likely to be put and thus behaves more like an option-free bond. C is also incorrect because the effective convexity of a putable bond is always positive. It is the effective convexity of a callable bond that will change from positive to negative if interest rates fall and the call option is near the money.

4 A is correct:



5 C is correct:

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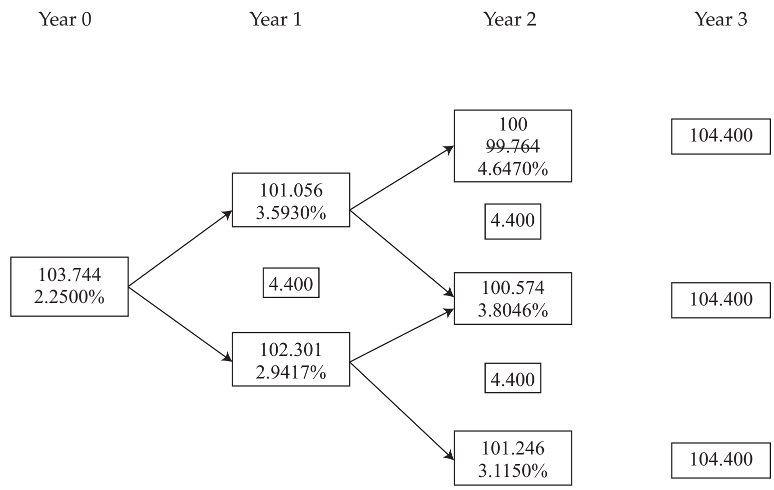

6 C is correct. Bond #3 is a putable bond, and the value of a put option increases as interest rates rise. At higher interest rates, the value of the underlying option-free bond (straight bond) declines, but the decline is offset partially by the increase in the value of the embedded put option, which is more likely to be exercised.

7 C is correct. Regardless of the type of option, an increase in interest rate volatility results in an increase in option value. Because the value of a putable bond is equal to the value of the straight bond *plus* the value of the embedded put option, Bond #3 will increase in value if interest rate volatility increases. Put another way, an increase in interest rate volatility will most likely result in more scenarios where the put option is exercised, which increases the values calculated in the interest rate tree and, thus, the value of the putable bond.

8 C is correct. Bond #2 is a callable bond, and the value of the embedded call option increases as the yield curve flattens. When the yield curve is upward sloping, the one-period forward rates on the interest rate tree are high and opportunities for the issuer to call the bond are fewer. When the yield curve flattens or inverts, many nodes on the tree have lower forward rates, which increases the opportunities to call and, thus, the value of the embedded call option.

9 B is correct. The conversion price of a convertible bond is equal to the par value divided by the conversion ratio—that is, \$1,000/31= \$32.26 per share.

10 B is correct. The conversion value of the bond is 31 × \$37.50 or \$1,162.50, which represents its minimum value. Thus, the convertible bond exhibits mostly stock risk-return characteristics, and a fall in the stock price will result in a fall in the convertible bond price. However, the change in the convertible bond price is less than the change in the stock price because the convertible bond has a floor—that floor is the value of the straight (option-free) bond.

11 C is correct. The option-adjusted spread (OAS) is the constant spread added to all the one-period forward rates that makes the arbitrage-free value of a risky bond equal to its market price. The OAS approach is often used to assess bond relative values. If two bonds have the same characteristics and credit quality, they should have the same OAS. If this is not the case, the bond with the largest OAS (i.e., Bond #2) is likely to be underpriced (cheap) relative to the bond with the smallest OAS (Bond #1).

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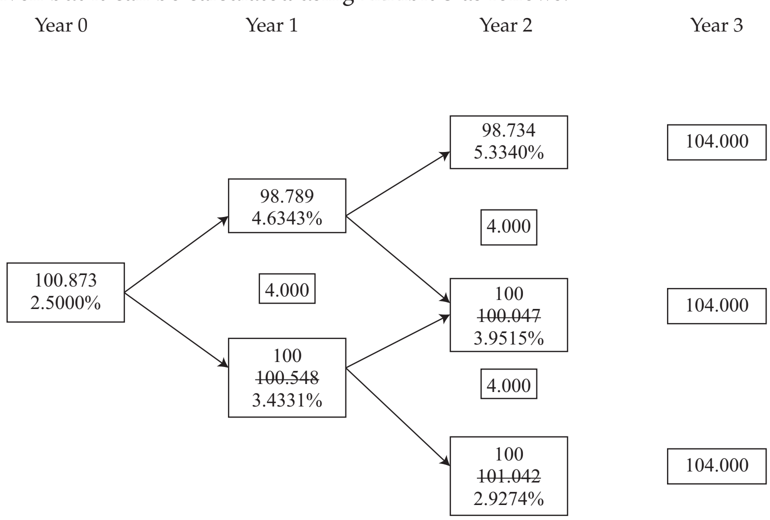
12 A is correct. The effective duration of a floating-rate bond is close to the time to next reset. As the reset for Bond #6 is annual, the effective duration of this bond is lower than or equal to 1.

13 B is correct. Effective duration indicates the sensitivity of a bond's price to a 100 bps parallel shift of the benchmark yield curve assuming no change in the bond's credit spread. The effective duration of an option-free bond such as Bond #3 changes very little in response to interest rate movements. As interest rates rise, a call option moves out of the money, which increases the value of the callable bond and lengthens its effective duration. In contrast, as interest rates rise, a put option moves into the money, which limits the price depreciation of the putable bond and shortens its effective duration. Thus, the bond whose effective duration will lengthen if interest rates rise is the callable bond, i.e., Bond #4.

14 B is correct. The effective duration of Bond #4 can be calculated using Equation 3 from the reading, where ΔCurve is 20 bps, PV_- is 101.238, and PV_+ is 100.478. PV_0, the current full price of the bond (i.e., with no shift), is not given but it can be calculated using Exhibit 3 as follows:



Thus, the effective duration of Bond #4 is:

$$\text{Effective duration} = \frac{101.238 - 100.478}{2 \times (0.0020) \times (100.873)} = 1.88$$

15 A is correct:

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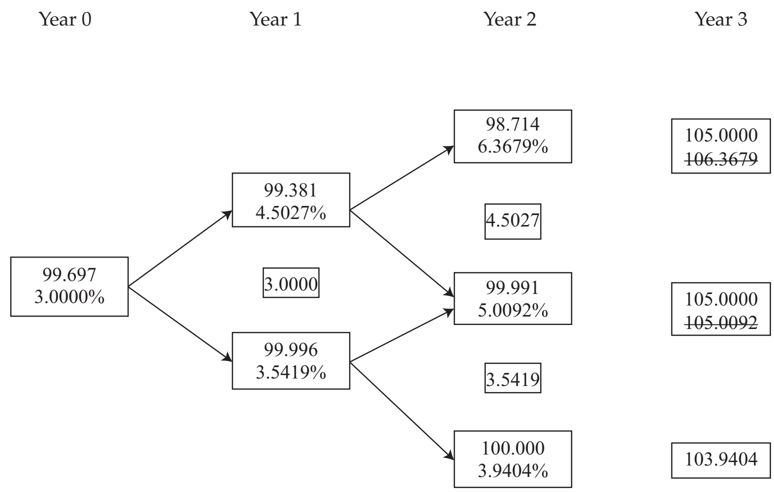
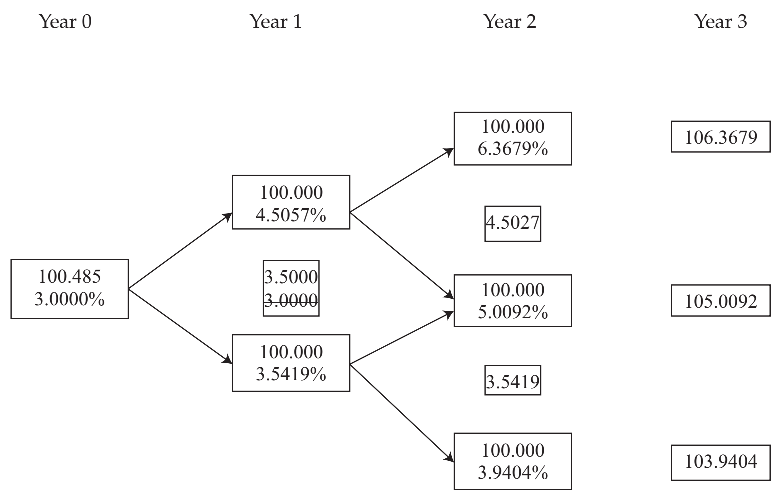


16 C is correct:



17 B is correct. A convertible bond includes a conversion option, which is a call option on the issuer's common stock. This conversion option gives the bondholders the right to convert their debt into equity. Thus, the value of Bond #9, the convertible bond, is equal to the value of Bond #10, the underlying option-free bond (straight bond), plus the value of a call option on Whorton's common stock.

18 A is correct. The minimum value of a convertible bond is equal to the greater of the conversion value of the convertible bond (i.e., Bond #9) and the current value of the straight bond (i.e., Bond #10).

19 C is correct. The risk-return characteristics of a convertible bond depend on the market price of the issuer's common stock (underlying share price) relative to the bond's conversion price. When the underlying share price is well below the conversion price, the convertible bond exhibits mostly bond risk-return characteristics. In this case, the price of the convertible bond is mainly affected by interest rate movements and the issuer's credit spreads. In contrast, when the underlying share price is above the conversion price, the convertible bond exhibits mostly stock risk-return characteristics. In this case, the price of the convertible bond is mainly affected by the issuer's common stock price

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movements. The underlying share price ($30) is lower than the conversion price of Bond #9 ($50). Thus, Bond #9 exhibits mostly bond risk-return characteristics and is least affected by Whorton's common stock price movements.

20 C is correct. If the central bank takes actions that lead to lower interest rates, the yields on Alpha's bonds are likely to decrease. If the yield to maturity on Bond 4 (callable) falls below the 1.55% coupon rate, the call option will become valuable and Alpha may call the bond because it is in the money.

A is incorrect because if the equity market declines, the market value of Alpha stock will also likely decrease. Therefore, Bond 2 (convertible) would have a lower conversion value, and hence, the conversion option likely would not be exercised. Because Bond 2 is currently trading out of the money, it will likely trade further out of the money once the price of Alpha stock decreases.

B is incorrect because Bond 3 (putable) is more likely to be exercised in an increasing rather than a decreasing interest rate environment.

21 C is correct. All four bonds in Exhibit 2 issued by Alpha Corporation offer the same coupon rate and have the same remaining term to maturity. Bond 4 (callable) most likely has a current price that is less than Bond 1 (straight or option free) because investors are short the call option and must be compensated for bearing call risk. Bond 2 (convertible) most likely has a current price that is greater than Bond 1 because investors are paying for the conversion option embedded in Bond 2 and the option has time value associated with it, even though the option is trading out of the money. Similarly, Bond 3 (putable) most likely has a current price that is greater than Bond 1 because investors are paying for the put option.

22 C is correct. The consensus economic forecast is for interest rates to decrease. In an environment of decreasing interest rates, all bond prices should rise ignoring any price impact resulting from any embedded options. When interest rates fall, the value of the embedded call option in Bond 4 (callable) increases, causing an opposing effect on price. The put option of putable bonds, by contrast, increases in value when interest rates rise rather than decline.

23 C is correct. Bond 4 is a callable bond. Value of an issuer call option = Value of straight bond – Value of callable bond. The value of the straight bond may be calculated using the spot rates or the one-year forward rates.

Value of an option-free (straight) bond with a 1.55% coupon using spot rates:

$$1.55/(1.0100)^1 + 1.55/(1.012012)^2 + 101.55/(1.012515)^3 = 100.8789.$$

The value of a callable bond (at par) with no lockout period and a 1.55% coupon rate is 100.5446, as shown in the following table:

	Today	Year 1	Year 2	Year 3
Cash flow		1.55	1.55	100 + 1.55
One-year forward		1.0000%	1.4028%	1.3522%
Value of bond	101.55/1.010000 = 100.5446	101.55/1.014028 = 100.1452 Called at 100	101.55/1.013522 = 100.1952 Called at 100	

The value of the call option = 100.8789 – 100.5446 = 0.3343.

24 B is correct. An increase in interest rate volatility will cause the value of the put and call options embedded in Bond 3 and Bond 4 to increase. Bond 3 (putable) would experience an increase in price because the increased value of the put option increases the bond's value. In contrast, Bond 4 (callable) will experience a price decrease because the increased value of the call option reduces the

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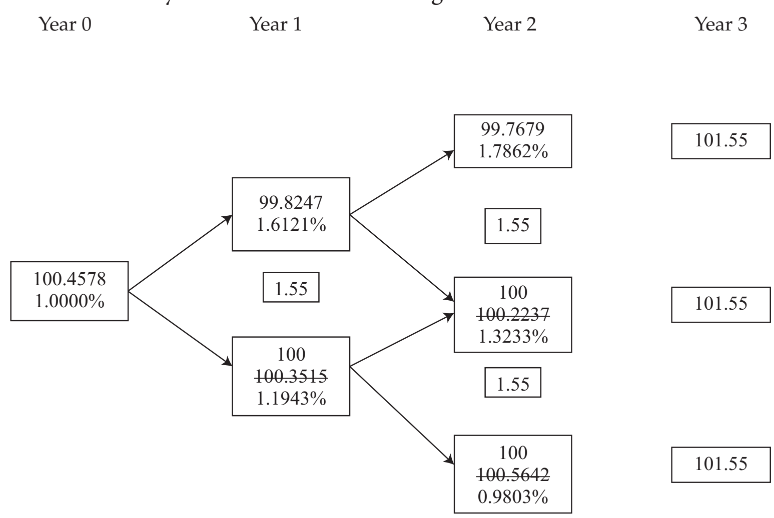
callable bond's value. Bond 2, an out-of-the-money convertible, will resemble the risk–return characteristics of a straight bond and will thus be unaffected by interest rate volatility.

25 A is correct. All else being equal, the value of a put option decreases as the yield curve moves from being upward sloping to flat to downward sloping (inverted). Alternatively, a call option's value increases as the yield curve flattens and increases further if the yield curve inverts. Therefore, if the yield curve became inverted, the value of the embedded option in Bond 3 (putable) would decrease and the value of the embedded option in Bond 4 (Callable) would increase.

26 A is correct. The market price of Bond 4 using the binomial interest rate tree is 100.4578.

The valuation of Bond 4 (Callable) with a 1.55% coupon, no lockout periods, and 15% volatility is shown in the following table.



27 B is correct. A bond with a larger option-adjusted spread (OAS) than that of a bond with similar characteristics and credit quality means that the bond is likely underpriced (cheap). Bond 7 (OAS 85 bps) is relatively cheaper than Bond 6 (OAS 65 bps).

C is incorrect because Bond 8 (CCC) has a lower credit rating than Bond 7 (B) and the OAS alone cannot be used for the relative value comparison. The larger OAS (105 bps) incorporates compensation for the difference between the B and CCC bond credit ratings. Therefore, there is not enough information to draw a conclusion about relative value.

28 B is correct. The AI bond's value if interest rates shift down by 30 bps (PV_) is 100.78:

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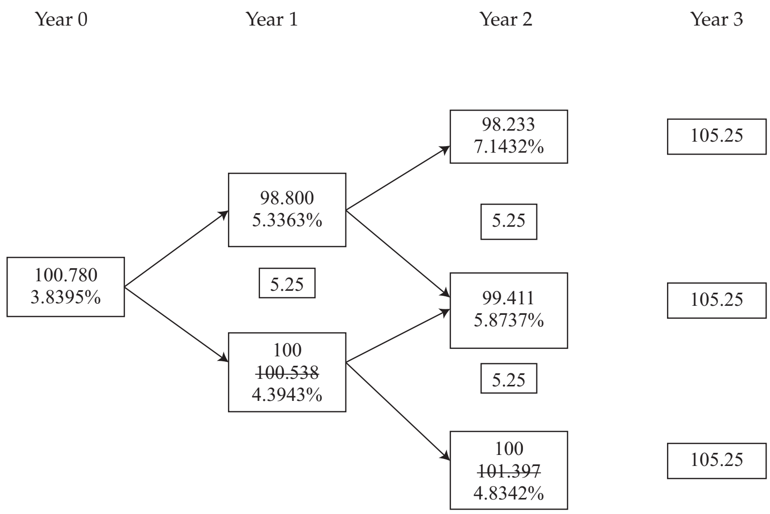
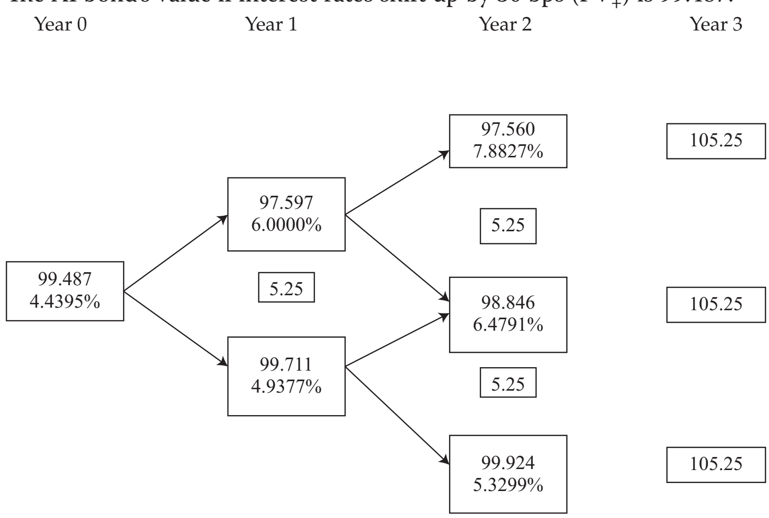


The AI bond's value if interest rates shift up by 30 bps (PV_+) is 99.487:



$$\text{Effective duration} = \frac{(PV_-) - (PV_+)}{2 \times (\Delta\text{Curve}) \times (PV_0)} = \frac{100.780 - 99.487}{2 \times 0.003 \times 100.200} = 2.15$$

29 A is correct. The AI bond is a callable bond and the effective duration of a callable bond decreases when interest rates fall. The reason is because a decline in interest rates may result in the call option moving into the money, which limits the price appreciation of the callable bond. Exhibit 1 also shows that the price of the AI bond is 100.200 and that it is callable at par in one year and two years. Thus, the call option is already in the money and would likely be exercised in response to increases in the AI bond's price.

30 C is correct. The BI bond is an option-free bond and one-sided up-duration and one-sided down-duration will be about equal for option-free bonds.

31 C is correct. The BI bond is an option-free bond. Its longest key rate duration will be in the year of its maturity because the largest cash flow (payment of both coupon and principal) occurs in that year.

32 A is correct. All else being equal, a callable bond will have lower effective convexity than an option-free bond when the call option is in the money. Similarly, when the call option is in the money, a callable bond will also have lower

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effective convexity than a putable bond if the put option is out of the money. Exhibit 1 shows that the callable AI bond is currently priced slightly higher than its call price of par value, which means the embedded call option is in the money. The put option embedded in the CE bond is not in the money; the bond is currently priced 2.1% above par value. Thus, at the current price, the putable CE bond is more likely to behave like the option-free BI bond. Consequently, the effective convexity of the AI bond will likely be lower than the option-free BI bond and the putable CE bond.

33 A is correct. The conversion price would be adjusted downward because Gillette's expected dividend payment of €0.70 is greater than the threshold dividend of €0.50.

34 B is correct. The market conversion premium per share is equal to the market conversion price minus the underlying share price. The market conversion price is calculated as follows:

$$\text{Market conversion price} = \frac{\text{Convertible bond price}}{\text{Conversion ratio}}$$

$$= \frac{€1{,}123}{€1{,}000/€10 \text{ per share}} = €11.23 \text{ per share}$$

The market conversion premium per share is then calculated as follows:

Market conversion premium per share = Market conversion price − Underlying share price
= €11.23 − €9.10 = €2.13

35 C is correct. The value of a convertible bond with both an embedded call option and a put option can be determined using the following formula:

Value of callable putable convertible bond = Value of straight bond + Value of call option on the issuer's stock − Value of issuer call option + Value of investor put option.

Value of callable putable bond = €978 + €147 − €43 + €26 = €1,108

36 A is correct. Over the next year, Gillette believes that Raffarin's share price will continue to increase towards the conversion price but not exceed it. If Gillette's forecast becomes true, the return on the RI bond will increase but at a lower rate than the increase in Raffarin's share price because the conversion price is not expected to be reached.

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READING

35

Credit Analysis Models

by James F. Adams, PhD, CFA, and Donald J. Smith, PhD

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LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** explain expected exposure, the loss given default, the probability of default, and the credit valuation adjustment;
☐	**b.** explain credit scores and credit ratings;
☐	**c.** calculate the expected return on a bond given transition in its credit rating;
☐	**d.** explain structural and reduced-form models of corporate credit risk, including assumptions, strengths, and weaknesses;
☐	**e.** calculate the value of a bond and its credit spread, given assumptions about the credit risk parameters;
☐	**f.** interpret changes in a credit spread;
☐	**g.** explain the determinants of the term structure of credit spreads and interpret a term structure of credit spreads;
☐	**h.** compare the credit analysis required for securitized debt to the credit analysis of corporate debt.

1 INTRODUCTION

This reading covers important concepts, tools, and applications of credit analysis. The topic of Section 2 is modeling credit risk. The inputs to credit risk modeling are the expected exposure to default loss, the loss given default, and the probability of default. We explain these terms and use a numerical example to illustrate the calculation of the credit valuation adjustment for a corporate bond and its credit spread over a government bond yield taken as a proxy for a default risk-free rate (or default-free rate).

Section 3 discusses credit scoring and credit ratings. Credit scoring is a measure of credit risk used in retail loan markets, and ratings are used in the wholesale bond market. Section 4 explains two types of credit analysis models used in practice—structural

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models and reduced-form models. Both models are highly mathematical and beyond the scope of this reading. Therefore, we provide only an overview to highlight the key ideas and similarities and differences between them.

Section 5 uses the arbitrage-free framework and a binomial interest rate tree to value risky fixed-rate and floating-rate bonds for different assumptions about interest rate volatility. Section 6 builds on the credit risk model to interpret changes in credit spreads that arise from changes in the assumed probability of default, the recovery rate, or the exposure to default loss. The term structure of credit spreads is explained in Section 7. Section 8 compares the credit analysis required for securitized debt to the credit analysis of corporate bonds.

2 MODELING CREDIT RISK AND THE CREDIT VALUATION ADJUSTMENT

The difference between the yields to maturity on a corporate bond and a government bond with the same maturity is the most commonly used measure of credit risk. It is called the *credit spread* and is also known in practice as the G-spread. It reveals the compensation to the investor for bearing the default risk of the issuer—the possibility that the issuer fails to make a scheduled payment in full on the due date—and for losses incurred in the event of default.

The terms "default risk" and "credit risk" may be used interchangeably in practice, but we will distinguish between the two in this reading. Default risk is the narrower term because it addresses the likelihood of an event of default. Credit risk is the broader term because it considers both the default probability and how much is expected to be lost if default occurs. For example, it is possible that the default risk on a collateralized loan is high while the credit risk is low, especially if the value of the collateral is high relative to the amount that is owed.

We assume for this reading that the corporate bond and the default risk-free government bond have the same taxation and liquidity. This is a simplifying assumption, of course. In reality, government bonds typically are more liquid than corporate bonds. Also, differences in liquidity within the universe of corporate bonds is great. Government bonds are available in greater supply than even the most liquid corporates and have demand from a wider set of institutional investors. In addition, government bonds can be used more readily as collateral in repo transactions and for centrally cleared derivatives. There also are differences in taxation in some markets. For example, interest income on US corporate bonds is taxable by both the federal and state governments. Government debt, however, is exempt from taxes at the state level. Disregarding tax and liquidity differences allows us to focus on default risk and expected loss as the determining factors for the credit spread.

The first factor to consider in modeling credit risk is the **expected exposure** to default loss. This quantity is the projected amount of money the investor could lose if an event of default occurs, before factoring in possible recovery. Although the most common event of default is nonpayment leading to bankruptcy proceedings, the bond prospectus might identify other events of default, such as the failure to meet a different obligation or the violation of a financial covenant.

Consider a 1-year, 4% annual payment corporate bond priced at par value. The expected exposure to default loss at the end of the year is simply 104 (per 100 of par value). Later in this reading we include multiple time periods and volatility in interest rates. That complicates the calculation of expected exposure because we will need

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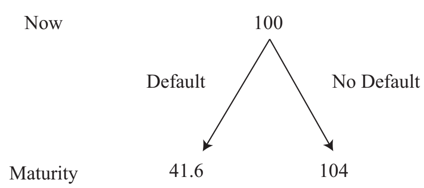
to consider the probability that the bond price varies as interest rates vary. In this initial example, the exposure is simply the final coupon payment plus the redemption of principal.

The second factor is the assumed **recovery rate**, which is percentage of the loss recovered from a bond in default. The recovery rate varies by industry, the degree of seniority in the capital structure, the amount of leverage in the capital structure in total, and whether a particular security is secured or otherwise collateralized. We assume a 40% recovery rate for this corporate bond, which is a common baseline assumption in practice. Given the recovery rate assumption, we can determine the assumed **loss given default** (the amount of loss if a default occurs). This is 62.4 per 100 of par value: 104 × (1 – 0.40) = 62.4. A related term is *loss severity*; if the recovery rate is 40%, the assumed loss severity is 60%.

Exhibit 1 illustrates the projected cash flows on the corporate bond. If there is no default, the investor receives 104. If default occurs, the investor receives 41.6: 104 – 62.4 = 41.6. We assume instantaneous recovery, which surely is another simplifying assumption. In practice, lengthy time delays can occur between the event of default and eventual recovery of cash. Notice that we assume in this reading that the recovery rate applies to interest as well as principal. One last note is that in the exhibits in this reading, calculations may slightly differ on occasion due to rounding at intermediate steps.

Exhibit 1 A Simple Credit Risk Example



The third factor is the assumed **probability of default**, which is the probability that a bond issuer will not meet its contractual obligations on schedule. It is important in credit risk modeling to distinguish *risk-neutral* probabilities of default and *actual* (or historical) default probabilities. "Risk-neutral" follows the usage of the term in option pricing. In the risk-neutral option pricing methodology, the expected value for the payoffs is discounted using the risk-free interest rate. The key point is that in getting the expected value, the risk-neutral probabilities associated with the payoffs need to be used. The same idea applies to valuing corporate bonds.

Suppose that a credit rating agency has collected an extensive data set on the historical default experience for 1-year corporate bonds issued by companies having the same business profile as the issuer in this example. It is observed that 99% of the bonds survive and make the full coupon and principal payment at maturity. Just 1% of the bonds default, resulting in an average recovery rate of 40%. Based on these data, the actual default probability for the corporate bond can reasonably be assumed as 1%.

If the actual probability of default is used to get the expected future value for the corporate bond, the result is 103.376: (104 × 0.99) + (41.6 × 0.01) = 103.376. Discounting that at an assumed risk-free rate of 3% gives a present value of 100.365: 103.376/1.03 = 100.365. Note that in risk-neutral valuation, the expected value is discounted using the risk-free rate and not the bond's yield to maturity. The key point is that 100.365 overstates the observed value of the bond, which is 100. The issue is to determine the default probability that does produce a value of 100.

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Denote the risk-neutral default probability to be P^*. The probability of survival is $1 - P^*$. Given that the corporate bond is priced at 100, $P^* = 1.60\%$. This is found as the solution to P^* in this equation:[1]

$$100 = \frac{\left[104 \times \left(1 - P^*\right)\right] + \left(41.6 \times P^*\right)}{1.03}$$

One reason for the difference between actual (or historical) and risk-neutral default probabilities is that actual default probabilities do not include the default risk premium associated with uncertainty over the timing of possible default loss. Another reason is that the observed spread over the yield on a risk-free bond in practice also includes liquidity and tax considerations in addition to credit risk.

To further see the interaction between the credit risk parameters—the expected exposure, the loss given default, and the probability of default—we consider a 5-year, zero-coupon corporate bond. Our goal is to determine the fair value for the bond given its credit risk, its yield to maturity, and its spread over a maturity-matching government bond.

Exhibit 2 displays the calculation of the **credit valuation adjustment** (CVA). The CVA is the value of the credit risk in present value terms. In Exhibit 2, LGD stands for the loss given default, POD for the probability of default on the given date, POS for the probability of survival as of the given date, DF for the discount factor, and PV for the present value.

Exhibit 2 A 5-Year, Zero-Coupon Corporate Bond

Date (1)	Exposure (2)	Recovery (3)	LGD (4)	POD (5)	POS (6)	Expected Loss (7)	DF (8)	PV of Expected Loss (9)
0								
1	88.8487	35.5395	53.3092	1.2500%	98.7500%	0.6664	0.970874	0.6470
2	91.5142	36.6057	54.9085	1.2344%	97.5156%	0.6778	0.942596	0.6389
3	94.2596	37.7038	56.5558	1.2189%	96.2967%	0.6894	0.915142	0.6309
4	97.0874	38.8350	58.2524	1.2037%	95.0930%	0.7012	0.888487	0.6230
5	100.0000	40.0000	60.0000	1.1887%	93.9043%	0.7132	0.862609	0.6152
				6.0957%			CVA =	3.1549

The first step is to get the exposures to default loss. These are shown in column 2 of Exhibit 2. We assume a flat government bond yield curve at 3.00%. Also, we assume that default occurs only at year-end—on dates 1, 2, 3, 4, and 5—and that default will not occur on date 0, the current date. The exposure on date 5 is 100. For the other dates, we discount using the risk-free rate and the remaining number of years until maturity.

$100/(1.0300)^4 = 88.8487$
$100/(1.0300)^3 = 91.5142$
$100/(1.0300)^2 = 94.2596$
$100/(1.0300)^1 = 97.0874$

1 This example is based on a similar one in Duffie and Singleton (2003).

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Note that there is no interest rate volatility in this example. In Section 5, we use the arbitrage-free framework to build a binomial interest rate tree for a specified level of volatility. Then, knowing the probability of attaining each node in the tree, we calculate the *expected exposure* for each date.

Column 3 of Exhibit 2 projects the assumed recovery if default occurs. Here the recovery rate is a percentage of the exposure. In general, it will be a percentage of the expected exposure, including coupon interest payments, when the model allows for interest rate volatility. We assume for this example that the recovery rate is 40%. The amounts shown in Column 3 are the exposures times 0.40:

$$88.8487 \times 0.40 = 35.5395$$
$$91.5142 \times 0.40 = 36.6057$$
$$94.2596 \times 0.40 = 37.7038$$
$$97.0874 \times 0.40 = 38.8350$$
$$100.0000 \times 0.40 = 40.0000$$

Column 4 shows the loss given default (LGD). It is the exposure for each date minus the assumed recovery. If the issuer defaults on date 4, the investor's loss is projected to be 58.2524 (= 97.0874 – 38.8350) per 100 of par value.

The next parameter is the risk-neutral probability of default (POD) for each date. In Column 5 of Exhibit 2, we assume that the given POD on date 1 is 1.25%. We assume *conditional probabilities of default*, meaning that each year-by-year POD assumes no prior default. This initial POD, which is called the hazard rate in statistics, is used to calculate the remaining PODs. Column 6 reports the probability of survival (POS) for each year. The probability of surviving past date 1 and arriving at date 2 is 98.75% (= 100% – 1.25%). Therefore, the POD for date 2 is 1.2344% (= 1.25% × 98.75%) and the POS is 97.5156% (= 98.75% – 1.2344%). The POD for date 3 is 1.2189% (= 1.25% × 97.5156%), and the POS is 96.2967% (= 97.5156% – 1.2189%). The cumulative probability of default over the 5-year lifetime of the corporate bond is 6.0957%, the sum of the PODs in Column 5. The probability of the bond surviving until maturity is 93.9043%. Note that 6.0957% plus 93.9043% equals 100%.

Another method to calculate the POS for each year, a method that is used later in this reading, is 100% minus the hazard rate raised to the power of the number of years:

$$(100\% - 1.25\%)^1 = 98.7500\%$$
$$(100\% - 1.25\%)^2 = 97.5156\%$$
$$(100\% - 1.25\%)^3 = 96.2967\%$$
$$(100\% - 1.25\%)^4 = 95.0930\%$$
$$(100\% - 1.25\%)^5 = 93.9043\%$$

The assumed hazard rate does not need to be the same each year. Later in this reading, we will show some examples of it changing over the lifetime of the bond.

Column 7 gives the *expected loss* for each date. This is the LGD times the POD. For example, if default occurs on date 3, the expected loss is 0.6894 per 100 of par value. The exposure is 94.2596. At 40% recovery, the LGD is 56.5558. Assuming no prior default, the POD for that date is 1.2189%. The expected loss of 0.6894 is calculated as 56.5558 times 1.2189%.

Column 8 presents the default risk-free *discount factors* based on the flat government bond yield curve at 3.00%. The date-5 discount factor is 0.862609 [$=1/(1.0300)^5$]. Finally, Column 9 shows the present value (PV) of the expected loss for each year. These are the expected loss times the discount factor. The present value of the expected date-5 loss is 0.6152 per 100 of par value, the expected loss of 0.7132 times 0.862609.

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The sum of Column 9 is 3.1549. This amount is known as the credit valuation adjustment (CVA). It allows us to calculate the *fair value* of the 5-year, zero-coupon corporate bond. If the bond were default free, its price would be 86.2609—that is, the par value of 100 times the date-5 discount factor. Subtracting the CVA from this amount gives a fair value of 83.1060 (= 86.2609 – 3.1549).

We can now calculate the credit spread on the corporate bond. Given a price of 83.1060, its yield to maturity is 3.77%. The solution for *yield* in this expression is:

$$\frac{100}{(1 + \text{yield})^5} = 83.1060$$

The yield on the 5-year, zero-coupon government bond is 3.00%. Therefore, the credit spread is 77 basis points: 3.77% – 3.00% = 0.77%.[2] A key point is that the compensation for credit risk received by the investor can be expressed in two ways: (1) as the CVA of 3.1549 in terms of a present value per 100 of par value on date 0, and (2) as a credit spread of 77 basis points in terms of an annual percentage rate for five years.

Exhibit 3 provides a display of the projected cash flows and annual rates of return depending on when and if default occurs. On date 0 the 5-year, zero-coupon corporate bond is worth its fair value, 83.1060 per 100 of par value. If on date 1 the issuer defaults, the investor gets the recoverable amount of 35.5395. The annual rate of return is –57.24%, the solution for the internal rate of return (IRR):

$$83.1060 = \frac{35.5395}{1 + \text{IRR}}$$

$$\text{IRR} = -0.5724$$

If there is no default, the investor receives the coupon payment on that date, which in this case is zero.

2 An approximation for the credit spread, commonly used in practice, is the initial default probability (which is called the annual hazard rate) times one minus the recovery rate. In this case, the approximate credit spread is 0.75% [= 1.25% × (1 – 0.40)].

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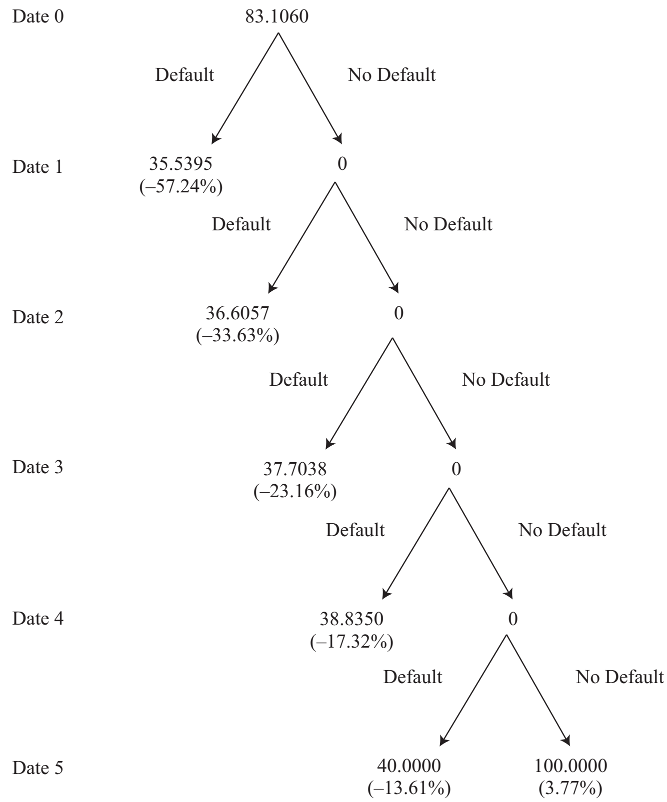
Exhibit 3 Projected Annual Rates of Return



If the issuer defaults on date 2, the annual rate of return is –33.63%.

$$83.1060 = \frac{0}{(1+\text{IRR})^1} + \frac{36.6057}{(1+\text{IRR})^2}$$

$$\text{IRR} = -0.3363$$

If the default occurs on the maturity date, the annual rate of return "improves" to –13.61%.

$$83.1060 = \frac{0}{(1+\text{IRR})^1} + \frac{0}{(1+\text{IRR})^2} + \frac{0}{(1+\text{IRR})^3} + \frac{0}{(1+\text{IRR})^4} + \frac{40.0000}{(1+\text{IRR})^5}$$

$$\text{IRR} = -0.1361$$

If there is no default, which is most likely because the probability of survival to date 5 is 93.9043%, the realized rate of return is 3.77%. This reminds us that a yield to maturity on a risky bond is a measure of return to the investor, assuming no default.

The key observation from this example is that the investor faces a wide range of outcomes on the bond depending critically on the *timing* of default. This is a source of the default risk premium that typically is built into the pricing of the bond. Said differently, the probability of default in credit risk models incorporates the likely time of incidence of default events as well as uncertainty over the timing of the events.

Although this is clearly a simple example of a credit risk model, it does serve to illustrate the interaction between the exposure to default loss for each date, the recovery rate, the loss given default, the probability of default, the expected loss, and the present value of expected loss. It can be made more complex and realistic. Here, the initial probability of default (the hazard rate) used to calculate the conditional

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PODs and the recovery rate are the same for each year, but these parameters could vary year by year. The government bond yield curve is flat, but it could be upward or downward sloping. Then, the discount factors would need to be calculated sequentially by a process known as "bootstrapping." An example of this is included in Section 5.

In this example, we assume a default probability and a recovery rate to get the fair value for the risky corporate bond. This could be reversed. Suppose that we observe that the market price for the 5-year, zero-coupon bond is 83.1060 and its credit spread is 77 basis points. Then the same table could be used to get by trial-and-error search the probability of default that is consistent with the bond price and a recovery rate of 40%. That initial default probability, which is used to calculate the year-by-year PODs, would be 1.25%. Another possibility is to change the assumed recovery rate. Suppose it is 30% of the exposure. Given the observed bond price and credit spread, the default probability would turn out to be 1.0675%. In that case, the lower recovery rate is offset by the lower probability of default. A higher recovery rate would need to be offset by a higher default probability. In general, for a given price and credit spread, the assumed probability of default and the recovery rate are positively correlated.

EXAMPLE 1

Analysis of Credit Risk (1)

A fixed-income analyst is considering the credit risk over the next year for three corporate bonds currently held in her bond portfolio. Her assessment for the exposure, probability of default, and recovery is summarized in this table:

Corporate Bond	Exposure (per 100 of par value)	Probability of Default	Recovery (per 100 of par value)
A	104	0.75%	40
B	98	0.90%	35
C	92	0.80%	30

Although all three bonds have very similar yields to maturity, the differences in the exposures arise because of differences in their coupon rates.

Based on these assumptions, how would she rank the three bonds in terms of credit risk over the next year, highest to lowest?

Solution:

She needs to get the loss given default (LGD) for each bond and multiply that by the probability of default (POD) to get the expected loss. The LGD is the exposure minus the assumed recovery.

Corporate Bond	LGD (per 100 of par value)	POD	Expected Loss
A	64	0.75%	0.480
B	63	0.90%	0.567
C	62	0.80%	0.496

Based on the expected losses, Bond B has the highest credit risk and Bond A the lowest. The ranking is: B, C, and A. Note that there is not enough information to recommend a trading strategy because the current prices of the bonds are not given.

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EXAMPLE 2

Analysis of Credit Risk (2)

A fixed-income trader at a hedge fund observes a 3-year, 5% annual payment corporate bond trading at 104 per 100 of par value. The research team at the hedge fund determines that the risk-neutral probability of default used to calculate the conditional POD for each date for the bond is 1.50% given a recovery rate of 40%. The government bond yield curve is flat at 2.50%.

Based on these assumptions, does the trader deem the corporate bond to be overvalued or undervalued? By how much? If the trader buys the bond at 104, what are the projected annual rates of return?

Solution:

The trader needs to build a table similar to that shown in Exhibit 2; this table is presented in Exhibit 4:

Exhibit 4 CVA Calculation for Example 2

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	109.8186	43.9274	65.8911	1.5000%	98.5000%	0.9884	0.975610	0.9643
2	107.4390	42.9756	64.4634	1.4775%	97.0225%	0.9524	0.951814	0.9066
3	105.0000	42.0000	63.0000	1.4553%	95.5672%	0.9169	0.928599	0.8514
				4.4328%			CVA =	2.7222

The exposures are the values for the bond plus the coupon payment for each date assuming a yield to maturity of 2.50%. The exposure is 109.8186 for date 1 when two years to maturity remain.

$$5+\frac{5}{(1.0250)^1}+\frac{105}{(1.0250)^2}=109.8186$$

The assumed recovery for date 1 is 43.9274 (= 109.8186 × 0.40) for a loss given default of 65.8911 (= 109.8186 – 43.9274). [Note: All calculations in this reading are carried out on spreadsheets to preserve precision. The rounded results are reported in the text]. The expected loss is 0.9884 (= 65.8911 × 0.0150). The discount factor for date 1 is 0.975610 $=1/(1.0250)^1$. The present value of the expected loss is 0.9643 (= 0.9884 × 0.975610).

The credit valuation adjustment (CVA) for the bond is 2.7222, the sum of the present values of expected loss. If this 5-year, 5% bond were default-free, its price would be 107.1401.

$$\frac{5}{(1.0250)^1}+\frac{5}{(1.0250)^2}+\frac{105}{(1.0250)^3}=107.1401$$

Therefore, the fair value of the bond given the assumed credit risk parameters is 104.4178 (= 107.1401 – 2.7222). The fixed-income trader at the hedge fund would deem this corporate bond to be undervalued by 0.4178 per 100 of par value if it is trading at a price of 104.

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The projected annual rates of return for default on dates 1, 2, and 3 are –57.76%, –33.27%, and –22.23%, respectively. If there is no default, the rate of return is 3.57%, which is the yield to maturity. Note that these rates of return neglect coupon reinvestment risk because internal rate of return calculations implicitly assume reinvestment at the same rate. These are the calculations:

$$104 = \frac{43.9274}{(1 + \text{IRR})^1}$$

$$\text{IRR} = -0.5776$$

$$104 = \frac{5}{(1 + \text{IRR})^1} + \frac{42.9756}{(1 + \text{IRR})^2}$$

$$\text{IRR} = -0.3327$$

$$104 = \frac{5}{(1 + \text{IRR})^1} + \frac{5}{(1 + \text{IRR})^2} + \frac{42.0000}{(1 + \text{IRR})^3}$$

$$\text{IRR} = -0.2223$$

$$104 = \frac{5}{(1 + \text{IRR})^1} + \frac{5}{(1 + \text{IRR})^2} + \frac{105}{(1 + \text{IRR})^3}$$

$$\text{IRR} = 0.0357$$

3 CREDIT SCORES AND CREDIT RATINGS

Credit scores and ratings are used by lenders in deciding to extend credit to a borrower and in determining the terms of the contract. Credit scores are used primarily in the retail lending market for small businesses and individuals. Credit ratings are used in the wholesale market for bonds issued by corporations and government entities as well as for asset-backed securities (ABS).

Credit scoring methodologies can vary. In some countries, only negative information, such as delinquent payments or outright default, is included. Essentially, everyone has a good credit score until proven otherwise. In other countries, a broader set of information is used to determine the score. A score reflects actual observed factors. In general, credit scoring agencies are national in scope because of differences in legal systems and privacy concerns across countries.

The FICO score, which is the federally registered trademark of the Fair Isaac Corportion, is used in the United States by about 90% of lenders to retail customers.[3] FICO scores are computed using data from consumer credit files collected by three national credit bureaus: Experian, Equifax, and TransUnion. Five primary factors are included in the proprietary algorithm used to get the score:

- 35% for the payment history—this includes the presence or lack of such information as delinquency, bankruptcy, court judgments, repossessions, and foreclosures.
- 30% for the debt burden—this includes credit card debt-to-limit ratios, the number of accounts with positive balances, and the total amount owed.

3 The Fair Isaac Corporation has released FICO scores since 1989. William Fair, a mathematician, and Earl Isaac, an engineer, started the company in 1956 to use multivariate analysis on the immense amount of data being collected on credit cards. An interesting history of the company is available at www.fundinguniverse.com.

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- 15% for the length of credit history—this includes the average age of accounts on the credit file and the age of the oldest account.
- 10% for the types of credit used—this includes the use of installment payments, consumer finance, and mortgages.
- 10% for recent searches for credit—this includes "hard" credit inquiries when consumers apply for new loans but not "soft" inquiries, such as for employee verification or self-checking one's score.

Fair Isaac Corporation, on its website, notes items that are not included in the FICO credit score: race, color, national origin, sex, marital status, age, salary, occupation, employment history, home address, and child/family support obligations. The company also reports from time to time the distribution across scores, which range from a low of 300 to a perfect score of 850. Exhibit 5 shows the distribution for three particular months: October 2005 before the financial crisis, October 2009 at the depth of the crisis, and, considerably after the crisis, April 2015. It is evident that the percentage of weaker scores increased as economic conditions worsened but has gone down since then. Using straight-line interpolation, the median FICO has increased from 709.4, to 710.0, to 713.5 over these observations.

Exhibit 5 Distribution of FICO Scores

FICO Score	October 2005	October 2009	April 2015
300–499	6.6%	7.3%	4.9%
500–549	8.0%	8.7%	7.6%
550–599	9.0%	9.1%	9.4%
600–649	10.2%	9.5%	10.3%
650–699	12.8%	11.9%	13.0%
700–749	16.4%	15.9%	16.6%
750–799	20.1%	19.4%	18.2%
800–850	16.9%	18.2%	19.9%

Source: Fair Isaac Corporation.

EXAMPLE 3

Credit Scoring

Tess Waresmith is a young finance professional who plans to eventually buy a two-family house, living in one unit and renting the other to help cover the mortgage payments. She is a careful money manager and every year checks her FICO credit score. She is pleased to see that it has improved from 760 last year to 775 this year. Which of these factors can explain the improvement?

A She is now one year older and has not had any late payments on credit cards during the year.

B Her bank on its own raised her limit on a credit card from $1,000 to $2,500, but she has maintained the same average monthly balance.

C She applied for and received a new car loan from her credit union.

D She refrained from checking her FICO score monthly, like some of her friends do.

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Solution:

Factors A, B, and C help explain the improvement. Going down the list:

A Age itself is not a factor used by Fair Isaac to determine the credit score. However, the average age of the accounts is a factor as is the age of the oldest account. Therefore, other things being equal, the passage of time tends to improve the score. In general, age and credit score are highly correlated.

B The credit card debt-to-limit ratio is a component of the debt burden. Having a higher limit for the same average balance reduces the ratio and improves the credit score.

C Because the car loan is a new type of credit usage and thus does not have any late payments, it has a positive impact on the score.

D Refraining from self-checking one's credit score has no impact. Self-checking is deemed to be a "soft inquiry" and does not factor into the calibration of the FICO score.

Although credit scores are the primary measure of credit risk in retail lending, credit ratings are widely used in corporate and sovereign bond markets. The three major global credit rating agencies are Moody's Investors Service, Standard & Poor's, and Fitch Ratings. Each provides quality ratings for issuers as well as specific issues. These, like credit scores, are ordinal ratings focusing on the probability of default. Exhibit 6 displays historical default experience by rating category from 1995 to 2014. Notice that defaults on corporate debt that were rated as investment grade at the time of default are rare events. However, the "high-yield" sector does experience defaults, especially when the securities have been downgraded below B.

Exhibit 6 Historical Corporate Default Experience by Rating (entries are %)

	AAA	AA	A	BBB	BB	B	CCC/CC/C
1995	0.00	0.00	0.00	0.17	0.99	4.58	28.00
1996	0.00	0.00	0.00	0.00	0.45	2.91	8.00
1997	0.00	0.00	0.00	0.25	0.19	3.51	12.00
1998	0.00	0.00	0.00	0.41	0.82	4.63	42.86
1999	0.00	0.17	0.18	0.20	0.95	7.29	33.33
2000	0.00	0.00	0.27	0.37	1.15	7.67	35.96
2001	0.00	0.00	0.27	0.34	2.94	11.52	45.45
2002	0.00	0.00	0.00	1.02	2.88	8.20	44.44
2003	0.00	0.00	0.00	0.23	0.58	4.06	32.73
2004	0.00	0.00	0.08	0.00	0.43	1.45	16.18
2005	0.00	0.00	0.00	0.07	0.31	1.74	9.09
2006	0.00	0.00	0.00	0.00	0.30	0.82	13.33
2007	0.00	0.00	0.00	0.00	0.20	0.25	15.24
2008	0.00	0.38	0.39	0.49	0.81	4.08	27.00
2009	0.00	0.00	0.22	0.55	0.75	10.92	49.46
2010	0.00	0.00	0.00	0.00	0.58	0.85	22.73
2011	0.00	0.00	0.00	0.07	0.00	1.66	16.42
2012	0.00	0.00	0.00	0.00	0.30	1.56	27.33

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Exhibit 6 (Continued)

	AAA	AA	A	BBB	BB	B	CCC/CC/C
2013	0.00	0.00	0.00	0.00	0.09	1.63	24.18
2014	0.00	0.00	0.00	0.00	0.00	0.77	17.03

Source: Standard & Poor's, "2014 Annual Global Corporate Default Study and Rating Transitions," Table 3 (30 April 2015).

The credit rating agencies consider the expected loss given default by means of *notching*, which is an adjustment to the issuer rating to reflect the priority of claim for specific debt issues of that issuer and to reflect any subordination. The issuer rating is typically for senior unsecured debt. The rating on subordinated debt is then adjusted, or "notched," by lowering it one or two levels—for instance, from A+ down to A or further down to A–. This inclusion of loss given default in addition to the probability of default explains why they are called "credit ratings" and not just "default ratings."

In addition to the "letter grade," the rating agencies provide an outlook (positive, stable, or negative) for the issuer as well as when the issuer is under "watch." For example, what follows is the history of Standard & Poor's issuer rating for RadioShack Corporation as it moves from BBB– in 1969, to BB+ in 1978, to AAA in 1983, to BB in 2006, and finally to default in 2015:[4]

• 02 May 1969	BBB–
• 13 October 1978	BB+
• 12 December 1980	BB
• 01 April 1981	BBB+
• 07 January 1982	A
• 10 January 1983	AAA
• 28 November 1984	A+/Watch Negative
• 08 August 1991	A/Stable
• 04 January 1993	A/Watch Negative
• 25 February 1993	A–/Stable
• 27 May 1993	A–/Watch Positive
• 17 January 1994	A–/Stable
• 17 October 1996	A–/Negative
• 24 February 1999	A–/Stable
• 13 May 2005	A–/Watch Negative
• 08 August 2005	BBB+/Stable
• 21 April 2006	BBB–/Stable
• 24 July 2006	BBB–/Negative
• 25 October 2006	BB/Negative
• 12 August 2008	BB/Stable
• 21 November 2011	BB–/Stable
• 02 March 2012	B+/Negative
• 30 July 2012	B–/Negative

(continued)

4 This history is from Standard & Poor's "2014 Annual Global Corporate Default Study and Rating Transitions," Table 54 (30 April 2015).

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• 21 November 2012	CCC+/Negative
• 01 August 2013	CCC/Negative
• 20 December 2013	CCC+/Negative
• 16 June 2014	CCC/Negative
• 11 September 2014	CCC–/Negative
• 06 February 2015	D

The history of RadioShack illustrates the rating can remain the same for prolonged periods of time. The company was A+ from 1984 to 1991 and A– from 1993 to 2005. The rating agencies report *transition matrixes* based on their historical experience. Exhibit 7 is a representative example. It shows the probabilities of a particular rating transitioning to another over the course of the following year. An A-rated issuer has an 87.50% probability of remaining at that level, a 0.05% probability of moving up to AAA (such as RadioShack did in 1983), a 2.50% probability of moving up to AA, an 8.40% probability of moving down to BBB, 0.75% down to BB, 0.60% to B, 0.12% to CCC, CC, or C, and 0.08% to D, where it is in default.

Exhibit 7 Representative One-Year Corporate Transition Matrix (entries are %)

From/To	AAA	AA	A	BBB	BB	B	CCC,CC,C	D
AAA	**90.00**	9.00	0.60	0.15	0.10	0.10	0.05	0.00
AA	1.50	**88.00**	9.50	0.75	0.15	0.05	0.03	0.02
A	0.05	2.50	**87.50**	8.40	0.75	0.60	0.12	0.08
BBB	0.02	0.30	4.80	**85.50**	6.95	1.75	0.45	0.23
BB	0.01	0.06	0.30	7.75	**79.50**	8.75	2.38	1.25
B	0.00	0.05	0.15	1.40	9.15	**76.60**	8.45	4.20
CCC,CC,C	0.00	0.01	0.12	0.87	1.65	18.50	**49.25**	29.60
Credit Spread	0.60%	0.90%	1.10%	1.50%	3.40%	6.50%	9.50%	

Exhibit 7 also shows representative credit spreads for a 10-year corporate bond. The credit transition matrix and the credit spreads allow a fixed-income analyst to estimate a 1-year rate of return given the possibility of credit rating migration but still no default. Assume that an A-rated 10-year corporate bond will have a modified duration of 7.2 at the end of the year given stable yields and spreads. For each possible transition, the analyst can calculate the expected percentage price change as the product of the modified duration and the change in the spread:

From A to AAA:	$-7.2 \times (0.60\% - 1.10\%) = +3.60\%$
From A to AA:	$-7.2 \times (0.90\% - 1.10\%) = +1.44\%$
From A to BBB:	$-7.2 \times (1.50\% - 1.10\%) = -2.88\%$
From A to BB:	$-7.2 \times (3.40\% - 1.10\%) = -16.56\%$
From A to B:	$-7.2 \times (6.50\% - 1.10\%) = -38.88\%$
From A to CCC,CC,C:	$-7.2 \times (9.50\% - 1.10\%) = -60.48\%$

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The probabilities of migration now can be used to calculate the expected percentage change in the bond value over the year. The expected percentage change in bond value for an A-rated corporate bond is found by multiplying each expected percentage price change for a possible credit transition by its respective transition probability and summing the products: [5]

$$(0.0005 \times 3.60\%) + (0.0250 \times 1.44\%) + (0.8750 \times 0\%) + (0.0840 \times -2.88\%) + (0.0075 \times -16.56\%) + (0.0060 \times -38.88\%) + (0.0012 \times -60.48\%) = -0.6342\%$$

Therefore, the expected return on the bond over the next year is its yield to maturity minus 0.6342%, assuming no default. Credit spread migration typically reduces the expected return for two reasons. First, the probabilities for change are not symmetrically distributed around the current rating. They are skewed toward a downgrade rather than an upgrade. Second, the increase in the credit spread is much larger for downgrades than the decrease in the spread for upgrades.

EXAMPLE 4

The Impact of Credit Migration on Expected Return

Manuel Perello is a wealth manager for several Latin American families that seek to keep a portion of their assets in very high-quality corporate bonds. Mr. Perello explains that the yields to maturity on the bonds should be adjusted for possible *credit spread widening* to measure the expected rate of return over a given time horizon. In his presentation to one of the families, he uses a 10-year, AAA-rated corporate bond that would have a modified duration of 7.3 at the end of the year. Using the corporate transition matrix in Exhibit 7, Mr. Perello concludes that the expected return on the bond over the next year can be approximated by the yield to maturity less 32.5 basis points to account for a possible credit downgrade even if there is no default. Demonstrate how he arrives at that conclusion.

Solution:

First, calculate the expected percentage price change using the modified duration for the bond and the change in the credit spread:

From AAA to AA:	$-7.3 \times (0.90\% - 0.60\%) = -2.19\%$
From AAA to A:	$-7.3 \times (1.10\% - 0.60\%) = -3.65\%$
From AAA to BBB:	$-7.3 \times (1.50\% - 0.60\%) = -6.57\%$
From AAA to BB:	$-7.3 \times (3.40\% - 0.60\%) = -20.44\%$
From AAA to B:	$-7.3 \times (6.50\% - 0.60\%) = -43.07\%$
From AAA to CCC,CC,C:	$-7.3 \times (9.50\% - 0.60\%) = -64.97\%$

Second, calculate the expected percentage change in the bond value over the year using the probabilities in the corporate transition matrix:

$$(0.9000 \times 0\%) + (0.0900 \times -2.19\%) + (0.0060 \times -3.65\%) + (0.0015 \times -6.57\%) + (0.0010 \times -20.44\%) + (0.0010 \times -43.07\%) + (0.0005 \times -64.97\%) = -0.3249\%$$

5 In this calculation, we neglect the small probability of migration to the default state. That would need to be taken into consideration if the bond was not investment-grade.

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4 STRUCTURAL AND REDUCED-FORM CREDIT MODELS

Credit analysis models fall into two broad categories—structural models and reduced-form models.[6] Structural models of credit risk go back to the 1970s and the seminal contributions to finance theory by Fisher Black, Myron Scholes, and Robert Merton.[7] Their key insights were that a company defaults on its debt if the value of its assets falls below the amount of its liabilities and that the probability of that event has the features of an option.

Reduced-form varieties emerged in the 1990s with significant contributions from Robert Jarrow, Stuart Turnbull, Darrell Duffie, and Kenneth Singleton.[8] Reduced-form models avoid a fundamental problem with the structural models. The Black–Scholes–Merton option pricing model explicitly assumes that the assets on which the options are written (i.e., the shares of a company) are actively traded. That assumption is fine for stock options; however, the assets of the company typically do not trade. Reduced-form models get around this problem by not treating default as an endogenous (internal) variable. Instead, the default is an exogenous (external) variable that occurs randomly. Unlike structural models that aim to explain *why* default occurs (i.e., when the asset value falls below the amount of liabilities), reduced-form models aim to explain statistically *when*. This is known as the *default time* and can be modeled using a Poisson stochastic process. The key parameter in this process is the *default intensity*, which is the probability of default over the next time increment. Reduced-form credit risk models are thus also called *intensity-based* and *stochastic default rate* models.

Both types of credit risk models have advantages and disadvantages. Structural models provide insight into the nature of credit risk but can be burdensome to implement. The modeler needs to determine the value of the company, its volatility, and the default barrier that is based on the liabilities of the company. In the model, the company defaults when the value of its assets dips below this default barrier. Although straightforward in theory, it can be difficult in practice because of limitations in available data. Examples of companies hiding debt (Enron, Tyco, WorldCom, Parmalat, and Lehman, to name a few) highlight the challenge to measure the default barrier, especially in times when knowing changes in default probabilities would be most beneficial to investors.[9]

Reduced-form models have the advantage that the inputs are observable variables, including historical data. The default intensity is estimated using regression analysis on *company-specific* variables (e.g., leverage ratio, net income-to-assets ratio, and cash-to-assets ratio) and *macroeconomic* variables (e.g., unemployment rate, GDP growth rate, measures of stock market volatility). This flexibility allows the model to directly reflect the business cycle in the credit risk measure.

A disadvantage to reduced-form models is that they, unlike structural models, do not explain the economic reasons for default. Also, reduced-form models assume that default comes as a "surprise" and can occur at any time. In reality, default is rarely a surprise because the issuer usually has been downgraded several times before the final event, as we saw with the RadioShack experience in the previous section.

Exhibit 8 depicts a structural model of default. The vertical axis measures the asset value of the company. This is called a structural model because it depends on the structure of the company's balance sheet—its assets, liabilities, and equity. It also can be called a *company-value* model because the key variable is the asset value of

6 This section draws on the "Credit Risk Modeling" chapter in Fabozzi (2013).
7 See Black and Scholes (1973): 637–654, and Merton (1974): 449–470.
8 See Jarrow and Turnbull (1995): 53–86, and Duffie and Singleton (1999): 687–720.
9 See Smith (2011): 15–22.

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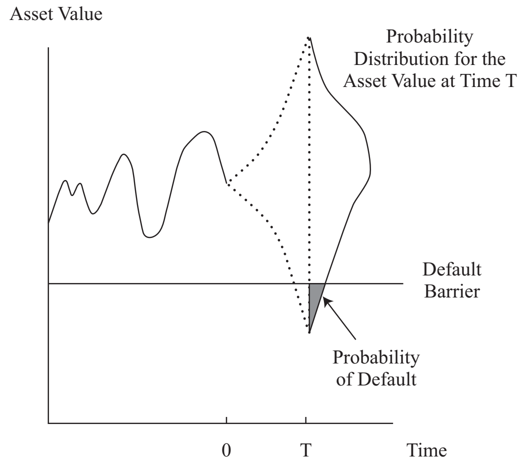
the company. In Exhibit 8 the asset value has been volatile prior to now, time 0, but has remained above the horizontal line that represents the default barrier. If the asset value falls below the barrier, the company defaults on the debt.

Exhibit 8 A Structural Model of Default



Source: This exhibit is adapted from Duffie and Singleton, 2003, page 54.

There is a probability distribution for the asset value as of some future date, time T. The probability of default is endogenous to this structural model. It is the portion of the probability distribution that lies below the default barrier. This default probability increases with the variance of the future asset value, with greater time to T and with greater financial leverage. Less debt in the capital structure lowers the horizontal line and reduces the probability of default. These factors indicate that credit risk is linked to option pricing theory.

An important feature of the structural credit models is that they allow interpretation of debt and equity values in terms of options. Let $A(T)$ be the random asset value as of time T. To simplify, we can assume that the debt liabilities are zero-coupon bonds that mature at time T. These bonds have a face value of K, which represents the default barrier in Exhibit 8. The values for debt and equity at time T are denoted $D(T)$ and $E(T)$ and depend on the relationship between $A(T)$ and K:

$$D(T) + E(T) = A(T) \tag{1}$$

$$E(T) = \text{Max}[A(T) - K,0] \tag{2}$$

$$D(T) = A(T) - \text{Max}[A(T) - K,0] \tag{3}$$

Equation 1 is the balance sheet identity—the market values of debt and equity at time T equal the asset value. Equation 2 indicates that equity is essentially a purchased call option on the assets of the company whereby the strike price is the face value of the debt. It is a long position in a call option because the value of equity goes up when the asset value goes up. Moreover, like an option, equity does not take on negative values. Equation 3 shows that in this formulation, the debtholders own the assets of the company and have written the call option held by the shareholders. We can interpret the premium that the debtholders receive for writing the option as the value of having priority of claim in the event that the asset value falls below K. In that case, the value of equity falls to zero and the debtholders own the remaining assets.

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Suppose that at time T, $A(T) > K$ so that the call option is in the money to the shareholders. Then, $E(T) = A(T) - K$ and $D(T) = A(T) - [A(T) - K] = K$. Instead, suppose that $A(T) < K$ so that the call option is out of the money and the debt is in default. In this case, $E(T) = 0$ and $D(T) = A(T) - 0 = A(T)$. In both situations, as well as when $A(T) = K$, the balance sheet identity holds. Notice that *limited liability* is an inherent assumption in this model. Equity, like an option, does not take on negative values.

EXAMPLE 5

An Equivalent Option Interpretation of Debt and Equity

Carol Feely is a junior credit analyst at one of the major international credit rating agencies. She understands that in the standard structural models equity is interpreted as a call option on the asset value of the company. However, she is not comfortable with the assumption that it is the debtholders who implicitly own the assets and write a call option on them. She claims that the model should start with the understanding that the shareholders own the net value of the company, which is $A(T) - K$, and that their limited liability is essentially the value of a long position in a put option at a strike price of K. Furthermore, the debtholders own a "risk-free" bond having a value of K at time T and a short position in the put that is held by the shareholders.

Demonstrate that Ms. Feely's "embedded put option" interpretation provides the same values for debt and equity at time T as does the more customary call option structural model.

Solution:

A long position in a put option on the asset value at a strike price of K takes the form: $\text{Max}[K - A(T),0]$. This put option has intrinsic value to its holder when $K > A(T)$ and is worthless when $K \le A(T)$. The values for $E(T)$ and $D(T)$ according to Ms. Feely at time T are:

$$E(T) = A(T) - K + \text{Max}[K - A(T),0]$$

$$D(T) = K - \text{Max}[K - A(T),0]$$

If $A(T) > K$ at time T, the put option is out of the money, $E(T) = A(T) - K + 0 = A(T) - K$ and $D(T) = K - 0 = K$. If $A(T) < K$, the put is in the money, $E(T) = A(T) - K + [K - A(T)] = 0$ and $D(T) = K - [K - A(T)] = A(T)$. This interpretation indicates that the value of limited liability to shareholders is the value of the put option that they purchase from the debtholders. Ms. Feely is correct in that the same payoffs as the embedded call option interpretation are obtained.

Although observing that credit risk is inherently linked to option pricing, it is the implementation of structural models that has provided practical value to fixed-income analysis. Many credit rating agencies and consultancies, most notably Moody's KMV Corporation, use option pricing methodologies to estimate such credit risk parameters as the probability of default and the loss given default. Building on the classic Black–Scholes–Merton model and later variants, the model builders use historical data on the company's equity price to estimate volatility, which is a key element in option pricing models.

These advantages and disadvantages indicate that the choice of credit risk model depends on how it is to be used and by whom. Structural models require information best known to the managers of the company (and perhaps their commercial bankers and the credit rating agencies). Therefore, they can be used for internal risk management, for banks' internal credit risk measures, and for publicly available credit

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ratings. Reduced-form models only require information generally available in financial markets, which suggests that they should be used to value risky debt securities and credit derivatives.

VALUING RISKY BONDS IN AN ARBITRAGE-FREE FRAMEWORK

5

In this section, we use the arbitrage-free framework to analyze the credit risk of a corporate bond in the context of volatile interest rates.[10] In Section 2, we solved for the credit valuation adjustment and the credit spread under the assumptions of no interest rate volatility and a flat government bond yield curve. Section 5 shows that a binomial interest rate tree for benchmark bond yields allows us to calculate the *expected exposure* to default loss. In addition, we have an upward-sloping yield curve for benchmark bonds. We take the risk-neutral probability of default as given, as if it has been determined using a structural or reduced-form credit model. We also assume a recovery rate if default were to occur that conforms to the seniority of the debt issue and the nature of the issuer's assets.

The first step is to build the binomial interest rate tree under the assumption of no arbitrage. Exhibit 9 displays the data on annual payment benchmark government bonds that are used to build the binomial interest rate tree. This is the *par curve* because each bond is priced at par value. The coupon rates are equal to the yields to maturity because the years to maturity are whole numbers (integers) so that there is no accrued interest. The 1-year government bond has a negative yield to reflect the conditions seen in some financial markets. Note that the actual 1-year security is likely to be a zero-coupon bond priced at a premium, at 100.2506 per 100 of par value: (100/100.2506) – 1 = –0.0025. However, on a par curve whereby all the bonds are priced at 100, it is shown as having a negative coupon rate.

Exhibit 9 Par Curve for Annual Payment Benchmark Government Bonds, Spot Rates, Discount Factors, and Forward Rates[11]

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	–0.25%	100	1.002506	–0.2500%	
2	0.75%	100	0.985093	0.7538%	1.7677%
3	1.50%	100	0.955848	1.5166%	3.0596%
4	2.25%	100	0.913225	2.2953%	4.6674%
5	2.75%	100	0.870016	2.8240%	4.9664%

The discount factors and spot rates are bootstrapped using the cash flows on the underlying benchmark bonds in this sequence of equations:

$$100 = (100 - 0.25) \times DF_1$$

$$DF_1 = 1.002506$$

10 Sections 5 and 6 are based on Smith (2017).

11 In this and subsequent exhibits, all calculations were completed on a spreadsheet and rounded results are reported in the text.

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$100 = (0.75 \times 1.002506) + (100.75 \times DF_2)$

$DF_2 = 0.985093$

$100 = (1.50 \times 1.002506) + (1.50 \times 0.985093) + (101.50 \times DF_3)$

$DF_3 = 0.955848$

$100 = (2.25 \times 1.002506) + (2.25 \times 0.985093) + (2.25 \times 0.955848) + (102.25 \times DF_4)$

$DF_4 = 0.913225$

$100 = (2.75 \times 1.002506) + (2.75 \times 0.985093) + (2.75 \times 0.955848) + (2.75 \times 0.913225) + (102.75 \times DF_5)$

$DF_5 = 0.870016$

The spot (i.e., implied zero-coupon) rates are calculated from the discount factors. For instance, the 2-year spot rate is 0.7538%:

$$\left(\frac{1}{0.985093}\right)^{1/2} - 1 = 0.007538$$

The 4-year spot rate is 2.2953%:

$$\left(\frac{1}{0.913225}\right)^{1/4} - 1 = 0.022953$$

The forward rates are calculated as the ratios of the discount factors. The 1-year forward rate two years into the future is 3.0596%: 0.985093/0.955848 – 1 = 0.030596. The 1-year forward rate four years into the future is 4.9665%: 0.913225/0.870016 – 1 = 0.049665.[12]

Following the methodology detailed in the “Arbitrage-Free Valuation Framework” reading, we build a binomial interest rate tree for 1-year forward rates consistent with the pricing of the benchmark government bonds and an assumption of future interest rate volatility. Here we assume 10% volatility. The resulting binomial interest rate tree is presented in Exhibit 10. Below each rate is the probability of attaining that node in the tree. The current (date 0) 1-year rate of –0.25% will rise to 1.9442% or “fall” to 1.5918% by the end of the year (date 1) with equal probability. On date 2 at the end of the second year, the 1-year rate will be 3.7026%, 3.0315%, or 2.4820% with probabilities of 0.25, 0.50, and 0.25. On date 4, the forward rate will fall within the range of a high of 7.2918% to a low of 3.2764%. For each date, the possible rates are spread out around the forward rates shown in Exhibit 9.

12 Note that when the numbers from Exhibit 9 are entered in a spreadsheet, the 5-year forward rate is actually 4.9664%, as shown in the exhibit.

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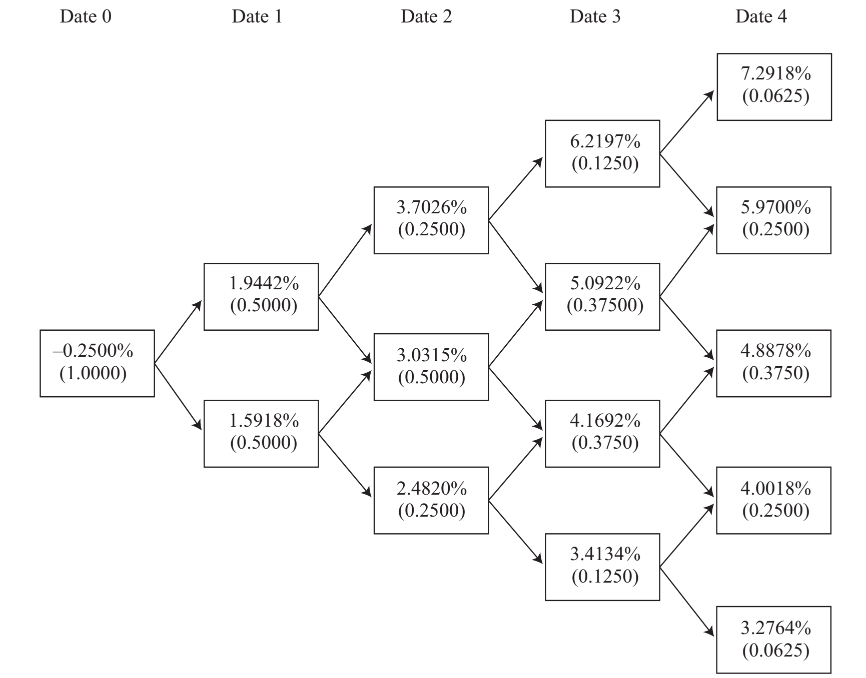
Exhibit 10 1-Year Binomial Interest Rate Tree for 10% Volatility



To demonstrate that this is an arbitrage-free binomial interest rate tree, we calculate the date-0 value of a 2.75% annual payment government bond. We know from Exhibit 9 that this bond is priced at par value. Exhibit 11 shows that the date-0 value is indeed 100.0000. Notice that the scheduled year-end coupon and principal payments are placed to the right of each forward rate in the tree.

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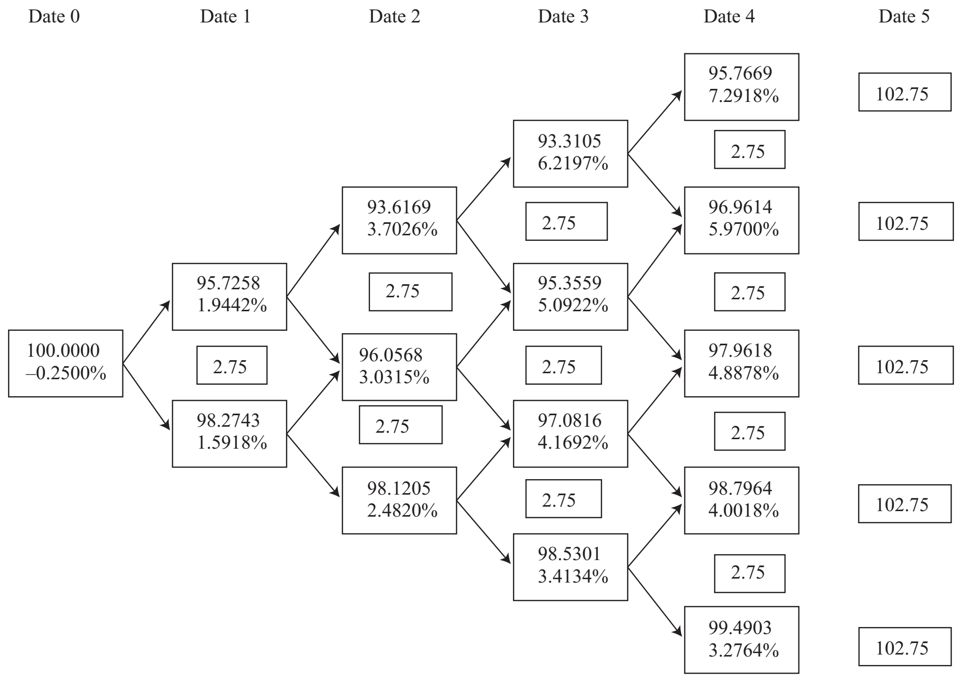
Exhibit 11 Valuation of a 2.75% Annual Payment Government Bond



These are the five date-4 values for the government bond, shown above the interest rate at each node:

102.75/1.072918 = 95.7669
102.75/1.059700 = 96.9614
102.75/1.048878 = 97.9618
102.75/1.040018 = 98.7964
102.75/1.032764 = 99.4903

These are the four date-3 values:

$$\frac{\left[(0.5 \times 95.7669) + (0.5 \times 96.9614)\right] + 2.75}{1.062197} = 93.3105$$

$$\frac{\left[(0.5 \times 96.9614) + (0.5 \times 97.9618)\right] + 2.75}{1.050922} = 95.3559$$

$$\frac{\left[(0.5 \times 97.9618) + (0.5 \times 98.7964)\right] + 2.75}{1.041692} = 97.0816$$

$$\frac{\left[(0.5 \times 98.7964) + (0.5 \times 99.4903)\right] + 2.75}{1.034134} = 98.5301$$

Continuing with backward induction, the date-0 value turns out to be 100.0000, confirming that the binomial interest rate tree has been correctly calibrated.

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Now consider a 5-year, 3.50% annual payment corporate bond. A fixed-income analyst assigns an annual default probability of 1.25% (the hazard rate) and a recovery rate of 40% to this bond and assumes 10% volatility in benchmark interest rates. The problem at hand for the analyst is to assess the fair value for the bond under these assumptions. This is done in two steps:

- First, determine the value for the corporate bond assuming no default (VND).
- Second, calculate the credit valuation adjustment (CVA).

The fair value of the bond is the VND minus the CVA.

The binomial interest rate tree for benchmark rates in Exhibit 10 can be used to calculate the VND for the bond. Exhibit 12 shows that the VND is 103.5450 per 100 of par value. This could also have been obtained more directly using the benchmark discount factors:

$$(3.50 \times 1.002506) + (3.50 \times 0.985093) + (3.50 \times 0.955848) + (3.50 \times 0.913225) + (103.50 \times 0.870016) = 103.5450$$

The advantage to using the binomial interest rate tree to get the VND is that the same tree is used to calculate the expected exposure to default loss, which is a key element in the credit risk model.

Exhibit 12 Value of a 3.50% Annual Payment Corporate Bond Assuming No Default (VND)

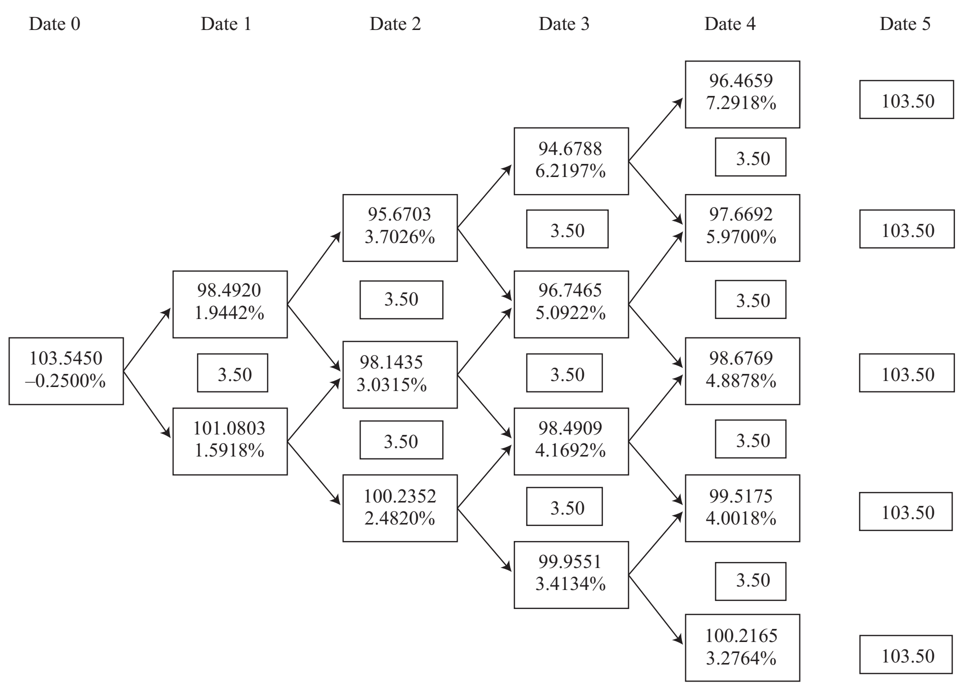

Exhibit 13 shows that the credit valuation adjustment to the value assuming no default is 3.5394 per 100 of par value. The expected exposure for date 4 is 102.0931, calculated using the bond values at each node, the probability of attaining the node, and the coupon payment:

$$[0.0625 \times 96.4659) + (0.25 \times 97.6692) + (0.375 \times 98.6769) + (0.25 \times 99.5175) + (0.0625 \times 100.2165)] + 3.50 = 102.0931$$

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[Note again that all calculations are done on a spreadsheet to maintain precision; only the rounded results are reported in the text]. The loss given default (LGD) for date 4 is 61.2559 [= 102.0931 × (1 – 0.40)] because the assumed recovery rate is 40% of the exposure. The POD (probability of default) at date 4 is 1.2037% assuming no prior default. As described in Section 2, this is based on the probability of survival into the fourth year. It is calculated as:

$$1.25\% \times (100\% - 1.25\%)^3 = 1.2037\%$$

$(100\% - 1.25\%)^3$ is the probability of survival after date 3, and 1.25% is the probability of default on date 4 (the hazard rate). The product of the LGD and the POD is the expected loss. The present value of the expected loss, 0.6734, is the contribution to total CVA for date 4. The sum of the CVAs for each year is the overall CVA.

Exhibit 13 Credit Valuation Adjustment (CVA) for the 3.50% Annual Payment Corporate Bond

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	103.2862	61.9717	1.2500%	1.002506	0.7766
2	101.5481	60.9289	1.2344%	0.985093	0.7409
3	101.0433	60.6260	1.2189%	0.955848	0.7064
4	102.0931	61.2559	1.2037%	0.913225	0.6734
5	103.5000	62.1000	1.1887%	0.870016	0.6422
			6.0957%	CVA =	3.5394

The fixed-income analyst concludes that the fair value of the corporate bond is 100.0056 per 100 of par value: 103.5450 – 3.5394 = 100.0056. Depending on the current market price for the bond, the analyst might recommend a buy or sell decision.

The yield to maturity (YTM) for the corporate bond given a fair value of 100.0056 is 3.4988%.

$$100.0056 = \frac{3.50}{(1+\text{YTM})^1} + \frac{3.50}{(1+\text{YTM})^2} + \frac{3.50}{(1+\text{YTM})^3} + \frac{3.50}{(1+\text{YTM})^4} + \frac{103.50}{(1+\text{YTM})^5}$$

$$\text{YTM} = 0.034988$$

The 5-year par yield for the government bond is 2.75% in Exhibit 9. Therefore, the credit spread over the benchmark bond is 0.7488% (= 3.4988% – 2.75%). In practice, the credit spread is typically measured against the actual yield on the comparable maturity government bond, which might be trading at a premium or a discount.

We can say that the credit risk on this corporate bond is captured by a CVA of 3.5394 per 100 in par value as of date 0 or as an annual spread of 74.88 basis points per year for five years. This conclusion, however, assumes that the observed credit spread is based entirely on credit risk. In fact, there usually are liquidity and tax differences between government and corporate bonds. Those differences are neglected in this analysis to focus on credit risk. Said differently, the liquidity and tax differences are represented in the credit spread.

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EXAMPLE 6

Using Credit Analysis in Decision Making

Lori Boller is a fixed-income money manager specializing in taking long positions on high-yield corporate bonds that she deems to be under-valued. In particular, she looks for bonds for which the credit spread over government securities appears to indicate too high a probability of default or too low a recovery rate if default were to occur. Currently, she is looking at a 3-year, 4.00% annual payment bond that is priced at 104 (per 100 of par value). In her opinion, this bond should be priced to reflect an annual default probability of 2.25% (the hazard rate) given a recovery rate of 40%. Ms. Boller is comfortable with an assumption of 10% volatility in government bond yields over the next few years. Should she consider buying this bond for her portfolio? Use the government par curve in Exhibit 9 and the binomial interest rate tree in Exhibit 10 in the solution.

Solution:

Ms. Boller needs to calculate the fair value of the 3-year, 4% annual payment corporate bond given her assumptions about the credit risk parameters. The results are shown in Exhibit 14.

Exhibit 14 Fair Value of the 3-Year, 4% Annual Payment Corporate Bond

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	107.0902	64.2541	2.2500%	1.002506	1.4493
2	104.9120	62.9472	2.1994%	0.985093	1.3638
3	104.0000	62.4000	2.1499%	0.955848	1.2823
			6.5993%	CVA =	4.0954

The VND for the bond is 107.3586. The calculations for the bond values in the binomial interest rate tree are:

104/1.037026 = 100.2868

104/1.030315 = 100.9400

104/1.024820 = 101.4812

$$\frac{\left[(0.5 \times 100.2868) + (0.5 \times 100.9400)\right] + 4}{1.019442} = 102.6183$$

$$\frac{\left[(0.5 \times 100.9400) + (0.5 \times 101.4812)\right] + 4}{1.015918} = 103.5621$$

$$\frac{\left[(0.5 \times 102.6183) + (0.5 \times 103.5621)\right] + 4}{0.997500} = 107.3586$$

The CVA for the bond is 4.0954 given the assumption of an annual default probability of 2.25% and a recovery rate of 40% of the expected exposure. These are calculations for the date-1 and date-2 expected exposures:

[(0.50 × 102.6183) + (0.50 × 103.5621)] + 4 = 107.0902

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$$[(0.25 \times 100.2868) + (0.50 \times 100.9400) + (0.25 \times 101.4812)] + 4 = 104.9120$$

The calculations for the LGD are:

$$107.0902 \times (1 - 0.40) = 64.2541$$

$$104.9120 \times (1 - 0.40) = 62.9472$$

$$104 \times (1 - 0.40) = 62.4000$$

The calculations for the POD for date 2 and date 3 are:

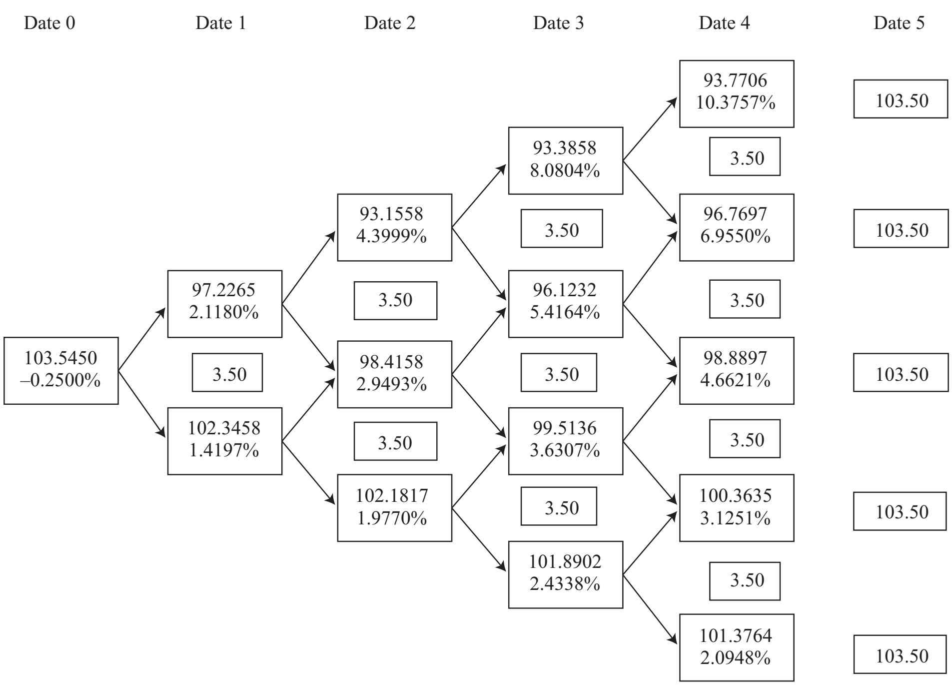
$$2.25\% \times (100\% - 2.25\%) = 2.1994\%$$

$$2.25\% \times (100\% - 2.25\%)^2 = 2.1499\%$$

Ms. Boller determines, based on her assumed credit risk parameters, that the fair value for the high-yield corporate bond is 103.2632 (= 107.3586 – 4.0954). Given that the bond is trading at 104, she would likely decline to purchase because in her opinion the bond is over-valued.

A change in the assumed level of interest rate volatility can be shown to have a small impact on the fair value of the corporate bond. Usually the effect of a change in volatility is demonstrated with a bond having an embedded option, such as a callable or putable bond. Here we see an impact of the calculation of CVA on a bond having no embedded options. This is illustrated with Exhibits 15 and 16, which use a no-arbitrage binomial interest rate tree for 20% volatility to value the 5-year, 3.50% annual payment corporate bond using the same credit risk parameters as in the previous calculations.

Exhibit 15 VND Calculation for the 3.50% Corporate Bond Assuming No Default and 20% Volatility



Notice in Exhibit 15 that with 20% volatility, the range in forward rates for each date is now wider. With 10% volatility, the date-4 rates go from a low of 3.2764% to a high of 7.2918%. Now with 20% volatility, the range is from 2.0948% to 10.3757%.

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The key point is that changing all the bond values still results in a VND of 103.5450. This confirms that the tree has been correctly calibrated and that the assumed level of future interest rate volatility has no impact on the value of a default risk-free government bond. Changes in the fair value of a corporate bond arising from a change in the assumed rate volatility occur only when there are embedded options and, as demonstrated in Exhibit 16, when there is credit risk.

Exhibit 16 CVA Calculation for the 3.50% Corporate Bond Assuming 20% Volatility

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	103.2862	61.9717	1.2500%	1.002506	0.7766
2	101.5423	60.9254	1.2344%	0.985093	0.7408
3	101.0233	60.6140	1.2189%	0.955848	0.7062
4	102.0636	61.2382	1.2037%	0.913225	0.6732
5	103.5000	62.1000	1.1887%	0.870016	0.6422
			6.0957%	CVA =	3.5390

Exhibit 16 presents the table to calculate the CVA for 20% volatility. The expected exposures to default loss are slightly lower for dates 2, 3, and 4 compared to Exhibit 13 for 10% volatility. These small changes feed through the table, reducing the loss given default and the contribution to total CVA for those dates. Overall, the CVA is 3.5390 per 100 of par value. The fair value of the bond is now slightly higher at 100.0060 (= 103.5450 – 3.5390) compared to the value for 10% volatility of 100.0056 (= 103.5450 – 3.5394).

The reason for the small volatility impact on the fair value is the asymmetry in the forward rates produced by the log-normality assumption in the interest rate model. In building the tree, rates are spread out around the implied forward rate for each date, the more so the greater the given level of volatility. However, the range is not symmetric about the implied forward rate. For example, the 1-year forward rate four years into the future is 4.9665% in Exhibit 9. With 20% volatility, the date-4 rate at the top of the tree is higher by 5.4092% (= 10.3757% – 4.9665%), while the rate at the bottom of the tree is lower by 2.8717% (= 4.9665% – 2.0948%). The net effect is to reduce the expected exposure to default loss. The top of the tree shows less potential loss because the current value of the bond is lower, which more than offsets the greater exposure to loss at the bottom of the tree.

The arbitrage-free framework can be adapted to value a risky floating-rate note. Consider a 5-year "floater" that pays annually the 1-year benchmark rate plus 0.50%. This 50 basis point addition to the index rate is called the *quoted margin* and typically is fixed over the lifetime of the security. Exhibit 17 demonstrates that the VND for the floater is 102.3633 per 100 of par value, using the binomial interest rate tree for 10% interest rate volatility. Notice that the interest payment is "in arrears," meaning that the rate is set at the beginning of the period and paid at the end of the period. That is why the interest payments set to the right of each rate vary depending on the realized rate in the tree. The interest payment for date 1 is 0.25 because the date-0 reference rate is –0.25%: (–0.25% + 0.50%) × 100 = 0.25. The final payment on date 5 when the floater matures is 105.3878 if the 1-year rate is 4.8878% on date 4: (4.8878% + 0.50%) × 100 + 100 = 105.3878.

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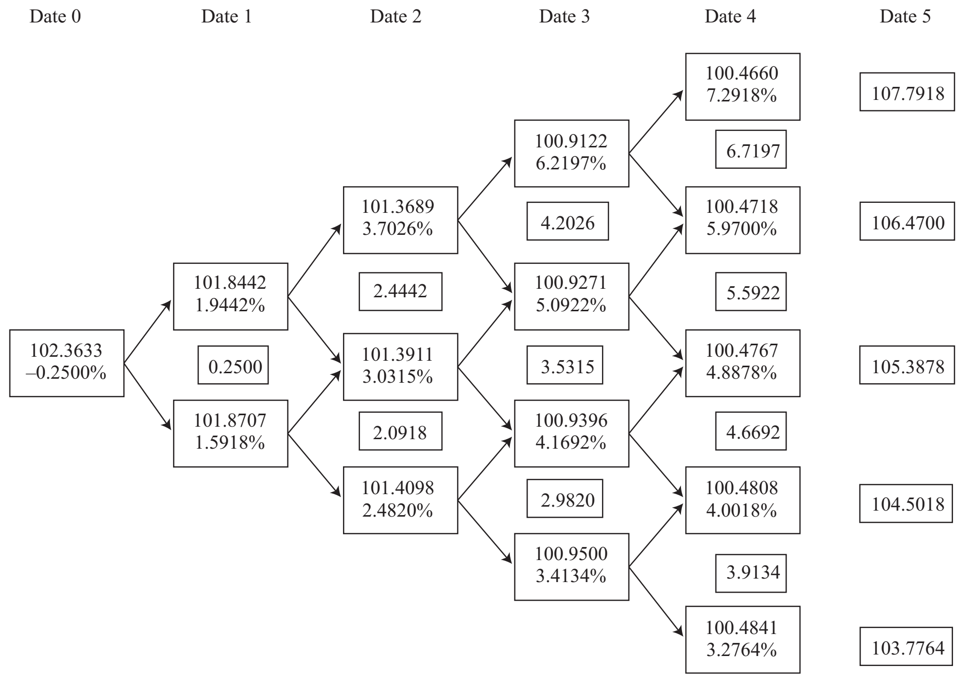
Exhibit 17 Value of a Floating-Rate Note Paying the Benchmark Rate Plus 0.50% Assuming No Default and 10% Volatility



Notice that the bond values for each date are very similar for the various forward rates. That, of course, is the intent of a floating-rate note. The bond values would all be exactly 100.0000 if the note paid the benchmark rate "flat," meaning a quoted margin of zero. The VND of 102.3633 is obtained via backward induction (i.e., beginning at maturity and working backwards in time). These are the calculations for the bond values for date 4:

107.7918/1.072918 = 100.4660

106.4700/1.059700 = 100.4718

105.3878/1.048878 = 100.4767

104.5018/1.040018 = 100.4808

103.7764/1.032764 = 100.4841

These are the calculations for date 3:

$$\frac{\left[(0.5 \times 100.4660) + (0.5 \times 100.4718)\right] + 6.7197}{1.062197} = 100.9122$$

$$\frac{\left[(0.50 \times 100.4718) + (0.5 \times 100.4767)\right] + 5.5922}{1.050922} = 100.9271$$

$$\frac{\left[(0.5 \times 100.4767) + (0.5 \times 100.4808)\right] + 4.6692}{1.041692} = 100.9396$$

$$\frac{\left[(0.5 \times 100.4808) + (0.5 \times 100.4841)\right] + 3.9134}{1.034134} = 100.9500$$

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These are the calculations for the bond values for date 2:

$$\frac{\left[(0.5 \times 100.9122) + (0.5 \times 100.9271)\right] + 4.2026}{1.037026} = 101.3689$$

$$\frac{\left[(0.5 \times 100.9271) + (0.5 \times 100.9396)\right] + 3.5315}{1.030315} = 101.3911$$

$$\frac{\left[(0.5 \times 100.9396) + (0.5 \times 100.9500)\right] + 2.9820}{1.024820} = 101.4098$$

These are the calculations for the bond values for date 1 and date 0:

$$\frac{\left[(0.5 \times 101.3689) + (0.5 \times 101.3911)\right] + 2.4442}{1.019442} = 101.8442$$

$$\frac{\left[(0.5 \times 101.3911) + (0.5 \times 101.4098)\right] + 2.0918}{1.015918} = 101.8707$$

$$\frac{\left[(0.5 \times 101.8442) + (0.5 \times 101.8707)\right] + 0.2500}{0.997500} = 102.3633$$

Exhibit 18 shows the credit risk table for the floating-rate note. For this example, we assume that for the first three years the annual default probability (the hazard rate) is 0.50% and the recovery rate 20%. The credit risk of the issuer then worsens: For the final two years the probability of default goes up to 0.75% and the recovery rate goes down to 10%. This is an example in which the assumed annual hazard rate changes over the lifetime of the bond.

Exhibit 18 CVA Calculation for the Value of a Floating-Rate Note Paying the Benchmark Rate Plus 0.50%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	102.1074	81.6859	0.5000%	1.002506	0.4095
2	103.6583	82.9266	0.4975%	0.985093	0.4064
3	104.4947	83.5957	0.4950%	0.955848	0.3955
4	105.6535	95.0881	0.7388%	0.913225	0.6416
5	105.4864	94.9377	0.7333%	0.870016	0.6057
			2.9646%	CVA =	2.4586

Notes: Credit risk parameter assumptions: for dates 1–3, hazard rate = 0.50% and recovery rate = 20%; for dates 4–5, hazard rate = 0.75% and recovery rate = 10%.

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The calculation for the expected exposure recognizes that the bond values for each date follow the probabilities of attaining those rates, whereas possible interest payments use the probabilities for the prior date. For example, the expected exposure to default loss for date 4 is 105.6535:

$$\begin{aligned}&\left[\begin{array}{l}(0.0625\times 100.4660)+(0.25\times 100.4718)+(0.375\times 100.4767)\\ +(0.25\times 100.4808)+(0.0625\times 100.4841)\end{array}\right]\\ &+\left[(0.125\times 6.7197)+(0.375\times 5.5922)+(0.375\times 4.6692)+(0.125\times 3.9134)\right]\\ &=105.6535\end{aligned}$$

The first term in brackets is the expected bond value using the date-4 probabilities for each of the five possible rates. The second term is the expected interest payment using the date-3 probabilities for each of the four possible rates.

The expected LGD for date 2 is 82.9266 [= 103.6583 × (1 – 0.20)]; for date 4 it is 95.0881 [= 105.6535 × (1 – 0.10)]. The PODs in Exhibit 18 reflect the probability of survival for each year. For date 2, the POD is 0.4975% conditional on no default on date 1: 0.50% × (100% – 0.50%) = 0.4975%. For date 3, the POD is 0.4950%: 0.50% × (100% – 0.50%)^2 = 0.4950%. The probability of survival into the fourth year is 98.5075%: (100% – 0.50%)^3 = 98.5075%. Therefore, the POD for date 4 increases to 0.7388% because of the assumed worsening credit risk: 0.75% × 98.5075% = 0.7388%. The probability of survival into the fifth year is 97.7687% (= 98.5075% – 0.7388%). The POD for date 5 is 0.7333% (= 0.75% × 97.7687%). The cumulative probability of default over the lifetime of the floater is 2.9646%.

Given these assumptions about credit risk, the CVA for the floater is 2.4586. The fair value is 99.9047, the VND of 102.3633 minus the CVA. Because the security is priced below par value, its *discount margin* (DM) must be higher than the quoted margin of 0.50%. The discount margin for a floating-rate note is a yield measure commonly used on floating-rate notes in the same manner that the credit spread is used with fixed-rate bonds.

The arbitrage-free framework can be used to determine the DM for this floater by trial-and-error search (or GoalSeek or Solver in Excel). We add a trial DM to benchmark rates that are used to get the bond values at each node in the tree. Then the trial DM is then changed until the date-0 value matches the fair value of 99.9047. Exhibit 19 shows that the DM for this floater is 0.52046%, slightly above the quoted margin because the security is priced at a small discount below par value.

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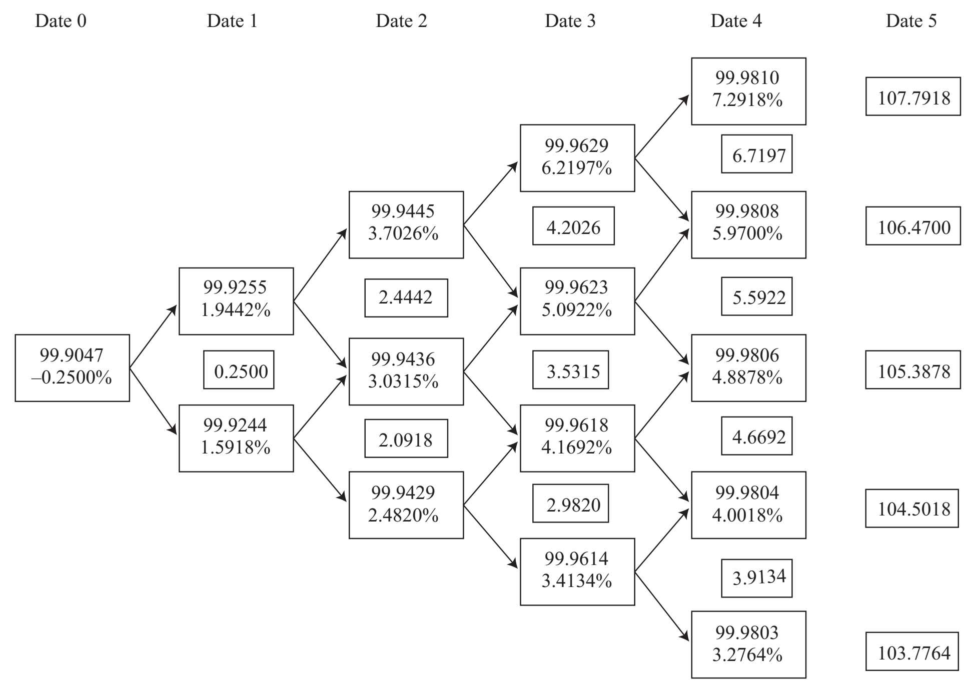
Exhibit 19 The Discount Margin for the Floating-Rate Note Paying the Benchmark Rate Plus 0.50% Assuming 10% Volatility



These are the calculations for the bond values for date 2:

$$\frac{\left[(0.5 \times 99.9629) + (0.5 \times 99.9623)\right] + 4.2026}{1 + 0.037026 + 0.0052046} = 99.9445$$

$$\frac{\left[(0.5 \times 99.9623) + (0.5 \times 99.9618)\right] + 3.5315}{1 + 0.030315 + 0.0052046} = 99.9436$$

$$\frac{\left[(0.5 \times 99.9618) + (0.5 \times 99.9614)\right] + 2.9820}{1 + 0.024820 + 0.0052046} = 99.9429$$

Throughout the binomial interest rate tree, the assumed DM is added to the benchmark rate to factor in credit risk. After trial-and-error search, a DM of 0.52046% gives the same date-0 value for the floating-rate note of 99.9047 as is obtained with the VND and CVA models.

EXAMPLE 7

Evaluating a Floating-Rate Note

Omar Yassin is an experienced credit analyst at a fixed-income investment firm. His current assignment is to assess potential purchases of distressed high-yield corporate bonds. One intriguing prospect is a 3-year, annual payment floating-rate note paying the 1-year benchmark rate plus 2.50%. The floater is rated CCC and is priced at 84 per 100 of par value. Based on various research reports and prices on the issuer's credit default swaps, Mr. Yassin believes the probability of default in the next year is about 30%. If the issuer does go into bankruptcy at any time, he expects the recovery rate to be at least 50%; it could be as high as

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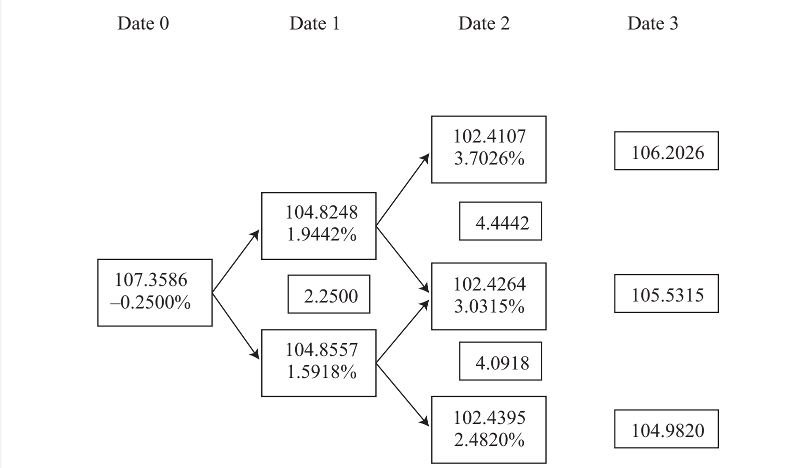
60% because of some valuable real estate holdings. He further believes that if the issuer is able to survive this next year, the default probability for the remaining two years will only be about 10% for each year. Based on these assumptions about the credit risk parameters and an expectation of 10% volatility for interest rates, should Mr. Yassin recommend purchasing the floating-rate note?

Solution:

Mr. Yassin calculates the fair value of the 3-year, annual payment floating-rate note given his assumptions about the default probabilities and the recovery rate ranging between 50% and 60%. The results are shown in Exhibit 20.

Exhibit 20 Fair Value of the 3-Year, Annual Payment Floating-Rate Note Paying the 1-Year Rate Plus 2.50%



Assumed 50% Recovery Rate:

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	107.0902	53.5451	30.0000%	1.002506	16.1038
2	106.6938	53.3469	7.0000%	0.985093	3.6786
3	105.5619	52.7810	6.3000%	0.955848	3.1784
			43.3000%	CVA =	22.9608

Fair Value = 107.3586 – 22.9608 = 84.3978

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Assumed 60% Recovery Rate:

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	107.0902	42.8361	30.0000%	1.002506	12.8830
2	106.6938	42.6775	7.0000%	0.985093	2.9429
3	105.5619	42.2248	6.3000%	0.955848	2.5427
			43.3000%	CVA =	18.3686

Fair Value = 107.3586 – 18.3686 = 88.9900

Each projected interest payment in the tree is the benchmark rate at the beginning of the year plus 2.50% times 100. The rate is –0.25% on date 0; the "in-arrears" interest payment on date 1 is 2.2500 [= (–0.25% + 2.50%) × 100]. If the rate is 2.4820% on date 2, the payment at maturity on date 3 is 104.9820 [= (2.4820% + 2.50%) × 100 + 100].

The VND for the floater is 107.3586. The calculations for the bond values in the binomial interest rate tree are:

$$106.2026/1.037026 = 102.4107$$

$$105.5315/1.030315 = 102.4264$$

$$104.9820/1.024820 = 102.4395$$

$$\frac{\left[(0.5 \times 102.4107) + (0.5 \times 102.4264)\right] + 4.4442}{1.019442} = 104.8248$$

$$\frac{\left[(0.5 \times 102.4264) + (0.5 \times 102.4395)\right] + 4.0918}{1.015918} = 104.8557$$

$$\frac{\left[(0.5 \times 104.8248) + (0.5 \times 104.8557)\right] + 2.2500}{0.997500} = 107.3586$$

These are the calculations for the expected exposures to default loss:

$$[(0.5 \times 104.8248) + (0.5 \times 104.8557)] + 2.2500 = 107.0902$$

$$[(0.25 \times 102.4107) + (0.5 \times 102.4264) + (0.25 \times 102.4395)] + [(0.5 \times 4.4442) + (0.5 \times 4.0918)] = 106.6938$$

$$[(0.25 \times 106.2026) + (0.5 \times 105.5315) + (0.25 \times 104.9820)] = 105.5619$$

The assumed default probability for the first year is 30%. The POD for date 2 is 7.00%, which is the probability of survival into the second year, 70%, times the 10% probability of default. The probability of survival into the third year is 63% (= 70% – 7%); the POD for date 3 is 6.30% (= 10% × 63%).

The decision to consider purchase of the floating-rate note comes down to the assumption about recovery. Exhibit 20 first shows the results for 50% recovery of the expected exposure. The LGD on date 2 is 53.3469 [= 106.6938 × (1 – 0.50)]. The overall CVA is 22.9608, giving a fair value of 84.3978 (=107.3586 – 22.9608). Exhibit 20 next shows the results for 60% recovery. With this assumption, the LGD for date 2 is just 42.6775 [= 106.6938 × (1 – 0.60)]. Stronger recovery reduces the overall CVA to 18.3686. The fair value for the floater is now 88.9900.

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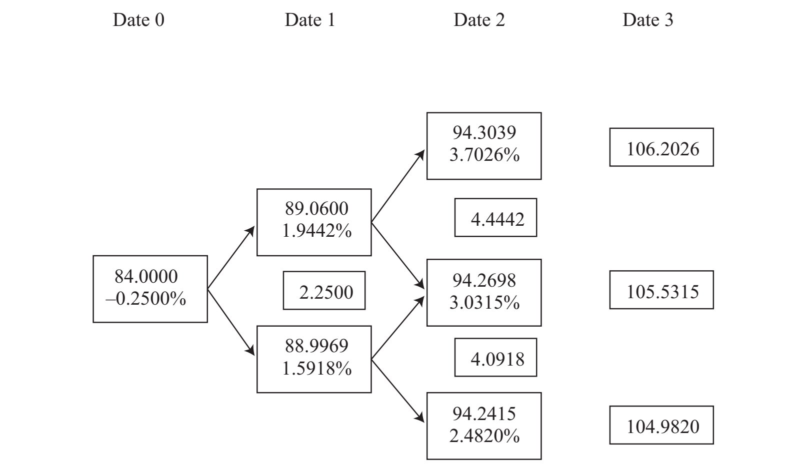
Mr. Yassin should recommend purchasing the distressed floating-rate note. Although there is a significant 43.3% probability of default at some point over the three years, the security appears to be fairly priced at 84 given a recovery rate of 50%. At 60% recovery, it is significantly undervalued.

In addition, there is still a 57.7% (= 100% – 43.3%) chance of no default. Exhibit 21 shows the calculation for the discount margin, which is a measure of the return to the investor assuming no default (like a yield to maturity on a fixed-rate bond). Found by trial-and-error search, the DM is 8.9148%, considerably higher than the quoted margin because the floater is priced at a deep discount.

Exhibit 21 Discount Margin on the 3-Year, Annual Payment Floating-Rate Note Paying the 1-Year Rate Plus 2.50%



These are the calculations for the bond values for date 1 and date 0:

$$\frac{\left[(0.5 \times 94.3039) + (0.5 \times 94.2698)\right] + 4.4442}{1 + 0.019442 + 0.089148} = 89.0600$$

$$\frac{\left[(0.5 \times 94.2698) + (0.5 \times 94.2415)\right] + 4.0918}{1 + 0.015918 + 0.089148} = 88.9969$$

$$\frac{\left[(0.5 \times 89.0600) + (0.5 \times 88.9969)\right] + 2.2500}{1 - 0.0025 + 0.089148} = 84.0000$$

6 INTERPRETING CHANGES IN CREDIT SPREADS

Corporate and benchmark bond yields, and the credit spread between them, change from day to day. The challenge to a fixed-income analyst is to understand and be able to explain *why* the yields and spreads change. Exhibit 22 offers a breakdown of the main components of bond yields. Benchmark bond yields, in general, capture the *macroeconomic* factors affecting all debt securities. These are the expected inflation rate and the expected real rate of return. Risk-averse investors in benchmark bonds also might require compensation for uncertainty regarding those variables.

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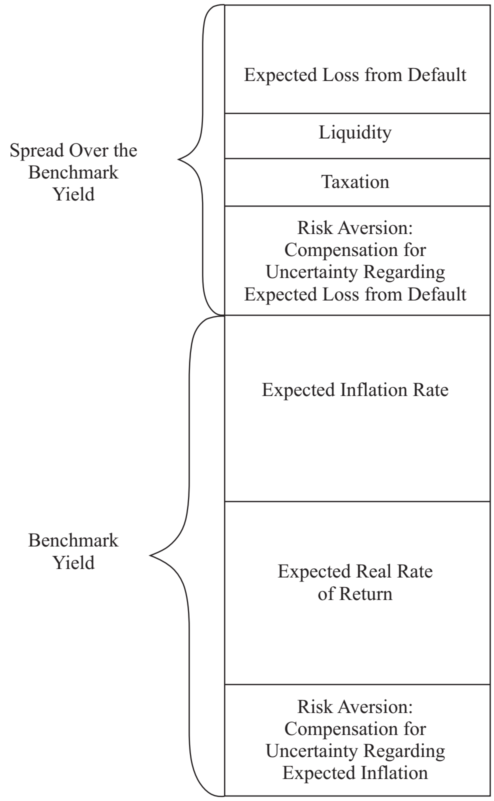
Exhibit 22 Components of a Corporate Bond Yield



Source: Smith (2017).

The spread over the benchmark bond yield captures the *microeconomic* factors that pertain to the corporate issuer and the specific issue itself. The chief microeconomic factor is the expected loss due to default. There also are liquidity and tax differences between the corporate and benchmark bonds. Moreover, it can be difficult to separate these factors. Securities for which it becomes more difficult for analysts to assess a probability of default and a recovery rate undoubtedly become less liquid. Similarly, an uncertain tax status on a bond's gains and losses will increase the time and cost to estimate value. That makes the bond less liquid. Another factor in the observed spread between the corporate and benchmark bond yields can be compensation to risk-averse investors for uncertainty regarding credit risk, as well as liquidity and tax factors.

Research groups at major banks and consultancies have been working on models to better include counterparty credit risk, funding costs, and liquidity and taxation effects in the valuations of derivatives. First, a value is obtained using benchmark discount factors, in practice, derived from rates on overnight indexed swaps (OIS). These are interest rate swaps that reference an average daily interest rate. For instance, in the United States this daily rate is the effective federal funds rate. Then this OIS value, which is comparable to the VND in the previous section, is adjusted for the other factors. These valuation adjustments collectively are known as the "XVA." The credit valuation adjustment (CVA) is the most developed and used in practice. Others include a funding valuation adjustment (FVA), a liquidity valuation adjustment (LVA), and a taxation valuation adjustment (TVA). In principle, the same ideas apply to debt

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securities in that these "XVA" comprise the observed spread between corporate and benchmark bond yields. In this reading, we focus only on the credit risk component, the CVA.

We can use the arbitrage-free framework and the credit risk model to examine the connections between the default probability, the recovery rate, and the credit spread. To be sure, this is a simple model to illustrate the much more complex models used in practice. These (which are called "XVA engines") typically use Monte Carlo simulations for thousands of possible paths for interest rates. Our binomial interest rate tree has only 16 paths for the five years; it's a model of the actual model.

Consider again the 5-year, 3.50% annual payment corporate bond examined in Section 5. In Exhibit 12, the value assuming no default (VND) was determined to be 103.5450 per 100 of par value. Now let us use the credit risk model to find the probabilities of default that would be consistent with various credit spreads and a recovery rate of 40%. Suppose, as in Exhibit 7, the credit spread for a triple A-rated bond is 0.60%. Using trial-and-error search, we find that an annual probability of default of 1.01% (the assumed hazard rate) produces a 60-basis point credit spread. The credit risk table is presented in Exhibit 23. Notice that the expected exposure to default loss and the loss given default are the same as in Exhibit 13. Only the default probabilities and the contributions to total CVA for each year change.

Exhibit 23 CVA Calculation for the 3.50% Corporate Bond Given a Default Probability of 1.01% and a Recovery Rate of 40%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	103.2862	61.9717	1.0100%	1.002506	0.6275
2	101.5481	60.9289	0.9998%	0.985093	0.6001
3	101.0433	60.6260	0.9897%	0.955848	0.5735
4	102.0931	61.2559	0.9797%	0.913225	0.5481
5	103.5000	62.1000	0.9698%	0.870016	0.5240
			4.9490%	CVA =	2.8731

The CVA for the bond is 2.8731 per 100 of par. The fair value is 100.6719 (= 103.5450 – 2.8731). This gives a yield to maturity of 3.35%.

$$100.6719 = \frac{3.50}{(1 + \mathrm{YTM})^1} + \frac{3.50}{(1 + \mathrm{YTM})^2} + \frac{3.50}{(1 + \mathrm{YTM})^3} + \frac{3.50}{(1 + \mathrm{YTM})^4} + \frac{103.50}{(1 + \mathrm{YTM})^5}$$

$$\mathrm{YTM} = 0.0335$$

Given that the yield on the 5-year benchmark bond is 2.75%, the credit spread is 0.60% (= 3.35% – 2.75%).

We can repeat this exercise for the other credit spreads and ratings shown in Exhibit 7. In each case, trial-and-error search is used to get the initial POD that corresponds to the CVA, the fair value, and the yield to maturity for each assumed spread. The results for the annual and cumulative default probabilities over the five years are shown in Exhibit 24.

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Exhibit 24 Default Probabilities Consistent with Given Credit Ratings and Spreads and 40% Recovery

Credit Rating	Credit Spread	Annual Default Probability	Cumulative Default Probability
AAA	0.60%	1.01%	4.95%
AA	0.90%	1.49%	7.23%
A	1.10%	1.83%	8.82%
BBB	1.50%	2.48%	11.80%
BB	3.40%	5.64%	25.19%
B	6.50%	10.97%	44.07%
CCC,CC,C	9.50%	16.50%	59.41%

The default probabilities illustrated in Exhibit 24 might seem high, especially given the historical experience presented in Exhibit 6. Since 1995, no AAA-rated company has defaulted; still, we model the likelihood to be over 1% for the first year and almost 5% for the next five years. However, as discussed in Section 2, these are *risk-neutral* probabilities of default and are higher than the actual probabilities because market prices reflect uncertainty over the timing of possible default. Investors are concerned about credit spread widening, especially if they do not intend to hold the bond to maturity. Credit rating migration from year to year, as illustrated in Exhibit 7, is a concern even for a high-quality investment-grade corporate bond. This is captured in the risk-neutral probability of default. Also, we must remember that observed credit spreads reflect more than just credit risk—there also are liquidity and tax differences. That further explains the difference between risk-neutral and actual default probabilities.

The relationship between the assumed recovery rate and the credit spread can be examined in the context of the credit risk model. Suppose that the 5-year, 3.50% annual payment corporate bond has an initial probability of default of 1.83% (the assumed annual hazard rate). In Exhibit 24, we see that for a 40% recovery rate, the credit spread is 1.10%. What if the recovery rate is expected to be only 30%? Exhibit 25 shows the credit risk table for that assumption.

Exhibit 25 CVA Calculation for the 3.50% Corporate Bond Given a Default Probability of 1.83% and a Recovery Rate of 30%

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	103.2862	72.3003	1.8300%	1.002506	1.3264
2	101.5481	71.0837	1.7965%	0.985093	1.2580
3	101.0433	70.7303	1.7636%	0.955848	1.1923
4	102.0931	71.4652	1.7314%	0.913225	1.1300
5	103.5000	72.4500	1.6997%	0.870016	1.0714
			8.8212%	CVA =	5.9781

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The reduction in the recovery rate from 40% to 30% has an impact on LGD and CVA for each year. The overall CVA is 5.9781 per 100 of par value. The fair value for the bond is 97.5670 (= 103.5450 – 5.9781) and the yield to maturity is 4.05%, giving a credit spread of 1.30% (= 4.05% – 2.75%).

$$97.5670 = \frac{3.50}{(1+\text{YTM})^1} + \frac{3.50}{(1+\text{YTM})^2} + \frac{3.50}{(1+\text{YTM})^3} + \frac{3.50}{(1+\text{YTM})^4} + \frac{103.50}{(1+\text{YTM})^5}$$

$$\text{YTM} = 0.0405$$

This example illustrates how a credit rating agency might use "notching" to combine the expected loss given default and the probability of default in setting the rating for a corporate bond. Suppose that the issuer rating for the company is single A. That is based on a default probability of 1.83% and a recovery rate of 40% on the company's senior unsecured debt. That debt has a credit spread of 1.10%, which is comparable to other A-rated companies. This particular bond is subordinated, and analysts at the rating agency believe that a recovery rate assumption of 30% is applicable. That could justify assigning a lower rating of A– or BBB+ on the subordinated debt.

EXAMPLE 8

Evaluating Changes in Credit Risk Parameters

Edward Kapili is a summer intern working on a fixed-income trading desk at a major money-center bank. His supervisor asks him to value a 3-year, 3% annual payment corporate bond using a binomial interest rate tree model for 20% volatility and the current par curve for benchmark government bond par curve. [This is the binomial tree in Exhibit 15]. The assumed annual probability of default (the hazard rate) is 1.50%, and the recovery rate is 40%.

The supervisor asks Mr. Kapili if the credit spread over the yield on the 3-year benchmark bond, which is 1.50% in Exhibit 9, is likely to go up more if the default probability doubles to 3.00% or if the recovery rate halves to 20%. Mr. Kapili's intuition is that doubling the probability of default has the larger impact on the credit spread. Is his intuition correct?

Solution:

Mr. Kapili first determines the fair value of the 3-year, 3% annual payment bond given the assumptions for the original credit risk parameters. The binomial interest rate tree and credit risk table are presented in Exhibit 26.

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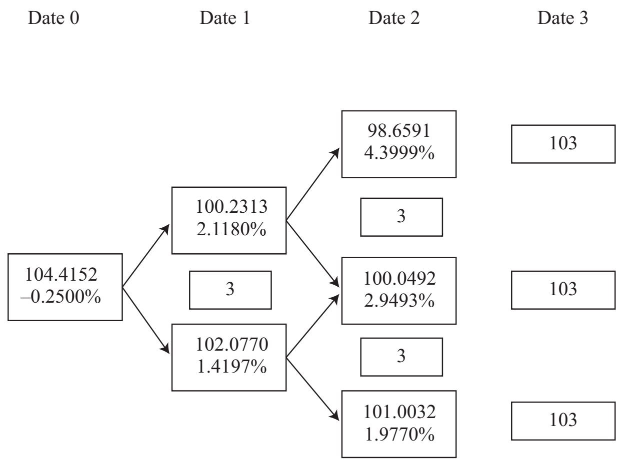
Exhibit 26 Fair Value of the 3-Year, 3% Annual Payment Corporate Bond Assuming 20% Volatility



Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.1541	62.4925	1.5000%	1.002506	0.9397
2	102.9402	61.7641	1.4775%	0.985093	0.8990
3	103.0000	61.8000	1.4553%	0.955848	0.8597
			4.4328%	CVA =	2.6984

Fair Value = 104.4152 – 2.6984 = 101.7168

The VND for the bond is 104.4152, the CVA is 2.6984, and the fair value is 101.7168 per 100 of par value. The yield to maturity is 2.40%, and the credit spread is 0.90% (= 2.40% – 1.50%).

$$101.7168 = \frac{3}{(1+\text{YTM})^1} + \frac{3}{(1+\text{YTM})^2} + \frac{103}{(1+\text{YTM})^3}$$

$$\text{YTM} = 0.0240$$

Next, Mr. Kapili calculates the fair values under the new credit risk parameters, first for doubling the default probability and second for halving the recovery rate. These tables are in Exhibit 27.

Exhibit 27 Fair Value Calculations for Doubling the Default Probability and Halving the Recovery Rate

3.00% Default Probability, 40% Recovery Rate

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.1541	62.4925	3.0000%	1.002506	1.8795
2	102.9402	61.7641	2.9100%	0.985093	1.7705

(continued)

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Exhibit 27 (Continued)

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
3	103.0000	61.8000	2.8227%	0.955848	1.6674
			8.7327%	CVA =	5.3174

Fair Value = 104.4152 − 5.3174 = 99.0978

1.50% Default Probability, 20% Recovery Rate

Date	Expected Exposure	LGD	POD	Discount Factor	CVA per Year
0					
1	104.1541	83.3233	1.5000%	1.002506	1.2530
2	102.9402	82.3522	1.4775%	0.985093	1.1986
3	103.0000	82.4000	1.4553%	0.955848	1.1463
			4.4328%	CVA =	3.5978

Fair Value = 104.4152 − 3.5978 = 100.8173

The fair value of the corporate bond falls to 99.0978 when the default probability is raised to 3.00% and the recovery rate stays at 40%. The VND is the same at 104.4152, while the CVA goes up to 5.3174. The yield to maturity increases to 3.32% and the credit spread to 1.82% (= 3.32% − 1.50%).

$$99.0978 = \frac{3}{(1 + \text{YTM})^1} + \frac{3}{(1 + \text{YTM})^2} + \frac{103}{(1 + \text{YTM})^3}$$

$$\text{YTM} = 0.0332$$

The fair value of the corporate bond falls to 100.8173 when the recovery rate is reduced by half to 20%, and the default probability is maintained at 1.50%. The VND is again the same at 104.4152 as the CVA goes up to 3.5978. The yield to maturity increases to 2.71% and the credit spread to 1.21% (= 2.71% − 1.50%).

$$100.8173 = \frac{3}{(1 + \text{YTM})^1} + \frac{3}{(1 + \text{YTM})^2} + \frac{103}{(1 + \text{YTM})^3}$$

$$\text{YTM} = 0.0271$$

Mr. Kapili's intuition is correct: Doubling the default probability has a greater impact on the credit spread than halving the recovery rate.

7 THE TERM STRUCTURE OF CREDIT SPREADS

In the same way that the yield curve is comprised of the interest rates on a single government issuer's debt across bond maturities, a credit curve shows the spread over a benchmark security for an issuer for outstanding fixed-income securities with

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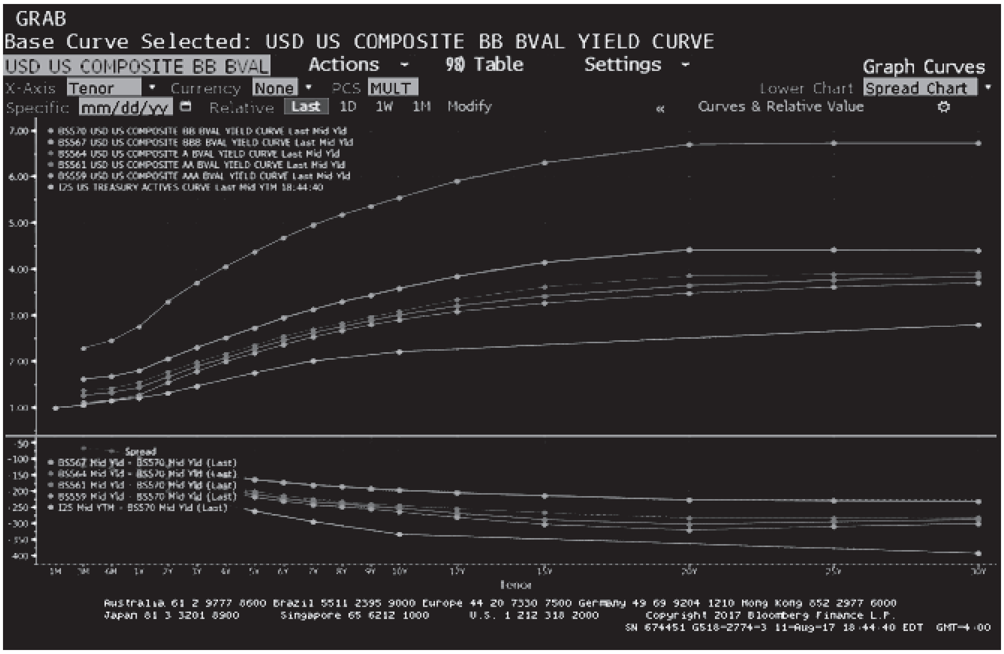
shorter to longer maturities. For example, Exhibit 28 shows the relationship between US Treasury yields of a specific maturity and bonds rated AA, A, BBB, and BB, respectively. The total yields of the bonds are shown in the upper half of the diagram, with spreads over the benchmark Treasury in the lower half.

Exhibit 28 Composite Yield Graphs



Source: Bloomberg.

The term structure of credit spreads is a useful gauge for issuers, underwriters, and investors in measuring the risk-versus-return tradeoff for a single issuer or set of issuers across ratings and/or sectors across maturities. Issuers often work with their underwriter to consider the terms of a new issuance or a tender for existing debt based on relative credit spreads across maturities. For example, an investment-grade bond portfolio manager might use the existing credit curve for a particular issuer to determine a bid for a new primary debt issuance as well as to inform trading decisions for secondary debt positions. In some cases, investors, issuers, or underwriters might use the credit spread term structure for a particular rating or corporate sector to derive either prospective pricing for a new issuance or to determine fair value spreads for outstanding securities, which is an extension of matrix pricing. A high-yield debt investor might employ the term structure of credit spreads to gauge the risk/reward tradeoffs between debt maturities. Given the impact of monetary and fiscal policies on risky debt markets, policymakers have extended their focus from default risk-free yield curve dynamics to the term structure of credit spreads.

There are several key drivers of the term structure of credit spreads. First, credit quality is a key factor. For investment-grade securities with the highest credit ratings and extremely low spreads, credit spread migration is only possible in one direction given the implied lower bound of zero on credit spreads. As a result, the credit term structure for the most highly rated securities tends to be either flat or slightly upward sloping. Securities with lower credit quality, however, face greater sensitivity to the

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credit cycle. The greater likelihood of default associated with high-yield securities generally results in a steeper credit spread curve, both in cases where a weaker economy suggests credit spread widening and when an inverted credit spread curve suggests tighter spreads for longer maturities. As a high-yield bond moves further down the credit spectrum into a more distressed scenario, the contractual cash flows through maturity become less certain—with the value of distressed debt converging to a dollar price equal to the recovery rate as default becomes more certain, regardless of the remaining time to maturity. Although such a scenario will result in a steep inverted credit spread term structure, we will review the implications of this scenario in more detail.

Financial conditions are another critical factor affecting the credit spread term structure. From a macroeconomic perspective, the credit risk of a bond is influenced by expectations for economic growth and inflation. A stronger economic climate is generally associated with higher benchmark yields but lower credit spreads for issuers whose default probability declines during periods of economic growth (cash flows tend to improve and profitability increases under such a scenario). The countercyclical relationship between spreads and benchmark rates is therefore commonly observed across the business cycle.

Market supply and demand dynamics are another critical factor influencing the credit curve term structure. Unlike default risk-free government securities in developed markets, the relative liquidity of corporate bonds varies widely, with the vast majority of securities not trading on a daily basis. Given that new and most recently issued securities tend to represent the largest proportion of trading volume and are responsible for much of the volatility in credit spreads, the credit curve will be most heavily influenced by the most frequently traded securities. For example, although one might expect the credit curve to steepen for a borrower refinancing near-term maturities with long-term debt, this effect may be partially offset by a tighter bid–offer spread for longer credit maturities. This flattening may also occur within a specific rating or if market participants anticipate significant supply in a particular tenor. Infrequently traded bonds trading with wider bid–offer spreads can also impact the shape of the term structure, so it is important to gauge the size and frequency of trades in bonds across the maturity spectrum to ensure consistency.

Finally, from a microeconomic perspective, company-value model results discussed in Section 4 are another key driver of the credit spread term structure. Under traditional credit analysis, the specific industry or industries within which an issuer operates is considered as well as key financial ratios, such as cash flow and leverage and profitability versus sector and ratings peers. This company-specific analysis based on fundamental data has been complemented by more probabilistic, forward-looking structural models for company valuation. These models take stock market valuation, equity volatility, and balance sheet information into account to derive the implied default probability for a company. Holding other factors constant, any microeconomic factor that increases the implied default probability, such as greater equity volatility, will tend to drive a steeper credit spread curve, while the reverse is true with a decline in equity volatility.

Practitioners will frequently employ these tools when analyzing the term structure of credit spreads to determine fair value. For example, the Bloomberg default risk screen (DRSK) shown in Exhibit 29 combines the company-value analysis with fundamental credit ratios for a composite analysis of TransCanada Corporation, a Canadian natural gas transmission and power services company.

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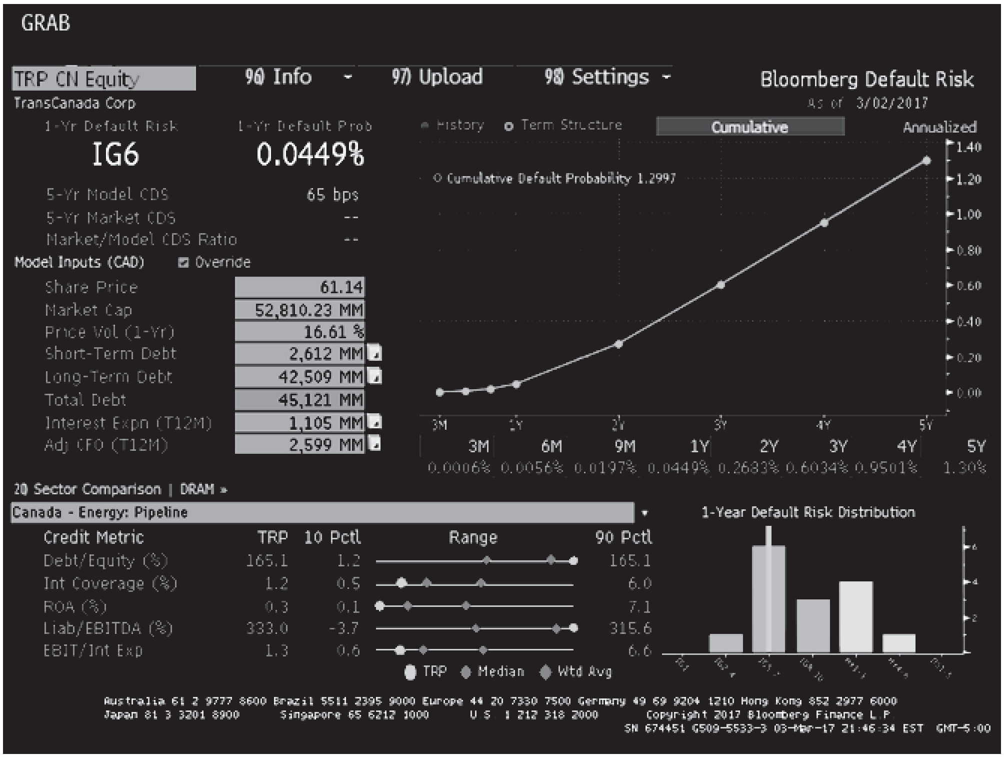
Exhibit 29 Default Risk Screen



Source: Bloomberg.

Two further considerations are important when analyzing the term structure of credit spreads. The first concerns the appropriate risk-free or benchmark rates used to determine spreads. A frequently traded government security with the nearest maturity to an outstanding corporate bond generally represents the lowest default risk for developed markets, so this is a logical benchmark choice. However, the duration and maturity of the most liquid or on-the-run government bonds rarely match that of corporate bonds trading in the secondary market, so it is often necessary to interpolate between yields of the two government securities with the closest maturity. As the interpolation may impact the analysis for less-liquid maturities, the benchmark swap curve based on interbank rates is often substituted for the government benchmark because of greater swap market liquidity for off-the-run maturities. For example, Exhibit 30 demonstrates the latter methodology on a Bloomberg screen for a composite of BBB-rated US industrial corporate issuers versus the benchmark US dollar swap curve, showing a positively sloped credit spread term structure across maturities.

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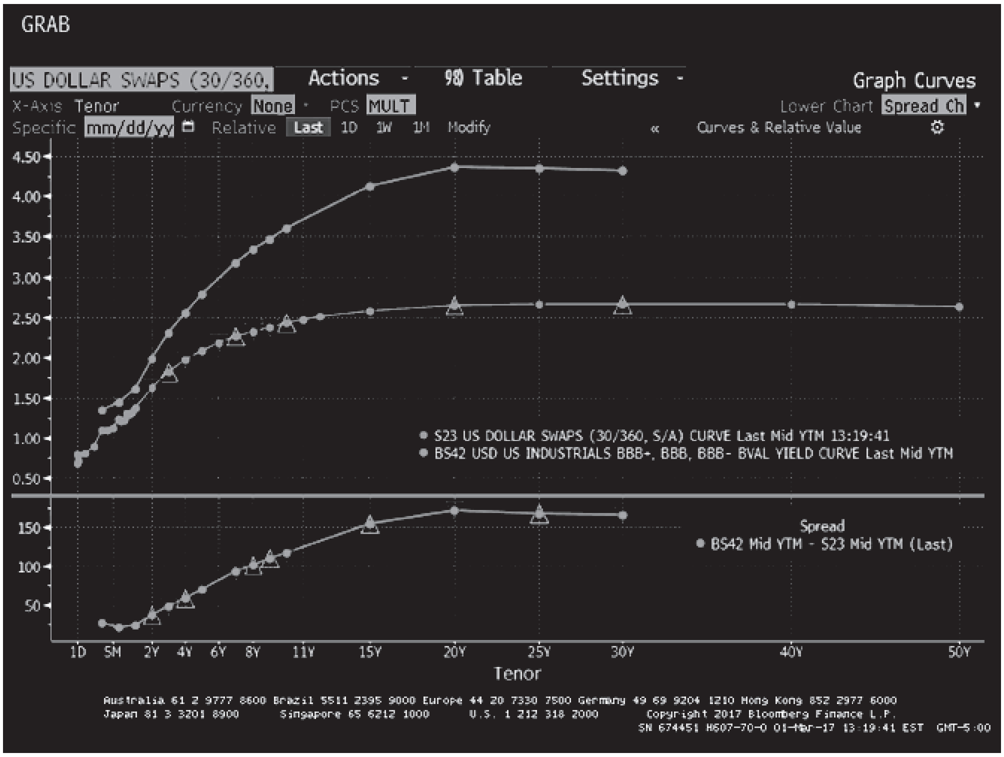
Exhibit 30 Credit Spreads over Swap Rates



Source: Bloomberg.

The second consideration concerns the all-in spread over the benchmark itself. Term structure analysis should include only bonds with similar credit characteristics, which are typically senior unsecured general obligations of the issuer. Any bonds of the issuer with embedded options, first or second lien provisions, or other unique provisions should be excluded from the analysis. It is also important to note that such securities typically include cross-default provisions so that all securities across the maturity spectrum of a single issuer will be subject to recovery in the event of bankruptcy.

Using the models presented in prior sections, we can demonstrate that the *change* in market expectations of default over time is a key determinant of the shape of the credit curve term structure. This may be shown using a simple extension of the zero-coupon corporate bond example in Exhibit 2 by changing the probability of default. Using a recovery rate of 40% and changing the probability of default from 1.25% to 1.50% raises the credit spread from 77 basis points in the original example to 92 basis points. These calculations are shown in Exhibit 31.

Exhibit 31 Raising the Default Probability of the 5-Year, Zero-Coupon Corporate Bond

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	88.8487	35.5395	53.3092	1.5000%	98.5000%	0.7996	0.970874	0.7763

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Exhibit 31 (Continued)

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
2	91.5142	36.6057	54.9085	1.4775%	97.0225%	0.8113	0.942596	0.7647
3	94.2596	37.7038	56.5558	1.4553%	95.5672%	0.8231	0.915142	0.7532
4	97.0874	38.8350	58.2524	1.4335%	94.1337%	0.8351	0.888487	0.7419
5	100.0000	40.0000	60.0000	1.4120%	92.7217%	0.8472	0.862609	0.7308
				7.2783%			CVA =	3.7670

Fair Value = 86.2609 – 3.7670 = 82.4939
Yield to Maturity = 3.9240%
Credit Spread = 3.9240% – 3.00% = 0.9240%

Flat credit spread curves imply a relatively stable expectation of default over time, while an upward-sloping credit curve implies that investors seek greater compensation for assuming issuer default over longer periods. For example, we can illustrate this in terms of a credit spread curve by holding the benchmark rate constant at 3.00% across 3-year, 5-year, and 10-year maturities while increasing the default probability over time. Although one could consider an increase in default probability each year, the following example in Exhibit 32 assumes a 1.00% default probability (annual hazard rate) for years 1, 2, and 3, a 2.00% probability of default in years 4 and 5, and a 3.00% default probability in years 6 through 10, with the recovery rate at a constant 40%. [Note that this is another example of the hazard rate changing over the lifetime of the bonds.] As shown in Exhibit 32, the credit spread rises from 62 basis points, to 86 basis points, to 132 basis points.

Exhibit 32 Increasing the Default Probability for Longer Times to Maturity

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	94.2596	37.7038	56.5558	1.0000%	99.0000%	0.5656	0.970874	0.5491
2	97.0874	38.8350	58.2524	0.9900%	98.0100%	0.5767	0.942596	0.5436
3	100.0000	40.0000	60.0000	0.9801%	97.0299%	0.5881	0.915142	0.5382
				2.9701%			CVA =	1.6308

Fair Value = 91.5142 – 1.6308 = 89.8833
Yield to Maturity = 3.6192%
Credit Spread = 3.6192% – 3.00% = 0.6192%

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	88.8487	35.5395	53.3092	1.0000%	99.0000%	0.5331	0.970874	0.5176
2	91.5142	36.6057	54.9085	0.9900%	98.0100%	0.5436	0.942596	0.5124
3	94.2596	37.7038	56.5558	0.9801%	97.0299%	0.5543	0.915142	0.5073
4	97.0874	38.8350	58.2524	1.9406%	95.0893%	1.1304	0.888487	1.0044

(continued)

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Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
5	100.0000	40.0000	60.0000	1.9018%	93.1875%	1.1411	0.862609	0.9843
				6.8125%			CVA =	3.5259

Fair Value = 86.2609 – 3.5259 = 82.7350
Yield to Maturity = 3.8633%
Credit Spread = 3.8633% – 3.00% = 0.8633%

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	76.6417	30.6567	45.9850	1.0000%	99.0000%	0.4599	0.970874	0.4465
2	78.9409	31.5764	47.3646	0.9900%	98.0100%	0.4689	0.942596	0.4420
3	81.3092	32.5237	48.7855	0.9801%	97.0299%	0.4781	0.915142	0.4376
4	83.7484	33.4994	50.2491	1.9406%	95.0893%	0.9751	0.888487	0.8664
5	86.2609	34.5044	51.7565	1.9018%	93.1875%	0.9843	0.862609	0.8491
6	88.8487	35.5395	53.3092	2.7956%	90.3919%	1.4903	0.837484	1.2481
7	91.5142	36.6057	54.9085	2.7118%	87.6801%	1.4890	0.813092	1.2107
8	94.2596	37.7038	56.5558	2.6304%	85.0497%	1.4876	0.789409	1.1744
9	97.0874	38.8350	58.2524	2.5515%	82.4982%	1.4863	0.766417	1.1391
10	100.0000	40.0000	60.0000	2.4749%	80.0233%	1.4850	0.744094	1.1050
				19.9767%			CVA =	8.9187

Fair Value = 74.4094 – 8.9187 = 65.4907
Yield to Maturity = 4.3235%
Credit Spread = 4.3235% – 3.00% = 1.3235%

Positively sloped credit spread curves may arise when a high-quality issuer with a strong competitive position in a stable industry has low leverage, strong cash flow, and a high profit margin. This type of issuer tends to exhibit very low short-term credit spreads rising with increasing maturity given greater uncertainty due to the macroeconomic environment, potential adverse changes in the competitive landscape, technological change, or other factors that drive a higher implied probability of default over time. Empirical academic studies also tend to support the view that the credit spread term structure is upward-sloping for investment-grade bond portfolios.[13]

Alternatively, high-yield issuers in cyclical industries sometimes face a downward-sloping credit term structure because of issuer- or industry-specific reasons. For example, an ownership change resulting from a leveraged buyout or private equity acquisition may often be accompanied by a significant increase in leverage. In such a case, an inverted credit curve may indicate investor expectations that the new owners will create efficiencies in the restructured organization, leading to improved future cash flow and profitability that will benefit debt investors. Another example of an inverted credit term structure might result when issuers in a historically cyclical

13 See, for example Bedendo, Cathcart, and El-Jahel (2007): 237–257.

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industry (such as oil and gas exploration or retail) find themselves at the bottom of an economic cycle, with investor expectations of a recovery in the industry tied to improving credit spreads over time.

That said, it is important to distinguish between scenarios where the contractual cash flows of a risky bond are likely to occur and distressed debt scenarios where investors expect to receive the recovery rate in a likely bankruptcy scenario. Bonds with a very high likelihood of default tend to trade on a price basis that converges toward the recovery rate rather than on a spread to benchmark rates. This scenario leads to credit spread term structures that may be considered more of an "optical" phenomenon rather than a true reflection of the relative risks and rewards of long-term versus short-term bonds from a single issuer, as illustrated in the following discussion.[14]

To demonstrate this using our zero-coupon bond example, let us shift to a scenario where bondholders with 5-year and 10-year bonds outstanding anticipate an imminent default scenario and both bonds trade at the recovery rate of 40%.

Note that if we solve for the fair value and resulting credit spread over the benchmark yield as in the instances where default probability was 1.25%, we end up with the same value given no default (VND) for the 5-year and 10-year bonds, respectively. However, when deriving a credit spread value for both securities assuming recovery in a bankruptcy scenario and cross-default provisions across maturities, the credit valuation adjustment (CVA) representing the sum of expected losses is simply the difference between the VND and the recovery rate.

For the 5-year example, we can thus calculate a VND of 86.2609, a CVA of 46.2609, and a fair value with recovery at 40. This results in a yield of 20.1124% and a credit spread over the government bond of 17.1124%. In the 10-year case, the VND may be shown as 74.4094, a CVA of 34.4094, and a fair value at 40. That gives a yield of 9.5958% and a credit spread of 6.5958%. We end up with a steep and inverted "credit spread" curve.

The interpretation of the credit spread term structure is important for investors seeking to capitalize on a market view that differs from that reflected in the credit curve. For example, if a portfolio manager disagrees with the market's expectation of a high near-term default probability that declines over time, she could sell short-term protection in the credit default swap market and buy longer-term protection. Under a scenario where the issuer does not default, the investor retains the premium on protection sold and may either retain or choose to sell back the longer-term credit default swap to realize a gain.

8 CREDIT ANALYSIS FOR SECURITIZED DEBT

Unlike the general obligation nature of most private or sovereign fixed-income securities, securitized debt allows issuers to finance a specific set of assets or receivables (e.g., mortgages, automobile loans, or credit card receivables) rather than an entire balance sheet. Issuers in securitized debt markets are frequently motivated to undertake financing using these more structured securities given their ability to increase debt capacity and reduce the originator's need to maintain regulatory capital or retain residual risk. The isolation of securitized assets generally decreases the relative financing cost for these assets on a stand-alone basis as compared to a general obligation financing of the debt originator. By freeing up capital, an originator is also able to continue to generate income from further originations. Investors, however, seek to benefit from greater diversification, more stable and predictable underlying

14 See Berd, Mashal, and Wang (2004).

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cash flows, and a return that is greater than that of securities of similar ratings, which provide a reward for accepting the greater complexity associated with collateralized debt. That said, the credit analysis of such structured finance instruments requires a fundamentally different approach to other risky bonds given the underlying collateral, parties associated with the origination or servicing of the portfolio over the life of the security, as well as the issuing entity and any structural and credit enhancement features typically present in these transactions.

It is important to distinguish first and foremost among the types of securitized debt issued globally as well as the various forms. In its summary of structured finance asset types shown in Exhibit 33, the German-based rating agency Scope Ratings AG provides its general approach to credit assessment based not only on the underlying time horizon and collateral but also on asset characteristics referred to as granularity and homogeneity.

Exhibit 33 Summary of Asset Types and Characteristics of Core Structured Finance Asset Classes

Deal Type	Underlying Collateral	Risk Horizon	Granularity	Homogeneity	Credit Analysis Approach
Asset-backed CP	Commercial discount credits or credit advances	Short-term	Granular	Homogeneous	Book
Auto ABS	Auto loans or leases	Medium-term	Granular	Homogeneous	Portfolio
CMBS	Commercial mortgages	Typically long-term	Non-granular	Heterogeneous	Loan by loan
Consumer ABS	Consumer loans	Medium-term	Granular	Homogeneous	Portfolio
CRE loans	Commercial real estate loans	Long-term	Non-granular	Heterogeneous	Loan by loan
Credit cards	Credit card balances	Short-term	Granular	Homogeneous	Book
Credit-linked notes / repackaging	Any financial assets	Typically medium-term	Typically single asset	N/A	Pass-through rating/asset by asset
LL CLOs	Leveraged corporate loans	Medium-term	Non-granular	Heterogeneous	Loan by loan
PF CLOs	Project finance debt	Long-term	Non-granular	Heterogeneous	Loan by loan
RMBS	Residential mortgages	Long-term	Granular	Homogeneous	Loan by loan or portfolio
SME ABS	Loans to small- and medium-sized businesses	Typically medium-term	Granular	Mixed	Loan by loan or portfolio
Trade receivables	Commercial credit	Short-term	Typically granular	Homogeneous	Book

Source: Adapted from Scope Ratings AG (2016b): 7–8.

The concept of homogeneity refers to the degree to which underlying debt characteristics within a structured finance instrument are similar across individual obligations. On the one hand, an investor or credit analyst might draw general conclusions about the nature of homogeneous credit card or auto loan obligations given that an individual obligation faces strict eligibility criteria to be included in a specific asset pool. On the other hand, heterogeneous leveraged loan, project finance, or real estate

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transactions require scrutiny on a loan-by-loan basis given their different characteristics. The granularity of the portfolio refers to the actual number of obligations that comprise the overall structured finance instrument. A highly granular portfolio may have hundreds of underlying creditors, suggesting it is appropriate to draw conclusions about creditworthiness based on portfolio summary statistics rather than investigating each borrower. Alternatively, an asset pool with fewer more-discrete or non-granular investments would warrant analysis of each individual obligation.

The combination of asset type and tenor as well as the relative granularity and homogeneity of the underlying obligations drive the approach to credit analysis for a given instrument type. For example, short-term structured finance vehicles with granular, homogeneous assets tend to be evaluated using a statistical-based approach to the existing book of loans. This changes to a portfolio-based approach for medium-term granular and homogeneous obligations because the portfolio is not static but changes over time. For discrete or non-granular heterogeneous portfolios, a loan-by-loan approach to credit analysis is more appropriate. The following example of a credit card securitization will provide further insight into the process.

Exhibit 34 provides a summary from the prospectus of the Synchrony Credit Card Master Note Trust $750,000,000 Series 2016-1 Asset Backed Notes issued in March 2016. As is spelled out in the prospectus, the Synchrony transaction is backed by credit card receivables having the given credit score distribution presented below:

Exhibit 34 A Structured Debt Example, Composition by FICO® Credit Score Range

FICO Credit Score Range	Receivables Outstanding	Percentage of Outstanding
Less than or equal to 599	$995,522,016	6.6%
600 to 659	$2,825,520,245	18.7%
660 to 719	$6,037,695,923	39.9%
720 and above	$5,193,614,599	34.4%
No score	$64,390,707	0.4%
Total	**$15,116,743,490**	**100%**

Source: Synchrony Credit Card Master Note Trust $750,000,000 Series 2016-1 Asset Backed Notes Prospectus, page 93 [available at investors.synchronyfinancial.com].

Investors in this type of ABS will base their probability of default on the mean default probability, recovery rate, and variance of a portfolio of borrowers reflecting the distribution of FICO scores within the pool rather than conducting an analysis of individual borrowers. The prospectus provides a broad set of details beyond the FICO scores of borrowers for further in-depth portfolio analysis, including age of the receivables, average outstanding balances, and delinquency rates.

A heterogeneous portfolio of fewer loans, however, requires a fundamentally different approach. In this instance, each obligation within the asset pool may warrant its own analysis to determine whether an individual commercial property or leveraged company is able to meet its financial obligations under the ABS contract. Here the expected default probability and recovery rate on an asset-by-asset basis is the best gauge of how the investment will perform under various scenarios.

A second critical aspect of the credit exposure associated with ABS relates to the origination and servicing of assets over the life of the transaction. The prospectus and other related documents determine the roles and responsibilities of these related parties over the life of an ABS transaction. Upon inception of the transaction, investors

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rely on the originator/servicer to establish and enforce loan eligibility criteria, secure and maintain proper documentation and records, and maximize timely repayment and contract enforceability in cases of delinquency. Once the asset pool has been identified, investors are also exposed to operational and counterparty risk over the life of an ABS transaction. That is, they remain exposed to the ability of the servicer to effectively manage and service the portfolio over the life of the transaction. For an auto ABS transaction, this may involve the ability to repossess and sell a vehicle at a price close to the residual value in a timely manner in the event that a borrower is unable to pay, while in a commercial real estate transaction, it may involve identifying and replacing a non-performing tenant. Investors in an asset portfolio whose composition changes over time also face exposure to the replacement of obligors over time. In all such instances, not only is the creditworthiness of the servicer of importance but also its track record in meeting these servicing obligations, which are frequently gauged by analyzing the performance of more seasoned transactions handled by the same servicer over the credit cycle.

For example, in the case of the Synchrony Credit Card Master Note Trust transaction, Synchrony Financial acts as servicer of the trust and Synchrony Bank, as sub-servicer, is primarily responsible for receiving and processing collections on the receivables. A potential investor might therefore not only evaluate the performance of other debt backed by credit card receivables, but also how outstanding notes serviced by Synchrony have performed over time versus their servicing competitors.

Finally, the structure of a collateralized or secured debt transaction is a critical factor in analyzing this type of investment. These structural aspects include both the nature of the obligor itself, which is often a special purpose entity (SPE) whose sole purpose is to acquire a specified pool of assets and issue ABS to finance the SPE, as well as any structural enhancements of the transaction, which may include overcollateralization, credit tranching (i.e., tiering the claim priorities of ownership or interest), or other characteristics.

A key question related to the issuer is its relationship to the originator, namely the degree to which the bankruptcy of the obligor is related to that of the originator. The bankruptcy remoteness is typically determined by whether the transfer of the assets from the originator to the SPE may be deemed a true sale, which otherwise allows for the ability to separate risk between the originator and SPE at a later date.

Second, additional credit enhancements are a key structural element to be evaluated in the context of credit risk. Credit enhancements for ABS take on several forms beyond the bankruptcy remoteness of the SPE. For example, ABS transactions frequently have payout or performance triggers that protect investors in the case of adverse credit events. Certain events related to the servicer or seller—such as failure to make deposits or payments or other adverse events—may trigger early repayment ("amortization") of the security. For consumer transactions such as credit card or automotive ABS, the primary protection against a decline in asset quality for investors is additional return built into the transaction that is greater than the expected or historical loss of the asset pool. This additional return is often called the excess spread. Issuers create subordinated tranches of debt that provide added protection to those rated higher and benefit from a greater excess spread cushion over the life of the financing.

One of the oldest forms of secured debt, **covered bonds** have some similarities to these structured finance investments, but they also have fundamental differences that warrant special consideration when conducting credit analysis on such transactions. Covered bonds have their origin in Germany in the 18th century, but they have since been adopted by issuers across Europe as well as Asia and Australia. A covered bond is a senior debt obligation of a financial institution that gives recourse to the originator/issuer as well as a predetermined underlying collateral pool. Each specific country or jurisdiction specifies the eligible collateral types as well as the specific

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structures permissible in the covered bond market. Covered bonds most frequently have either commercial or residential mortgages meeting specific criteria or public sector exposures as underlying collateral.

The dual recourse to both the issuing financial institution as well as the underlying asset pool has been a hallmark of covered bonds since their inception centuries ago in Europe, but it was also reinforced under the European Union Bank Recovery and Resolution Directive (BRRD).[15] Under the BRRD, covered bonds enjoy unique protection among bank liabilities in the event of restructuring or regulatory intervention. As a result, rating agencies often assign a credit rating to covered bonds that are several notches above that of the issuing financial institution.

Although the dual recourse principle is of central importance in evaluating the credit risk of a covered bond, the underlying collateral plays a role as well. These asset pools vary across jurisdictions but are generally comprised of residential mortgages or public sector assets. In the case of the former, delinquency rates based on region and asset type using standard criteria will dictate mean default probability and expected loss, while public sector asset performance depends on jurisdiction and asset type.

SUMMARY

This reading has covered several important topics in credit analysis. Among the points made are the following:

- Three factors important to modeling credit risk are the expected exposure to default, the recovery rate, and the loss given default.
- These factors permit the calculation of a credit valuation adjustment that is subtracted from the (hypothetical) value of the bond, if it were default risk free, to get the bond's fair value given its credit risk. The credit valuation adjustment is calculated as the sum of the present values of the expected loss for each period in the remaining life of the bond. Expected values are computed using risk-neutral probabilities, and discounting is done at the risk-free rates for the relevant maturities.
- The CVA captures investors' compensation for bearing default risk. The compensation can also be expressed in terms of a credit spread.
- Credit scores and credit ratings are third-party evaluations of creditworthiness used in distinct markets.
- Analysts may use credit ratings and a transition matrix of probabilities to adjust a bond's yield-to-maturity to reflect the probabilities of credit migration. Credit spread migration typically reduces expected return.
- Credit analysis models fall into two broad categories: structural models and reduced-form models.
- Structural models are based on an option perspective of the positions of the stakeholders of the company. Bondholders are viewed as owning the assets of the company; shareholders have call options on those assets.
- Reduced-form models seek to predict *when* a default may occur, but they do not explain the *why* as do structural models. Reduced-form models, unlike structural models, are based only on observable variables.

15 See Scope Ratings AG (2016a).

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- When interest rates are assumed to be volatile, the credit risk of a bond can be estimated in an arbitrage-free valuation framework.
- The discount margin for floating-rate notes is similar to the credit spread for fixed-coupon bonds. The discount margin can also be calculated using an arbitrage-free valuation framework.
- Arbitrage-free valuation can be applied to judge the sensitivity of the credit spread to changes in credit risk parameters.
- The term structure of credit spreads depends on macro and micro factors.
- As it concerns macro factors, the credit spread curve tends to become steeper and widen in conditions of weak economic activity. Market supply and demand dynamics are important. The most frequently traded securities tend to determine the shape of this curve.
- Issuer- or industry-specific factors, such as the chance of a future leverage-decreasing event, can cause the credit spread curve to flatten or invert.
- When a bond is very likely to default, it often trades close to its recovery value at various maturities; moreover, the credit spread curve is less informative about the relationship between credit risk and maturity.
- For securitized debt, the characteristics of the asset portfolio themselves suggest the best approach for a credit analyst to take when deciding among investments. Important considerations include the relative concentration of assets and their similarity or heterogeneity as it concerns credit risk.

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PRACTICE PROBLEMS

The following information relates to questions 1–15

Daniela Ibarra is a senior analyst in the fixed-income department of a large wealth management firm. Marten Koning is a junior analyst in the same department, and David Lok is a member of the credit research team.

The firm invests in a variety of bonds. Ibarra is presently analyzing a set of bonds with some similar characteristics, such as four years until maturity and a par value of €1,000. Exhibit 1 includes details of these bonds.

Exhibit 1 A Brief Description of the Bonds Being Analyzed

Bond	Description
B1	A zero-coupon, four-year corporate bond with a par value of €1,000. The wealth management firm's research team has estimated that the risk-neutral probability of default (the hazard rate) for each date for the bond is 1.50%, and the recovery rate is 30%.
B2	A bond similar to B1, except that it has a fixed annual coupon rate of 6% paid annually.
B3	A bond similar to B2 but rated AA.
B4	A bond similar to B2 but the coupon rate is the one-year benchmark rate plus 4%.

Ibarra asks Koning to assist her with analyzing the bonds. She wants him to perform the analysis with the assumptions that there is no interest rate volatility and that the government bond yield curve is flat at 3%.

Ibarra performs the analysis assuming an upward-sloping yield curve and volatile interest rates. Exhibit 2 provides the data on annual payment benchmark government bonds.[1] She uses this data to construct a binomial interest rate tree (shown in Exhibit 3) based on an assumption of future interest rate volatility of 20%.

Exhibit 2 Par Curve for Annual Payment Benchmark Government Bonds

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	–0.25%	€100	1.002506	–0.2500%	
2	0.75%	€100	0.985093	0.7538%	1.7677%
3	1.50%	€100	0.955848	1.5166%	3.0596%
4	2.25%	€100	0.913225	2.2953%	4.6674%

1 For simplicity, this exhibit uses data for the first four years from Exhibit 9 of the reading.
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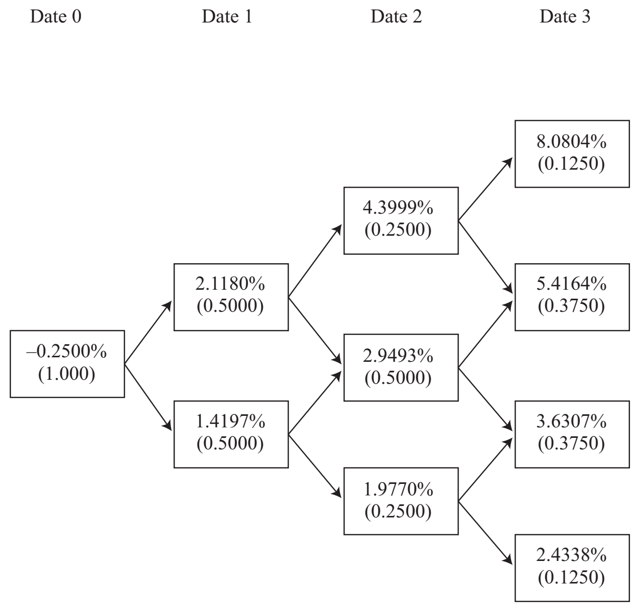
Exhibit 3 One-Year Binomial Interest Rate Tree for 20% Volatility



Answer the first five questions (1–4) based on the assumptions made by Marten Koning, the junior analyst. Answer questions (8–12) based on the assumptions made by Daniela Ibarra, the senior analyst.

Note: All calculations in this problem set are carried out on spreadsheets to preserve precision. The rounded results are reported in the solutions.

1 The market price of bond B1 is €875. The bond is:

  A fairly valued.

  B overvalued.

  C undervalued.

2 Koning realizes that an increase in the recovery rate would lead to an increase in the bond's fair value, whereas an increase in the probability of default would lead to a decrease in the bond's fair value. He is not sure which effect would be greater, however. So, he increases both the recovery rate and the probability of default by 25% of their existing estimates and recomputes the bond's fair value. The recomputed fair value is closest to:

  A €843.14.

  B €848.00.

  C €855.91.

3 The fair value of bond B2 is closest to:

  A €1,069.34.

  B €1,111.51.

  C €1,153.68.

4 The market price of bond B2 is €1,090. If the bond is purchased at this price and there is a default on Date 3, the rate of return to the bond buyer would be closest to:

  A –28.38%.

  B –41.72%.

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C −69.49%.

5 Bond B3 will have a modified duration of 2.75 at the end of the year. Based on the representative one-year corporate transition matrix in Exhibit 7 of the reading and assuming no default, how should the analyst adjust the bond's yield to maturity (YTM) to assess the expected return on the bond over the next year?

A Add 7.7 bps to YTM.

B Subtract 7.7 bps from YTM.

C Subtract 9.0 bps from YTM.

6 David Lok has estimated the probability of default of bond B1 to be 1.50%. He is presenting the approach the research team used to estimate the probability of default. Which of the following statements is Lok likely to make in his presentation if the team used a reduced-form credit model?

A Option pricing methodologies were used, with the volatility of the underlying asset estimated based on historical data on the firm's stock price.

B Regression analysis was used, with the independent variables including both firm-specific variables, such as the debt ratio and return on assets, and macroeconomic variables, such as the rate of inflation and the unemployment rate.

C The default barrier was first estimated followed by the estimation of the probability of default as the portion of the probability distribution that lies below the default barrier.

7 In the presentation, Lok is asked why the research team chose to use a reduced-form credit model instead of a structural model. Which statement is he likely to make in reply?

A Structural models are outdated having been developed in the 1970s; reduced-form models are more modern, having been developed in the 1990s.

B Structural models are overly complex because they require use of option pricing models, whereas reduced-form models use regression analysis.

C Structural models require "inside" information known to company management, whereas reduced-form models can use publicly available data on the firm.

8 As previously mentioned, Ibarra is considering a future interest rate volatility of 20% and an upward-sloping yield curve, as shown in Exhibit 2. Based on her analysis, the fair value of bond B2 is closest to:

A €1,101.24.

B €1,141.76.

C €1,144.63.

9 Ibarra wants to know the credit spread of bond B2 over a theoretical comparable-maturity government bond with the same coupon rate as this bond. The foregoing credit spread is closest to:

A 108 bps.

B 101 bps.

C 225 bps.

10 Ibarra is interested in analyzing how a simultaneous decrease in the recovery rate and the probability of default would affect the fair value of bond B2. She decreases both the recovery rate and the probability of default by 25% of their existing estimates and recomputes the bond's fair value. The recomputed fair value is closest to:

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A €1,096.59.

B €1,108.40.

C €1,111.91.

11 The wealth management firm has an existing position in bond B4. The market price of B4, a floating-rate note, is €1,070. Senior management has asked Ibarra to make a recommendation regarding the existing position. Based on the assumptions used to calculate the estimated fair value only, her recommendation should be to:

A add to the existing position.

B hold the existing position.

C reduce the existing position.

12 The issuer of the floating-rate note B4 is in the energy industry. Ibarra personally believes that oil prices are likely to increase significantly within the next year, which will lead to an improvement in the firm's financial health and a decline in the probability of default from 1.50% in Year 1 to 0.50% in Years 2, 3, and 4. Based on these expectations, which of the following statements is correct?

A The CVA will decrease to €22.99.

B The note's fair value will increase to €1,177.26.

C The value of the FRN, assuming no default, will increase to €1,173.55.

13 Floating-rate note B4 is currently rated BBB by Standard & Poor's and Fitch Ratings (and Baa by Moody's Investors Service). Based on the research department assumption about the probability of default in Question 10 and her own assumption in Question 11, which action does Ibarra *most likely* expect from the credit rating agencies?

A Downgrade from BBB to BB

B Upgrade from BBB to AAA

C Place the issuer on watch with a positive outlook

14 During the presentation about how the research team estimates the probability of default for a particular bond issuer, Lok is asked for his thoughts on the shape of the term structure of credit spreads. Which statement is he most likely to include in his response?

A The term structure of credit spreads typically is flat or slightly upward sloping for high-quality investment-grade bonds. High-yield bonds are more sensitive to the credit cycle, however, and can have a more upwardly sloped term structure of credit spreads than investment-grade bonds or even an inverted curve.

B The term structure of credit spreads for corporate bonds is always upward sloping, the more so the weaker the credit quality because probabilities of default are positively correlated with the time to maturity.

C There is no consistent pattern to the term structure of credit spreads. The shape of the credit term structure depends entirely on industry factors.

15 The final question to Lok is about covered bonds. The person asking says, "I've heard about them but don't know what they are." Which statement is Lok most likely to make to describe a covered bond?

A A covered bond is issued in a non-domestic currency. The currency risk is then fully hedged using a currency swap or a package of foreign exchange forward contracts.

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B A covered bond is issued with an attached credit default swap. It essentially is a "risk-free" government bond.

C A covered bond is a senior debt obligation giving recourse to the issuer as well as a predetermined underlying collateral pool, often commercial or residential mortgages.

The following information relates to Questions 16–22

Anna Lebedeva is a fixed-income portfolio manager. Paulina Kowalski, a junior analyst, and Lebedeva meet to review several positions in Lebedeva's portfolio.

Lebedeva begins the meeting by discussing credit rating migration. Kowalski asks Lebedeva about the typical impact of credit rating migration on the expected return on a bond. Lebedeva asks Kowalski to estimate the expected return over the next year on a bond issued by Entre Corp. The BBB rated bond has a yield to maturity of 5.50% and a modified duration of 7.54. Kowalski calculates the expected return on the bond over the next year given the partial credit transition and credit spread data in Exhibit 1. She assumes that market spreads and yields will remain stable over the year.

Exhibit 1 One-Year Transition Matrix for BBB Rated Bonds and Credit Spreads

	AAA	AA	A	BBB	BB	B	CCC, CC, C
Probability (%)	0.02	0.30	4.80	85.73	6.95	1.75	0.45
Credit spread	0.60%	0.90%	1.10%	1.50%	3.40%	6.50%	9.50%

Lebedeva next asks Kowalski to analyze a three-year bond, issued by VraiRive S.A., using an arbitrage-free framework. The bond's coupon rate is 5%, with interest paid annually and a par value of 100. In her analysis, she makes the following three assumptions:

- The annual interest rate volatility is 10%.
- The recovery rate is one-third of the exposure each period.
- The hazard rate, or conditional probability of default each year, is 2.00%.

Selected information on benchmark government bonds for the VraiRive bond is presented in Exhibit 2, and the relevant binomial interest rate tree is presented in Exhibit 3.

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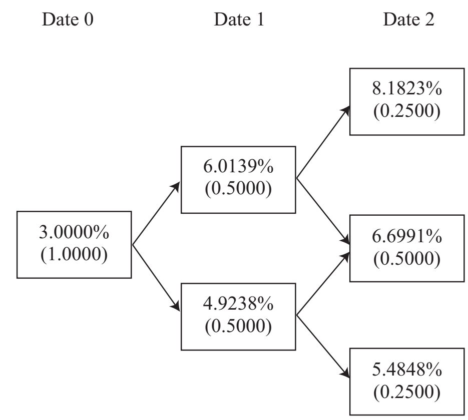
Exhibit 2 Par Curve Rates for Annual Payment Benchmark Government Bonds

Maturity	Coupon Rate	Price	Discount Factor	Spot Rate	Forward Rate
1	3.00%	100	0.970874	3.0000%	3.0000%
2	4.20%	100	0.920560	4.2255%	5.4656%
3	5.00%	100	0.862314	5.0618%	6.7547%

Exhibit 3 One-Year Binomial Interest Rate Tree for 10% Volatility (risk-neutral probabilities in parentheses)



Kowalski estimates the value of the VraiRive bond assuming no default (VND) as well as the fair value of the bond. She then estimates the bond's yield to maturity and the bond's credit spread over the benchmark in Exhibit 2. Kowalski asks Lebedeva, "What might cause the bond's credit spread to decrease?"

Lebedeva and Kowalski next discuss the drivers of the term structure of credit spreads. Kowalski tells Lebedeva:

Statement 1 The credit term structure for the most highly rated securities tends to be either flat or slightly upward sloping.

Statement 2 The credit term structure for lower-rated securities is often steeper, and credit spreads widen with expectations of strong economic growth.

Next, Kowalski analyzes the outstanding bonds of DLL Corporation, a high-quality issuer with a strong, competitive position. Her focus is to determine the rationale for a positively sloped credit spread term structure.

Lebedeva ends the meeting by asking Kowalski to recommend a credit analysis approach for a securitized asset-backed security (ABS) held in the portfolio. This non-static asset pool is made up of many medium-term auto loans that are homogeneous, and each loan is small relative to the total value of the pool.

16 The *most appropriate* response to Kowalski's question regarding credit rating migration is that it has:

A a negative impact.

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B no impact.

C a positive impact.

17 Based on Exhibit 1, the one-year expected return on the Entre Corp. bond is *closest* to:

A 3.73%.

B 5.50%.

C 7.27%.

18 Based on Kowalski's assumptions and Exhibits 2 and 3, the credit spread on the VraiRive bond is *closest* to:

A 0.6949%.

B 0.9388%.

C 1.4082%.

19 The *most appropriate* response to Kowalski's question relating to the credit spread is:

A an increase in the hazard rate.

B an increase in the loss given default.

C a decrease in the risk-neutral probability of default.

20 Which of Kowalski's statements regarding the term structure of credit spreads is correct?

A Only Statement 1

B Only Statement 2

C Both Statement 1 and Statement 2

21 DLL's credit spread term structure is *most* consistent with the firm having:

A low leverage.

B weak cash flow.

C a low profit margin.

22 Given the description of the asset pool of the ABS, Kowalski should recommend a:

A loan-by-loan approach.

B portfolio-based approach.

C statistics-based approach.

The following information relates to Questions 23–30

Lena Liecken is a senior bond analyst at Taurus Investment Management. Kristel Kreming, a junior analyst, works for Liecken in helping conduct fixed-income research for the firm's portfolio managers. Liecken and Kreming meet to discuss several bond positions held in the firm's portfolios.

Bonds I and II both have a maturity of one year, an annual coupon rate of 5%, and a market price equal to par value. The risk-free rate is 3%. Historical default experiences of bonds comparable to Bonds I and II are presented in Exhibit 1.

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Exhibit 1 Credit Risk Information for Comparable Bonds

Bond	Recovery Rate	Percentage of Bonds That Survive and Make Full Payment
I	40%	98%
II	35%	99%

Bond III

Bond III is a zero-coupon bond with three years to maturity. Liecken evaluates similar bonds and estimates a recovery rate of 38% and a risk-neutral default probability of 2%, assuming conditional probabilities of default. Kreming creates Exhibit 2 to compute Bond III's credit valuation adjustment. She assumes a flat yield curve at 3%, with exposure, recovery, and loss given default values expressed per 100 of par value.

Exhibit 2 Analysis of Bond III

Date	Exposure	Recovery	Loss Given Default	Probability of Default	Probability of Survival	Expected Loss	Present Value of Expected Loss
0							
1	94.2596	35.8186	58.4410	2.0000%	98.0000%	1.1688	1.1348
2	97.0874	36.8932	60.1942	1.9600%	96.0400%	1.1798	1.1121
3	100.0000	38.0000	62.0000	1.9208%	94.1192%	1.1909	1.0898
Sum				5.8808%		3.5395	3.3367

Bond IV

Bond IV is an AA rated bond that matures in five years, has a coupon rate of 6%, and a modified duration of 4.2. Liecken is concerned about whether this bond will be downgraded to an A rating, but she does not expect the bond to default during the next year. Kreming constructs a partial transition matrix, which is presented in Exhibit 3, and suggests using a model to predict the rating change of Bond IV using leverage ratios, return on assets, and macroeconomic variables.

Exhibit 3 Partial One-Year Corporate Transition Matrix (entries in %)

From/To	AAA	AA	A
AAA	92.00	6.00	1.00
AA	2.00	89.00	8.00
A	0.05	1.00	85.00
Credit Spread (%)	0.50	1.00	1.75

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Default Probabilities

Kreming calculates the risk-neutral probabilities, compares them with the actual default probabilities of bonds evaluated over the past 10 years, and observes that the actual and risk-neutral probabilities differ. She makes two observations regarding the comparison of these probabilities:

Observation 1	Actual default probabilities include the default risk premium associated with the uncertainty in the timing of the possible default loss.
Observation 2	The observed spread over the yield on a risk-free bond in practice includes liquidity and tax considerations, in addition to credit risk.

23 The expected exposure to default loss for Bond I is:

A less than the expected exposure for Bond II.

B the same as the expected exposure for Bond II.

C greater than the expected exposure for Bond II.

24 Based on Exhibit 1, the loss given default for Bond II is:

A less than that for Bond I.

B the same as that for Bond I.

C greater than that for Bond I.

25 Based on Exhibit 1, the expected future value of Bond I at maturity is *closest* to:

A 98.80.

B 103.74.

C 105.00.

26 Based on Exhibit 1, the risk-neutral default probability for Bond I is *closest* to:

A 2.000%.

B 3.175%.

C 4.762%.

27 Based on Exhibit 2, the credit valuation adjustment (CVA) for Bond III is *closest* to:

A 3.3367.

B 3.5395.

C 5.8808.

28 Based on Exhibit 3, if Bond IV's credit rating changes during the next year to an A rating, its expected price change would be *closest* to:

A –8.00%.

B –7.35%.

C –3.15%.

29 Kreming's suggested model for Bond IV is a:

A structural model.

B reduced-form model.

C term structure model.

30 Which of Kreming's observations regarding actual and risk-neutral default probabilities is correct?

A Only Observation 1

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B Only Observation 2

C Both Observation 1 and Observation 2

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SOLUTIONS

1 B is correct. The following table shows that the credit valuation adjustment (CVA) for the bond is €36.49, the sum of the present values of expected loss. The steps taken to complete the table are as follows.

Step 1 Exposure at Date T is $\frac{€1{,}000}{(1+r)^{4-T}}$, where r is 3%. That is, exposure is computed by discounting the face value of the bond using the risk-free rate and the number of years until maturity.

Step 2 Recovery = Exposure × Recovery rate

Step 3 Loss given default (LGD) = Exposure – Recovery

Step 4 Probability of default (POD) on Date 1 is 1.50%, the assumed hazard rate. The probability of survival (POS) on Date 1 is 98.50%.

For subsequent dates, POD is calculated as the hazard rate multiplied by the previous date's POS.

For example, to determine the Date 2 POD (1.4775%), the hazard rate of (1.50%) is multiplied by the Date 1 POS (98.50%).

Step 5 POS in Dates 2–4 = POS in the previous year – POD

(That is, POS in Year T = POS in year $[T - 1]$ – POD in Year T.)

POS can also be determined by subtracting the hazard rate from 100% and raising it to the power of the number of years:

$(100\% - 1.5000\%)^1 = 98.5000\%$

$(100\% - 1.5000\%)^2 = 97.0225\%$

$(100\% - 1.5000\%)^3 = 95.5672\%$

$(100\% - 1.5000\%)^4 = 94.1337\%$

Step 6 Expected loss = LGD × POD

Step 7 Discount factor (DF) for Date T is $\frac{1}{(1+r)^T}$, where r is 3%.

Step 8 PV of expected loss = Expected loss × DF

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	€915.14	€274.54	€640.60	1.5000%	98.5000%	€9.61	0.970874	€9.33
2	€942.60	€282.78	€659.82	1.4775%	97.0225%	€9.75	0.942596	€9.19
3	€970.87	€291.26	€679.61	1.4553%	95.5672%	€9.89	0.915142	€9.05
4	€1,000.00	€300.00	€700.00	1.4335%	94.1337%	€10.03	0.888487	€8.92
							CVA =	€36.49

Value of the bond if the bond were default free would be 1,000 × DF for Date 4 = €888.49.

Fair value of the bond considering CVA = €888.49 – CVA = €888.49 – €36.49 = €852.00.

Because the market price of the bond (€875) is greater than the fair value of €852, B is correct.

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A is incorrect because the market price of the bond differs from its fair value.
C is incorrect because although the bond's value if the bond were default free is greater than the market price, the bond has a risk of default, and CVA lowers its fair value to below the market price.

2 B is correct. The recovery rate to be used now in the computation of fair value is 30% × 1.25 = 37.5%, whereas the hazard rate to be used is 1.50% × 1.25 = 1.875%.

Using the steps outlined in the solution to Question 1, the following table is prepared, which shows that the bond's CVA increases to 40.49. Thus, Koning concludes that a change in the probability of default has a greater effect on fair value than a similar change in the recovery rate. The steps taken to complete the table are the same as those in the previous problem. There are no changes in exposures and discount factors in this table.

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	€915.14	€343.18	€571.96	1.8750%	98.1250%	€10.72	0.970874	€10.41
2	€942.60	€353.47	€589.12	1.8398%	96.2852%	€10.84	0.942596	€10.22
3	€970.87	€364.08	€606.80	1.8053%	94.4798%	€10.95	0.915142	€10.03
4	€1,000.00	€375.00	€625.00	1.7715%	92.7083%	€11.07	0.888487	€9.84
							CVA =	€40.49

Changes in the hazard and recovery rates do not affect the value of the default-free bond. So, it is the same as in the previous question: €888.49.

Fair value of the bond considering CVA = €888.49 – CVA = €888.49 – €40.49 = €848.00

3 A is correct. The following table shows that the CVA for the bond is €42.17, the sum of the present values of expected loss. The steps taken to complete the table are as follows.

Step 1 Exposure at Date 4 is €1,000 + Coupon amount = €1,000 + €60 = €1,060. Exposure at a date T prior to that is Coupon on Date T + PV at Date T of subsequent coupons + PV of €1,000 to be received at Date 4. For example, exposure at Date 2 is

$$€60+\frac{€60}{1+0.03}+\frac{€60}{(1+0.03)^2}+\frac{€1{,}000}{(1+0.03)^2}=€60+\frac{€60}{1+0.03}+\frac{€1{,}060}{(1+0.03)^2}$$
$$=€1{,}117.40$$

Steps 2 through 8 are the same as those in the solution to Question 1.

Date	Exposure	Recovery	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0								
1	€1,144.86	€343.46	€801.40	1.5000%	98.5000%	€12.02	0.970874	€11.67
2	€1,117.40	€335.22	€782.18	1.4775%	97.0225%	€11.56	0.942596	€10.89
3	€1,089.13	€326.74	€762.39	1.4553%	95.5672%	€11.10	0.915142	€10.15
4	€1,060.00	€318.00	€742.00	1.4335%	94.1337%	€10.64	0.888487	€9.45
							CVA =	€42.17

Value of the bond if the bond were default free would be €60 × DF_1 + €60 × DF_2 + €60 × DF_3 + €1,060 × DF_4 = €1,111.51.

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Fair value of the bond considering CVA = €1,111.51 – €42.17 = €1,069.34

4 A is correct. If default occurs on Date 3, the rate of return can be obtained by solving the following equation for internal rate of return (IRR):

$$€1{,}090 = \frac{€60}{1 + \text{IRR}} + \frac{€60}{(1 + \text{IRR})^2} + \frac{€326.74}{(1 + \text{IRR})^3}$$

In this equation, €60 is the amount of coupon received at Dates 1 and 2 prior to default at Date 3. The amount €326.74 is the recovery at Time 3 (from the CVA table in the solution to the previous question). The solution to the foregoing equation can be obtained using the cash flow IRR function on your calculator.

5 B is correct. For each possible transition, the expected percentage price change, computed as the product of the modified duration and the change in the spread as per Exhibit 7 of the reading, is calculated as follows:

From AA to AAA: –2.75 × (0.60% – 0.90%) = +0.83%

From AA to A: –2.75 × (1.10% – 0.90%) = –0.55%

From AA to BBB: –2.75 × (1.50% – 0.90%) = –1.65%

From AA to BB: –2.75 × (3.40% – 0.90%) = –6.88%

From AA to B: –2.75 × (6.50% – 0.90%) = –15.40%

From AA to C: –2.75 × (9.50% – 0.90%) = –23.65%

The expected percentage change in the value of the AA rated bond is computed by multiplying each expected percentage price change for a possible credit transition by its respective transition probability given in Exhibit 7 of the reading, and summing the products:

(0.0150 × 0.83%) + (0.8800 × 0%) + (0.0950 × –0.55%) + (0.0075 × –1.65%) + (0.0015 × –6.88%) + (0.0005 × –15.40%) + (0.0003 × –23.65%)= –0.0774%.

Therefore, the expected return on the bond over the next year is its YTM minus 0.0774%, assuming no default.

6 B is correct. Statement B is correct because a reduced-form credit model involves regression analysis using information generally available in the financial markets, such as the measures mentioned in the statement.

Statement A is incorrect because it is consistent with the use of a structural-form model and not a reduced-form model. It is a structural-form model that is based on the premise that a firm defaults on its debt if the value of its assets falls below its liabilities and that the probability of that event has the characteristics of an option.

Statement C is incorrect because it is consistent with the use of a structural-form model and not a reduced-form model. A structural-form model involves the estimation of a default barrier, and default occurs if the value of firm's assets falls below the default barrier.

7 C is correct. Structural models require information best known to the managers of the company. Reduced-form models only require information generally available in financial markets

A is literally true but when when models were developed is immaterial. Structural models are currently used in practice by commercial banks and credit rating agencies.

B is incorrect because computer technology facilities valuation using option pricing models as well as regression analysis.

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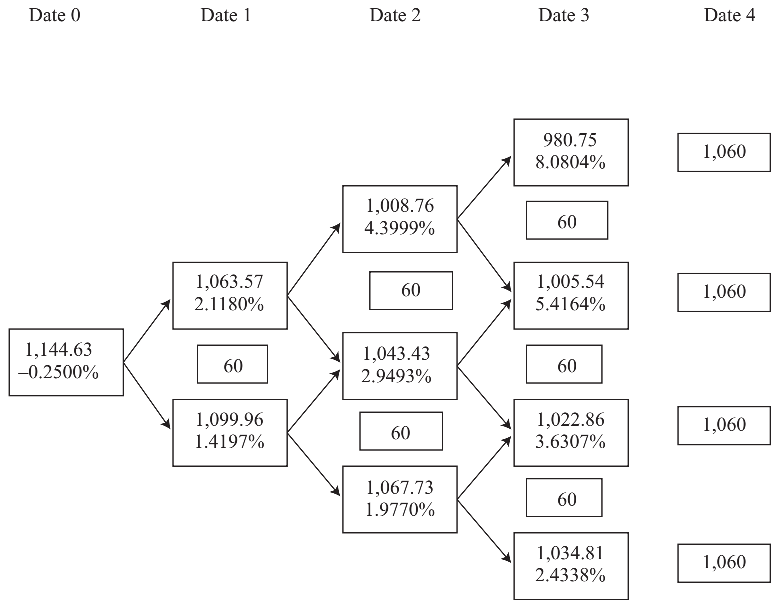
8 A is correct. The following tree shows the valuation assuming no default of bond B2, which pays a 6% annual coupon.



The scheduled year-end coupon and principal payments are placed to the right of each forward rate in the tree. For example, the Date 4 values are the principal plus the coupon of 60. The following are the four Date 3 values for the bond, shown above the interest rate at each node:

$$€1{,}060/1.080804 = €980.75$$

$$€1{,}060/1.054164 = €1{,}005.54$$

$$€1{,}060/1.036307 = €1{,}022.86$$

$$€1{,}060/1.024338 = €1{,}034.81$$

These are the three Date 2 values:

$$\frac{(0.5 \times €980.75 + 0.5 \times €1{,}005.54) + €60}{1.043999} = €1{,}008.76$$

$$\frac{(0.5 \times €1{,}005.54 + 0.5 \times €1{,}022.86) + €60}{1.029493} = €1{,}043.43$$

$$\frac{(0.5 \times €1{,}022.86 + 0.5 \times €1{,}034.81) + €60}{1.019770} = €1{,}067.73$$

These are the two Date 1 values:

$$\frac{(0.5 \times €1{,}008.76 + 0.5 \times €1{,}043.43) + €60}{1.021180} = €1{,}063.57$$

$$\frac{(0.5 \times €1{,}043.43 + 0.5 \times €1{,}067.73) + €60}{1.014197} = €1{,}099.96$$

This is the Date 0 value:

$$\frac{(0.5 \times €1{,}063.57 + 0.5 \times €1{,}099.96) + €60}{0.997500} = €1{,}144.63$$

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So, the value of the bond assuming no default (VND) is 1,144.63. This value could also have been obtained more directly using the benchmark discount factors from Exhibit 2:

€60 × 1.002506 + €60 × 0.985093 + €60 × 0.955848 + €1,060 × 0.913225 = €1,144.63.

The benefit of using the binomial interest rate tree to obtain the VND is that the same tree is used to calculate the expected exposure to default loss.

The credit valuation adjustment table is now prepared following these steps:

Step 1 Compute the expected exposures as described in the following, using the binomial interest rate tree prepared earlier.

The expected exposure for Date 4 is €1,060.

The expected exposure for Date 3 is

[(0.1250 × €980.75) + (0.3750 × €1,005.54) + (0.3750 × €1,022.86) + (0.1250 × €1,034.81)] + 60 = €1,072.60.

The expected exposure for Date 2 is

[(0.25 × €1,008.76) + (0.50 × €1,043.43) + (0.25 × €1,067.73)] + €60 = €1,100.84.

The expected exposure for Date 1 is

[(0.50 × €1,063.57) + (0.50 × €1,099.96)] + 60 = €1,141.76.

Step 2 LGD = Exposure × (1 – Recovery rate)

Step 3 The initial POD, also known as the hazard rate, is provided as 1.50%. For subsequent dates, POD is calculated as the hazard rate multiplied by the previous dates' POS.

For example, to determine the Date 2 POD (1.4775%), the hazard rate (1.5000%) is multiplied by the Date 1 POS (98.5000%).

Step 4 POS is determined by subtracting the hazard rate from 100% and raising it to the power of the number of years:

$(100\% - 1.5000\%)^1 = 98.5000\%$

$(100\% - 1.5000\%)^2 = 97.0225\%$

$(100\% - 1.5000\%)^3 = 95.5672\%$

$(100\% - 1.5000\%)^4 = 94.1337\%$

Step 5 Expected loss = LGD × POD

Step 6 Discount factors (DF) in Year T are obtained from Exhibit 2.

Step 7 PV of expected loss = Expected loss × DF

Date	Exposure	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0							
1	€1,141.76	€799.23	1.5000%	98.5000%	€11.99	1.002506	€12.02
2	€1,100.84	€770.58	1.4775%	97.0225%	€11.39	0.985093	€11.22
3	€1,072.60	€750.82	1.4553%	95.5672%	€10.93	0.955848	€10.44
4	€1,060.00	€742.00	1.4335%	94.1337%	€10.64	0.913225	€9.71
						CVA =	€43.39

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Fair value of the bond considering CVA = €1,144.63 – CVA = €1,144.63 – €43.39 = €1,101.24.

9 A is correct. The corporate bond's fair value is computed in the solution to Question 8 as €1,101.24 The YTM can be obtained by solving the following equation for IRR:

$$€1{,}101.24 = \frac{€60}{1+\text{IRR}} + \frac{€60}{(1+\text{IRR})^2} + \frac{€60}{(1+\text{IRR})^3} + \frac{€1{,}060}{(1+\text{IRR})^4}$$

The solution to this equation is 3.26%.

Valuation of a four-year, 6% coupon bond under no default (VND) is computed in the solution to Question 8 as 1,144.63. So, the YTM of a theoretical comparable-maturity government bond with the same coupon rate as the corporate bond B2 can be obtained by solving the following equation for IRR:

$$€1{,}144.63 = \frac{€60}{1+\text{IRR}} + \frac{€60}{(1+\text{IRR})^2} + \frac{€60}{(1+\text{IRR})^3} + \frac{€1{,}060}{(1+\text{IRR})^4}$$

The solution to this equation is 2.18%. So, the credit spread that the analyst wants to compute is 3.26% – 2.18% = 1.08%, or 108 bps.

B is incorrect, because that is the spread over the four-year government par bond that has a YTM of 2.25% in Exhibit 2: 3.26% – 2.25% = 1.01%, or 101 bps. Although this spread is commonly used in practice, the analyst is interested in finding the spread over a theoretical 6% coupon government bond.

C is incorrect, because that is the YTM of the coupon four-year government bond in Exhibit 2.

10 B is correct. The recovery rate to be used now in the computation of fair value is 30% × 0.75 = 22.500%, whereas the hazard rate to be used is 1.50% × 0.75 = 1.125%.

The tree that shows the valuation assuming no default of bond B2 in the solution to Question 8 will not be affected by the foregoing changes. Accordingly, VND remains €1,144.63.

Following the steps outlined in the solution to Question 8, the following table is prepared, which shows that the CVA for the bond decreases to €36.23. Thus, Ibarra concludes that a decrease in the probability of default has a greater effect on fair value than a similar decrease in the recovery rate. The steps taken to complete the table are the same as those in Question 8. There are no changes in exposures or discount factors in this table.

Date	Exposure	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0							
1	€1,141.76	€884.87	1.1250%	98.8750%	€9.95	1.002506	€9.98
2	€1,100.84	€853.15	1.1123%	97.7627%	€9.49	0.985093	€9.35
3	€1,072.60	€831.26	1.0998%	96.6628%	€9.14	0.955848	€8.74
4	€1,060.00	€821.50	1.0875%	95.5754%	€8.93	0.913225	€8.16
						CVA =	€36.23

Fair value of the bond considering CVA = €1,144.63 – CVA = €1,144.63 – €36.23 = €1,108.40

11 A is correct. The following tree shows the valuation assuming no default of floating-rate note (FRN) B4, which has a quoted margin of 4%.

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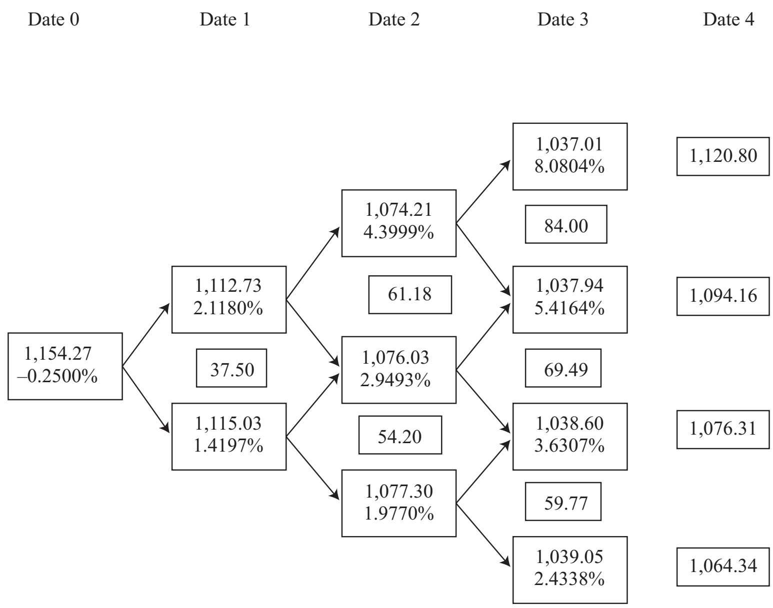

The scheduled year-end coupon and principal payments are placed to the right of each forward rate in the tree. For example, the four Date 4 values are the principal plus the coupon.

$$€1{,}000 \times (1 + 0.080804 + 0.04) = €1{,}120.80$$

$$€1{,}000 \times (1 + 0.054164 + 0.04) = €1{,}094.16$$

$$€1{,}000 \times (1 + 0.036307 + 0.04) = €1{,}076.31$$

$$€1{,}000 \times (1 + 0.024338 + 0.04) = €1{,}064.34$$

The following are the four Date 3 bond values for the note, shown above the interest rate at each node:

$$€1{,}120.80/1.080804 = €1{,}037.01$$

$$€1{,}094.16/1.054164 = €1{,}037.94$$

$$€1{,}076.31/1.036307 = €1{,}038.60$$

$$€1{,}064.34/1.024338 = €1{,}039.05$$

The three Date 3 coupon amounts are computed based on the interest rate at Date 2 plus the quoted margin of 4%:

$$€1{,}000 \times (0.043999 + 0.04) = €84.00$$

$$€1{,}000 \times (0.029493 + 0.04) = €69.49$$

$$€1{,}000 \times (0.019770 + 0.04) = €59.77$$

There are three Date 2 bond values:

$$\frac{(0.5 \times €1{,}037.01 + 0.5 \times €1{,}037.94) + €84.00}{1.043999} = €1{,}074.21$$

$$\frac{(0.5 \times €1{,}037.94 + 0.5 \times €1{,}038.60) + €69.49}{1.029493} = €1{,}076.03$$

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$$\frac{(0.5 \times €1{,}038.60 + 0.5 \times €1{,}039.05) + €59.77}{1.019770} = €1{,}077.30$$

The two Date 2 coupon amounts are computed based on the interest rate at Date 1 plus the quoted margin of 4%:

€1,000 × (0.021180 + 0.04) = €61.18

€1,000 × (0.014197 + 0.04) = €54.20

The Date 1 coupon amount is computed based on the interest rate at Date 0 plus the quoted margin of 4%:

€1,000 × (–0.0025 + 0.04) = €37.50

These are the calculations for the bond values for Date 1 and Date 0:

$$\frac{(0.5 \times €1{,}074.21 + 0.5 \times €1{,}076.03) + €61.18}{1.021180} = €1{,}112.73$$

$$\frac{(0.5 \times €1{,}076.06 + 0.5 \times €1{,}077.30) + €54.20}{1.014197} = €1{,}115.0$$

Then, the VND is calculated as follows:

$$\frac{(0.5 \times €1{,}112.73 + 0.5 \times €1{,}115.03) + €37.50}{0.9975} = €1{,}154.27$$

The expected exposures are then computed using the binomial interest rate tree prepared earlier. For example, the expected exposure for Date 4 is computed as follows:

[(0.125 × €1,120.80) + (0.375 × €1,094.16) + (0.375 × €1,076.31) + (0.125 × €1,064.34)] = €1,087.07

Similarly, the expected exposure for Date 3 is computed as follows:

[(0.125 × €1,037.01) + (0.375 × €1,037.94) + (0.375 × €1,038.60) + (0.125 × €1,039.05)] + [(0.250 × €84) + (0.500 × €69.49) + (0.250 × €59.77)] = €1,108.90

The expected exposures for Dates 2 and 1 are computed similarly, and the credit valuation adjustment table is completed following Steps 2–7 outlined in the solution to Question 8.

Date	Exposure	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0							
1	€1,151.38	€805.97	1.5000%	98.5000%	€12.09	1.002506	€12.12
2	€1,133.58	€793.51	1.4775%	97.0225%	€11.72	0.985093	€11.55
3	€1,108.90	€776.23	1.4553%	95.5672%	€11.30	0.955848	€10.80
4	€1,087.07	€760.95	1.4335%	94.1337%	€10.91	0.913225	€9.96
						CVA =	€44.43

Fair value of the FRN considering CVA = €1,154.27 – CVA = €1,154.27 – €44.43 = €1,109.84

Because the market price of €1,070 is less than the estimated fair value, the analyst should recommend adding to existing positions in the FRN.

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B and C are incorrect because the FRN is perceived to be undervalued in the market.

12 A is correct. The changing probability of default will not affect the binomial tree prepared in the solution to Question 11. The Date 1 value remains €1,154.27, which is also the VND. The expected exposures, loss given default, and discount factors are also unaffected by the changing probability of default. The following is the completed credit valuation adjustment table.

Date	Exposure	LGD	POD	POS	Expected Loss	DF	PV of Expected Loss
0							
1	€1,151.38	€805.97	1.5000%	98.5000%	€12.09	1.002506	€12.12
2	€1,133.58	€793.51	0.4925%	98.0075%	€3.91	0.985093	€3.85
3	€1,108.90	€776.23	0.4900%	97.5175%	€3.80	0.955848	€3.64
4	€1,087.07	€760.95	0.4876%	97.0299%	€3.71	0.913225	€3.39
						CVA =	€22.99

Thus, CVA decreases to €22.99.

13 C is correct. The credit rating agencies typically make incremental changes as seen in a transition matrix provided in Exhibit 7 of the reading. Ibarra believes the bond to be undervalued, in that her assessment of the probability of default and the recovery rate is more optimistic than that of the agencies. Therefore, she most likely expects the credit rating agencies to put the issuer on a positive watch.

A is incorrect because the bond is perceived to be undervalued, not overvalued. Ibarra is not expecting a credit downgrade.

B is incorrect because it is not the *most likely* expectation. The rating agencies rarely jump an issuer all the way from BBB to AAA. In Exhibit 7, the probability of a BBB rated issuer going from BBB to AAA is 0.02%, whereas it is 4.80% to go from BBB to A.

14 A is correct.

B is incorrect because, although generally true for investment-grade bonds, the statement neglects the fact that high-yield issuers sometimes face a downward-sloping credit term structure. Credit term structures are not *always* upward sloping.

C is incorrect because there is a consistent pattern to the term structure of credit spreads—typically it is upwardly sloped because greater time to maturity is associated with higher projected probabilities of default and lower recovery rates.

15 C is correct. A covered bond is a senior debt obligation of a financial institution that gives recourse to the originator/issuer as well as a predetermined underlying collateral pool. Each specific country or jurisdiction specifies the eligible collateral types as well as the specific structures permissible in the covered bond market. Covered bonds most frequently have either commercial or residential mortgages meeting specific criteria or public sector exposures as underlying collateral.

A is incorrect. The term "covered" is used in foreign exchange analysis, for instance, "covered interest rate parity." In the context of securitized debt, a covered bond is secured by specific assets in addition to the overall balance sheet of the issuer.

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B is incorrect because a covered bond does not involve a credit default swap. In addition, an issuer is not likely to sell a credit default swap on its own liability.

16 A is correct. Credit spread migration typically reduces the expected return for two reasons. First, the probabilities for rating changes are not symmetrically distributed around the current rating; they are skewed toward a downgrade rather than an upgrade. Second, the increase in the credit spread is much larger for downgrades than is the decrease in the spread for upgrades.

17 A is correct. The expected return on the Entre Corp. bond over the next year is its yield to maturity plus the expected percentage price change in the bond over the next year. In the table below, for each possible transition, the expected percentage price change is the product of the bond's modified duration of 7.54, multiplied by –1, and the change in the spread, weighted by the given probability:

$$\text{Expected percentage price change} = (0.0002 \times 6.786\%) + (0.0030 \times 4.524\%) + (0.0480 \times 3.016\%) + (0.8573 \times 0.000\%) + (0.0695 \times -14.326\%) + (0.0175 \times -37.700\%) + (0.0045 \times -60.320\%) = -1.76715\%.$$

So, the expected return on the Entre Corp. bond is its yield to maturity plus the expected percentage price change due to credit migration:

$$\text{Expected return} = 5.50\% - 1.77\% = 3.73\%.$$

	Expected % Price Change (1)	Probability (2)	Expected % Price Change × Probability (1 × 2)
From BBB to AAA	–7.54 × (0.60% – 1.50%) = 6.786%	0.0002	0.00136
From BBB to AA	–7.54 × (0.90% – 1.50%) = 4.524%	0.0030	0.01357
From BBB to A	–7.54 × (1.10% – 1.50%) = 3.016%	0.0480	0.14477
From BBB to BB	–7.54 × (3.40% – 1.50%) = –14.326%	0.0695	–0.99566
From BBB to B	–7.54 × (6.50% – 1.50%) = –37.700%	0.0175	–0.65975
From BBB to CCC, CC, C	–7.54 × (9.50% – 1.50%) = –60.320%	0.0045	–0.27144
		Total:	–1.76715

18 C is correct. The credit spread can be calculated in three steps:

Step 1 Estimate the value of the three-year VraiRive bond assuming no default. Based on Kowalski's assumptions and Exhibits 2 and 3, the value of the three-year VraiRive bond assuming no default is 100.0000.

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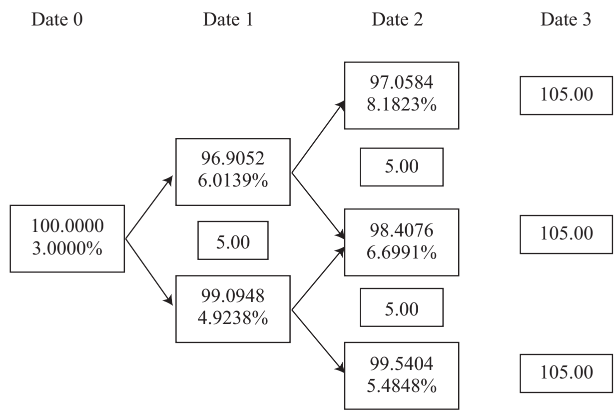

Supporting calculations:

The bond value in each node is the value of next period's cash flows discounted by the forward rate. For the three nodes on Date 2, the bond values are as follows:

105/1.081823 = 97.0584.

105/1.066991 = 98.4076.

105/1.054848 = 99.5404.

For the two nodes on Date 1, the two bond values are as follows:

[0.5 × (97.0584) + 0.5 × (98.4076) + 5.00]/1.060139 = 96.9052.

[0.5 × (98.4076) + 0.5 × (99.5404) + 5.00]/1.049238 = 99.0948.

Finally, for the node on Date 0, the bond value is

[0.5 × (96.9052) + 0.5 × (99.0948) + 5.00]/1.030000 = 100.0000.

Therefore, the VND for the VraiRive bond is 100.0000.

Step 2 Calculate the credit valuation adjustment (CVA), and then subtract the CVA from the VND from Step 1 to establish the fair value of the bond. The CVA equals the sum of the present values of each year's expected loss and is calculated as follows:

Date	Expected Exposure	Loss Given Default	Probability of Default	Discount Factor	Present Value of Expected Loss
1	103.0000	68.6667	2.0000%	0.970874	1.3333
2	103.3535	68.9023	1.9600%	0.920560	1.2432
3	105.0000	70.0000	1.9208%	0.862314	1.1594
				CVA =	3.7360

Supporting calculations:

The expected exposures at each date are the bond values at each node, weighted by their risk-neutral probabilities, plus the coupon payment:

Date 1: 0.5 × (96.9052) + 0.5 × (99.0948) + 5.00 = 103.0000.

Date 2: 0.25 × (97.0584) + 0.5 × (98.4076) + 0.25 × (99.5404) + 5.00 = 103.3535.

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Date 3: 105.0000

The loss given default (LGD) on each date is 2/3 of the expected exposure.

The probability of default (POD) on each date is as follows:

Date 1: 2%

Date 2: 2% × (100% – 2%) = 1.96%.

Date 3: 2% × (100% – 2%)^2 = 1.9208%.

The discount factor on each date is 1/(1 + spot rate for the date) raised to the correct power.

Finally, the credit valuation adjustment each year is the product of the LGD times the POD times the discount factor, as shown in the last column of the table. The sum of the three annual CVAs is 3.7360.

So, the fair value of the VraiRive bond is the VND less the CVA, or VND – CVA = 100 – 3.7360 = 96.2640.

Step 3 Based on the fair value from Step 2, calculate the yield to maturity of the bond, and solve for the credit spread by subtracting the yield to maturity on the benchmark bond from the yield to maturity on the VraiRive bond. The credit spread is equal to the yield to maturity on the VraiRive bond minus the yield to maturity on the three-year benchmark bond (which is 5.0000%). Based on its fair value of 96.2640, the VraiRive bond's yield to maturity (YTM) is

$$96.2640 = \frac{5}{(1 + \text{YTM})} + \frac{5}{(1 + \text{YTM})^2} + \frac{105}{(1 + \text{YTM})^3}$$

Solving for YTM, the yield to maturity is 6.4082%. Therefore, the credit spread on the VraiRive bond is 6.4082% – 5.0000% = 1.4082%.

19 C is correct. A decrease in the risk-neutral probability of default would decrease the credit valuation adjustment and decrease the credit spread. In contrast, increasing the bond's loss-given-default assumption and increasing the probability-of-default (hazard rate) assumption would increase the credit valuation adjustment and decrease the fair value of the bond (and increase the yield to maturity and the credit spread over its benchmark).

20 A is correct. For investment-grade bonds with the highest credit ratings, credit spreads are extremely low, and credit migration is possible only in one direction given the implied lower bound of zero on credit spreads. As a result, the credit term structure for the most highly rated securities tends to be either flat or slightly upward sloping. Securities with lower credit quality, however, face greater sensitivity to the credit cycle. Credit spreads would decrease, not increase, with the expectation of economic growth. There is a countercyclical relationship between credit spreads and benchmark rates over the business cycle. A strong economic climate is associated with higher benchmark yields but lower credit spreads because the probability of issuers defaulting declines in such good times.

21 A is correct. Positively sloped credit spread curves may arise when a high-quality issuer with a strong competitive position in a stable industry has low leverage, strong cash flow, and a high profit margin. This type of issuer tends to exhibit very low short-term credit spreads that rise with increasing maturity given greater uncertainty due to the macroeconomic environment, potential adverse changes in the competitive landscape, technological change, or other

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factors that drive a higher implied probability of default over time. Empirical academic studies also tend to support the view that the credit spread term structure is upward sloping for investment-grade bond portfolios.

22 B is correct. The auto ABS is granular, with many small loans relative to the size of the total portfolio. The auto loans are also homogeneous. These characteristics support using the portfolio-based approach. A loan-by-loan approach would be inefficient because of the large number of basically similar loans; this approach is best for a portfolio of discrete, large loans that are heterogeneous. A statistics-based approach would work for a static book of loans, whereas the auto loan portfolio would be dynamic and would change over time.

23 B is correct. The expected exposure is the projected amount of money that an investor could lose if an event of default occurs, before factoring in possible recovery. The expected exposure for both Bond I and Bond II is 100 + 5 = 105.

24 C is correct. The loss given default is a positive function of the expected exposure to default loss and a negative function of the recovery rate. Because Bond II has a lower recovery rate than Bond I and the same expected exposure to default loss (100 + 5 = 105), it will have a higher loss given default than Bond I will have. The loss given default for Bond I is 105 × (1 – 0.40) = 63.00. The loss given default for Bond II is 105 × (1 – 0.35) = 68.25.

25 B is correct. In the event of no default, the investor is expected to receive 105. In the event of a default, the investor is expected to receive 105 – [105 × (1 – 0.40)] = 42. The expected future value of the bond is, therefore, the weighted average of the no-default and default amounts, or (105 × 0.98) + (42 × 0.02) = 103.74.

26 B is correct. The risk-neutral default probability, P^*, is calculated using the current price, the expected receipt at maturity with no default (that is, 100 + 5 = 105), the expected receipt at maturity in the event of a default (that is, 0.40 × 105 = 42), and the risk-free rate of interest (0.03):

$$100 = \frac{\left[105 \times \left(1 - P^*\right)\right] + \left(42 \times P^*\right)}{1.03}$$

Solving for P^* gives 0.031746, or 3.1746%.

27 A is correct. The CVA is the sum of the present value of expected losses on the bond, which from Exhibit 2 is 3.3367.

28 C is correct. The expected percentage price change is the product of the negative of the modified duration and the difference between the credit spread in the new rating and the old rating:

Expected percentage price change = –4.2 × (0.0175 – 0.01) = –0.0315, or –3.15%.

29 B is correct. A reduced-form model in credit risk analysis uses historical variables, such as financial ratios and macroeconomic variables, to estimate the default intensity. A structural model for credit risk analysis, in contrast, uses option pricing and relies on a traded market for the issuer's equity.

30 B is correct. Observation 1 is incorrect, but Observation 2 is correct. The actual default probabilities do not include the default risk premium associated with the uncertainty in the timing of the possible default loss. The observed spread over the yield on a risk-free bond in practice does include liquidity and tax considerations, in addition to credit risk.

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READING

36

Credit Default Swaps

by Brian Rose and Don M. Chance, PhD, CFA

Brian Rose (USA). Don M. Chance, PhD, CFA, is at Louisiana State University (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** describe credit default swaps (CDS), single-name and index CDS, and the parameters that define a given CDS product;
☐	**b.** describe credit events and settlement protocols with respect to CDS;
☐	**c.** explain the principles underlying, and factors that influence, the market's pricing of CDS;
☐	**d.** describe the use of CDS to manage credit exposures and to express views regarding changes in shape and/or level of the credit curve;
☐	**e.** describe the use of CDS to take advantage of valuation disparities among separate markets, such as bonds, loans, equities, and equity-linked instruments.

1 INTRODUCTION

A **credit derivative** is a derivative instrument in which the underlying is a measure of a borrower's credit quality. Four types of credit derivatives are (1) total return swaps, (2) credit spread options, (3) credit-linked notes, and (4) credit default swaps, or CDS.[1] The first three are not frequently encountered. CDS have clearly emerged as the primary type of credit derivative and, as such, are the topic of this reading. In a CDS, one party makes payments to the other and receives in return the promise of compensation if a third party defaults.

1 We use the expression CDS in both singular and plural form, as opposed to CDSs or CDS's.

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In any derivative, the payoff is based on (*derived from*) the performance of an underlying instrument, rate, or asset that we call the underlying.[2] For a CDS, the underlying is the credit quality of a borrower. At its most fundamental level, a CDS provides protection against default, but it also protects against changes in the market's perception of a borrower's credit quality well in advance of default. The value of a CDS will rise and fall as opinions change about the likelihood of default. The actual event of default might never occur.

Derivatives are characterized as *contingent claims*, meaning that their payoffs are dependent on the occurrence of a specific event or outcome. For an equity option, the event is that the stock price is above (for a call) or below (for a put) the exercise price at expiration. For a CDS, the credit event is more difficult to identify. In financial markets, whether a default has occurred is sometimes not clear. Bankruptcy would seem to be a default, but many companies declare bankruptcy and some ultimately pay all of their debts. Some companies restructure their debts, usually with creditor approval but without formally declaring bankruptcy. Creditors are clearly damaged when debts are not paid, not paid on time, or paid in a form different from what was promised, but they are also damaged when there is simply an increase in the likelihood that the debt will not be paid. The extent of damage to the creditor can be difficult to determine. A decline in the price of a bond when investors perceive an increase in the likelihood of default is a very real loss to the bondholder. Credit default swaps are designed to protect creditors against such credit events. As a result of the complexity of defining what constitutes default, the industry has expended great effort to provide clear guidance on what credit events are covered by a CDS contract. As with all efforts to write a perfect contract, however, no such device exists and disputes do occasionally arise. We will take a look at these issues later.

This reading is organized as follows: Section 2 explores basic definitions and concepts, and Section 3 covers the elements of valuation and pricing. Section 4 discusses applications. Section 5 provides a summary.

2 BASIC DEFINITIONS AND CONCEPTS

We start by defining a **credit default swap**:

> *A credit default swap is a derivative contract between two parties, a credit protection buyer and credit protection seller, in which the buyer makes a series of cash payments to the seller and receives a promise of compensation for credit losses resulting from the default—that is, a pre-defined credit event—of a third party.*

In a CDS contract there are two counterparties, the **credit protection buyer** and the **credit protection seller**. The buyer agrees to make a series of periodic payments to the seller over the life of the contract (which are determined and fixed at contract initiation) and receives in return a promise that if default occurs, the protection seller will compensate the protection buyer. If default occurs, the periodic payments made by the protection buyer to the protection seller terminate. Exhibit 1 shows the structure of payment flows.

2 Consistent with industry practice, we use the word underlying as a noun even though it generally requires a follower, such as in underlying asset. Because derivatives exist on credit and other non-assets, the word underlying has taken on the properties of a noun in the world of derivatives.

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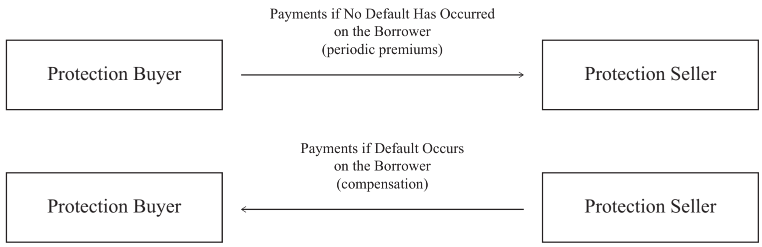
Exhibit 1 Payment Structure of a CDS



Credit default swaps are somewhat similar to put options. Put options effectively enable the option holder to sell (put) the underlying to the option seller if the underlying performs poorly relative to the exercise price. The option holder is thus compensated for the poor performance of the underlying. CDS act in a similar manner. If a default occurs, a loan or bond has clearly performed badly. The protection buyer is then compensated by the protection seller. How that compensation occurs and how much protection it provides are some points we will discuss.[3]

The majority of CDS are written on debt issued by corporate borrowers, which will be our focus in this reading. But note that CDS can be written on the debt of sovereign governments and state and local governments. In addition, CDS can be written on portfolios of loans, mortgages, or debt securities.

2.1 Types of CDS

There are three types of CDS; single-name CDS, index CDS, and tranche CDS.[4] A CDS on one specific borrower is called a **single-name CDS**. The borrower is called the **reference entity**, and the contract specifies a **reference obligation**, a particular debt instrument issued by the borrower that is the designated instrument being covered. The designated instrument is usually a senior unsecured obligation, which is often referred to as a senior CDS, but the reference obligation is not the only instrument covered by the CDS. Any debt obligation issued by the borrower that is *pari passu* (ranked equivalently in priority of claims) or higher relative to the reference obligation is covered. The payoff of the CDS is determined by the **cheapest-to-deliver** obligation, which is the debt instrument that can be purchased and delivered at the lowest cost but has the same seniority as the reference obligation.

EXAMPLE 1

Cheapest-to-Deliver Obligation

Assume that a company with several debt issues trading in the market files for bankruptcy (i.e., a credit event takes place). What is the cheapest-to-deliver obligation for a senior CDS contract?

A A subordinated unsecured bond trading at 20% of par

3 Note that a CDS does not eliminate credit risk. It eliminates the credit risk of one party but substitutes the credit risk of the CDS seller. Although there are no guarantees that the CDS seller will not default, as was seen with several large financial institutions in the crisis that started in 2007, most CDS sellers are relatively high-quality borrowers. If they were not, they could not be active sellers of CDS.

4 In addition to CDS, there are also options on CDS, which are called CDS swaptions. We will not cover this instrument here. Swaptions in general are covered elsewhere in the derivatives material.

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B A five-year senior unsecured bond trading at 50% of par

C A two-year senior unsecured bond trading at 45% of par

Solution:

C is correct. The cheapest-to-deliver, or lowest-priced, instrument is the two-year senior unsecured bond trading at 45% of par. Although the bond in A trades at a lower dollar price, it is subordinated and, therefore, does not qualify for coverage under the senior CDS. Note that even if the CDS holder also held the five-year bonds, he would still receive payment on the CDS based on the cheapest-to-deliver obligation, not the specific obligations he holds.

A second type of credit default swap, an **index CDS**, involves a combination of borrowers. These instruments have been created such that it is possible to trade indexes of CDS. This type of instrument allows participants to take positions on the credit risk of a combination of companies, in much the same way that investors can trade index or exchange-traded funds that are combinations of the equities of companies. Correlation of returns is a strong determinant of a portfolio's behavior. For index CDS, this concept takes the form of a factor called **credit correlation**, and it is a key determinant of the value of an index CDS. Analyzing the effects of those correlations is a highly specialized subject beyond the CFA Program, but the reader should be aware that much effort is placed on modeling how defaults by certain companies are connected to defaults by other companies. The more correlated the defaults, the more costly it is to purchase protection for a combination of the companies. In contrast, for a diverse combination of companies whose defaults have low correlations, it will be much less expensive to purchase protection.

A third type of CDS is the **tranche CDS**, which covers a combination of borrowers but only up to pre-specified levels of losses—much in the same manner that asset-backed securities are divided into tranches, each covering particular levels of losses. The tranche CDS is only a small portion of the CDS market, and we will not cover it any further.

2.2 Important Features of CDS Markets and Instruments

As we will describe in more detail later, the CDS market is large, global, and well organized. The unofficial industry governing body is the International Swaps and Derivatives Association (ISDA), which publishes industry-supported conventions that facilitate the functioning of the market. Parties to CDS contracts generally agree that their contracts will conform to ISDA specifications. These terms are specified in a document called the **ISDA Master Agreement**, which the parties to a CDS sign. In Europe, the standard CDS contract is called the Standard Europe Contract, and in the United States and Canada, it is called the Standard North American Contract. Other standardized contracts exist for Asia, Australia, Latin America, and a few specific countries.

Each CDS contract specifies a **notional amount**, or "notional" for short, which is the amount of protection being purchased. For example, if a company has a bond issue of €100 million, a CDS could be constructed for any amount up to €100 million. The notional amount can be thought of as the *size* of the contract. It is important to understand that the total amount of CDS notional can exceed the amount of debt

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outstanding of the reference entity.[5] As we will discuss later, the credit protection buyer does not have to be an actual creditor holding exposure (i.e., owning a loan, bond, or other debt instrument). It can be simply a party that believes that there will be a change in the credit quality of the reference entity.

As with all derivatives, the CDS contract has an expiration or maturity date, and coverage is provided up to that date. The typical maturity range is 1 to 10 years, with 5 years being the most common and actively traded maturity, but the two parties can negotiate any maturity. Maturity dates are typically the last day of March, June, September, or December, with June and December being the most popular. As with bonds, a CDS contract of a particular maturity is really that maturity only for an instant. For example, a five-year CDS is technically no longer a five-year CDS just a day later. As the maturity of that CDS decreases, a new five-year CDS is created, and it begins to be referred to as the five-year CDS. Of course, this point is no different from ordinary bonds.

The buyer of a CDS pays a periodic premium to the seller, referred to as the **CDS spread**, which is a return over Libor required to protect against credit risk. It is sometimes referred to as a credit spread. Conceptually, it is the same as the credit spread on a bond, the compensation for bearing credit risk. This premium is determined based on valuation models that are beyond the scope of the CFA program. Nonetheless, it is important to understand the concept of the credit spread on a CDS, and we will have much more to say about it later in this reading.

An important advancement in the development of CDS has been in establishing standard annual coupon rates on CDS contracts.[6] Formerly, the rate was set at the credit spread. If a CDS required a rate of 4% to compensate the protection seller for the assumption of credit risk, the protection buyer made quarterly payments amounting to 4% annually. Now CDS rates are standardized, with the most common coupons being either 1% or 5%. The 1% rate typically is used for a CDS on an investment-grade company or index, and the 5% rate is used for a CDS on a high-yield company or index. Obviously, either standardized rate might not be the appropriate rate to compensate the seller. Clearly, not all investment-grade companies have equivalent credit risk, and not all high-yield companies have equivalent credit risk. In effect, the standard rate may be too high or too low. This discrepancy is accounted for by an **upfront payment**, commonly called the **upfront premium**. The differential between the credit spread and the standard rate is converted to a present value basis. Thus, a protection buyer paying a standard rate that is insufficient to compensate the protection seller will make a cash upfront payment. Similarly, a credit spread less than the standard rate would result in a cash payment from the protection seller to the protection buyer.

Regardless of whether either party makes an upfront payment, the reference entity's credit quality could change during the life of the contract, thereby resulting in changes in the value of the CDS. These changes are reflected in the price of the CDS in the market. Consider a high-yield company with a 5% credit spread and its CDS bears a coupon of 5%. Therefore, there is no upfront payment. The protection buyer simply agrees to make 5% payments over the life of the CDS. Now suppose that at

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5 This point will be discussed in more detail later, but here we will address the obvious question of how the aggregate amount of protection can exceed the aggregate risk. As an analogy, consider the exercise of an option. Given the number of options created, at exercise the call option holders could have the right to buy more shares than exist or put option holders could have the right to sell more shares than exist. Such an event has never come close to happening. In the CDS market, the cash settlement feature, which is typically not used for options on stocks, solves this problem. We will describe how cash settlement works later.

6 The reader should be aware of the potential confusion over the term "coupon." The reference bond will make payments that are referred to collectively as the coupon. A CDS on the reference bond will have its own coupon rate, which is calculated based on the expected payoff. Furthermore, with standardization of CDS coupons, there is likely to be a third payment referred to as a coupon. The reader must be alert to the context.

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some later date, the reference entity experiences a decrease in its credit quality. The credit protection buyer is thus paying 5% for risk that now merits a rate higher than 5%. The coverage and cost of protection are the same, but the risk being covered is greater. The value of the CDS to the credit protection buyer has, therefore, increased, and if desired, he could unwind the position to capture the gain. The credit protection seller has experienced a loss in value of the instrument because he is receiving 5% to cover a risk that is higher than it was when the contract was initiated. It should be apparent that absent any other exposure to the reference entity, if the credit quality of the reference entity decreases, the credit protection buyer gains and the credit protection seller loses.[7] The market value of the CDS reflects these gains and losses.

Because of these CDS characteristics, there is potential confusion regarding which party is long and which is short. Normally, we think of buyers as being long and sellers as being short, but in the CDS world, it is the opposite. Because the credit protection buyer promises to make a series of future payments, it is regarded as being short. This is consistent with the fact that in the financial world, "shorts" are said to benefit when things go badly. Credit quality is based on the underlying debt obligation, and when it improves, the credit protection seller benefits. When credit quality deteriorates, the credit protection buyer benefits. Hence, the CDS industry views the credit protection seller as the long and the buyer as the short. This point can lead to confusion because we effectively say the credit protection buyer is short and the credit protection seller is long.

2.3 Credit and Succession Events

The **credit event** is what defines default by the reference entity—that is, the outcome that triggers a payment from the credit protection seller to the credit protection buyer. This event must be unambiguous: Did it occur, or did it not? For the market to function well, the answer to this question must be clear.

There are three general types of credit events: bankruptcy, failure to pay, and restructuring. **Bankruptcy** is a declaration provided for by a country's laws that typically involves the establishment of a legal procedure that forces creditors to defer their claims. Bankruptcy essentially creates a temporary fence around the company through which the creditors cannot pass. During the bankruptcy process, the defaulting party works with its creditors and the court to attempt to establish a plan for repaying the debt. If that plan fails, there is likely to be a full liquidation of the company, at which time the court determines the payouts to the various creditors. Until liquidation occurs, the company normally continues to operate. Many companies do not liquidate and are able to emerge from bankruptcy. A bankruptcy filing by the reference entity is universally regarded as a credit event in CDS contracts.

Another credit event recognized in standard CDS contracts is **failure to pay**, which occurs when a borrower does not make a scheduled payment of principal or interest on any outstanding obligations after a grace period, without a formal bankruptcy filing. The third type of event, **restructuring**, refers to a number of possible events, including reduction or deferral of principal or interest, change in seniority or priority of an obligation, or change in the currency in which principal or interest is scheduled to be paid. To qualify as a credit event, the restructuring must be involuntary, meaning that it is forced on the borrower by the creditors who must accept the restructured

7 A key element of this point is the absence of any other exposure to the reference entity. The credit protection buyer could be holding the debt itself, and the CDS might cover only a portion of the debt. Thus, the credit protection buyer might be gaining on the CDS, as described in the text, but be losing on its overall position.

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terms.[8] In the United States, restructuring is not considered a credit event because bankruptcy is typically the preferred route for US companies. Outside the United States, restructuring is more commonly used and is considered a credit event. The Greek debt crisis is a good example of a restructuring that triggered a credit event.

Determination of whether a credit event occurs is done by a 15-member group within the ISDA called the Determinations Committee (DC). Each region of the world has a Determinations Committee, which consists of 10 CDS dealer banks and 5 non-bank end users. To declare a credit event, there must be a supermajority vote of 12 members.

The determinations committees also play a role in determining whether a **succession event** occurred. A succession event arises when there is a change in the corporate structure of the reference entity, such as through a merger, divestiture, spinoff, or any similar action in which ultimate responsibility for the debt in question becomes unclear. For example, if a company acquires all of the shares of a target company, it ordinarily assumes the target company's debt as well. Many mergers, however, are more complicated and can involve only partial acquisition of shares. Spinoffs and divestitures can also involve some uncertainty about who is responsible for certain debts. When such a question arises, it becomes critical for CDS holders. The question is ordinarily submitted to a DC, and its resolution often involves complex legal interpretations of contract provisions and country laws. If a succession event is declared, the CDS contract is modified to reflect the DC's interpretation of whoever it believes becomes the obligor for the original debt. Ultimately, the CDS contract could be split among multiple entities.

2.4 Settlement Protocols

If the DC declares that a credit event has occurred, the two parties to a CDS have the right, but not the obligation, to settle. **Settlement** typically occurs 30 days after declaration of the credit event by the DC. CDS can be settled by **physical settlement** or by **cash settlement**. The former is less common and involves actual delivery of the debt instrument in exchange for a payment by the credit protection seller of the notional amount of the contract. In cash settlement, the credit protection seller pays cash to the credit protection buyer. Determining the amount of that payment is a critical factor because opinions can differ about how much money has actually been lost. The payment should essentially be the loss that the credit protection buyer has incurred, but determining that amount is not straightforward. Default on a debt does not mean that the creditor will lose the entire amount owed. A portion of the loss could be recovered. The percentage of the loss recovered is called the **recovery rate**. It then becomes the percentage received by the protection buyer relative to the amount owed. The complement is called the **payout ratio**, which is essentially an estimate of the expected credit loss. The **payout amount** is determined as the payout ratio multiplied by the notional.[9]

Payout ratio = 1 – Recovery rate (%)

Payout amount = Payout ratio × Notional

Actual recovery can be a very long process, however, and can occur much later than the payoff date of the CDS. To determine an appropriate payout ratio, the industry conducts an auction in which major banks and dealers submit bids and offers for the

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8 Although our focus is on corporate debt, sovereign and municipal governments sometimes declare a moratorium or, more drastically, a repudiation of debt, both of which typically qualify as credit events.

9 Do not confuse this payout ratio with the payout ratio in equity analysis, which is the percentage of earnings paid out as dividends.

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cheapest-to-deliver defaulted debt. This process identifies the market's expectation for the recovery rate and the complementary payout ratio, and the CDS parties agree to accept the outcome of the auction, even though the actual recovery rate can ultimately be quite different, which is an important point if the CDS protection buyer also holds the underlying debt.

EXAMPLE 2

Settlement Preference

A French company files for bankruptcy, triggering various CDS contracts. It has two series of senior bonds outstanding: Bond A trades at 30% of par, and Bond B trades at 40% of par. Investor X owns €10 million of Bond A and owns €10 million of CDS protection. Investor Y owns €10 million of Bond B and owns €10 million of CDS protection.

1 Determine the recovery rate for both CDS contracts.
2 Explain whether Investor X would prefer to cash settle or physically settle her CDS contract or whether she is indifferent.
3 Explain whether Investor Y would prefer to cash settle or physically settle his CDS contract or whether he is indifferent.

Solution to 1:

Bond A is the cheapest-to-deliver obligation, trading at 30% of par, so the recovery rate for both CDS contracts is 30%.

Solution to 2:

Investor X has no preference between settlement methods. She can cash settle for €7 million [(1 – 30%) × €10 million] and sell her bond for €3 million, for total proceeds of €10 million. Alternatively, she can physically deliver her entire €10 million face amount of bonds to the counterparty in exchange for €10 million in cash.

Solution to 3:

Investor Y would prefer a cash settlement because he owns Bond B, which is worth more than the cheapest-to-deliver obligation. He will receive the same €7 million payout on his CDS contract, but can sell Bond B for €4 million, for total proceeds of €11 million. If he were to physically settle his contract, he would receive only €10 million, the face amount of his bond.

2.5 CDS Index Products

So far, we have mostly been focusing on single-name CDS. As noted, there are also index CDS products. A company called Markit has been instrumental in producing CDS indexes. Of course, a CDS index is not in itself a traded instrument any more than a stock index is a traded product. As with the major stock indexes, however, the industry has created traded instruments based on the Markit indexes. These instruments are CDS that generate a payoff based on any default that occurs on any entity covered by the index.

The Markit indexes are classified by region and further classified (or divided) by credit quality. The two most commonly traded regions are North America and Europe. North American indexes are identified by the symbol CDX, and European, Asian, and Australian indexes are identified as iTraxx. Within each geographic category

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are investment-grade and high-yield indexes. The former are identified as CDX IG and iTraxx Main, each comprising 125 entities. The latter are identified as CDX HY, consisting of 100 entities, and iTraxx Crossover, consisting of up to 50 high-yield entities.[10] Investment-grade index CDS are typically quoted in terms of spreads, whereas high-yield index CDS are quoted in terms of prices. Both types of products use standardized coupons. All CDS indexes are equally weighted. Thus, if there are 125 entities, the settlement on one entity is 1/125 of the notional.[11]

Markit updates the components of each index every six months by creating new series while retaining the old series. The latest created series is called the **on-the-run** series, whereas the older series are called **off-the-run** series. When an investor moves from one series to a new one, the move is called a **roll**. When an entity within an index defaults, that entity is removed from the index and settled as a single-name CDS based on its relative proportion in the index. The index then moves forward with a smaller notional.

Index CDS are typically used to take positions on the credit risk of the sectors covered by the indexes as well as to protect bond portfolios that consist of or are similar to the components of the indexes. Standardization is generally undertaken to increase trading volume, which is somewhat limited in the single-name market with so many highly diverse entities. With CDS indexes on standardized portfolios based on the credit risk of well-identified companies, market participants have responded by trading them in large volumes. Indeed, index CDS are typically more liquid than single-name CDS with average daily trading volume several times that of single-name CDS.

EXAMPLE 3

Hedging and Exposure Using Index CDS

Assume that an investor sells $500 million of protection on the CDX IG index. Concerned about the creditworthiness of a few of the components, the investor hedges a portion of the credit risk in each. For Company A, he purchases $3 million of single-name CDS protection, and Company A subsequently defaults.

1 What is the investor's net notional exposure to Company A?
2 What proportion of his exposure to Company A has he hedged?
3 What is the remaining notional on his index CDS trade?

Solution to 1:

The investor is long $4 million notional ($500 million/125) through the index CDS and is short $3 million notional through the single-name CDS. His net notional exposure is $1 million.

Solution to 2:

He has hedged 75% of his exposure ($3 million out of $4 million).

Solution to 3:

His index CDS has $496 million remaining notional.

10 Markit also creates other categories of CDS indexes, including emerging markets, sovereigns, municipals, high-yield/high-beta companies, and high-volatility companies.

11 Some confusion might arise from quoting certain CDS as prices and some as spreads, but keep in mind that the bond market often quotes bonds as prices and sometimes as yields. For example, a Treasury bond can be described as having a price of 120 or a yield of 2¾%. Both terms, combined with the other characteristics of the bond, imply the same concept.

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2.6 Market Characteristics

Credit default swaps trade in the over-the-counter market in a network of banks and other financial institutions. To better understand this market, we will first review how credit derivatives and specifically CDS were started.

As financial intermediaries, banks draw funds from savings-surplus sectors, primarily consumers, and channel them to savings-deficit sectors, primarily businesses. Corporate lending is indeed the core element of banking. When a bank makes a corporate loan, it assumes two primary risks. One is that the borrower will not repay principal and interest, and the other is that interest rates will change such that the return the bank is earning is not commensurate with returns on comparable instruments in the marketplace. The former is called **credit risk** or **default risk**, and the latter is called **interest rate risk**. There are many ways to manage interest rate risk.[12] Until around the mid-1990s, credit risk could be managed only by using traditional methods, such as analysis of the borrower, its industry, and the macroeconomy, as well as control methods, such as credit limits, monitoring, and collateral. These two groups of techniques defined what amounted only to internal credit risk management. In effect, the only defenses against credit risk were to not make a loan, to lend but require collateral (the value of which is also at risk), or to lend and closely monitor the borrower, hoping that any problems could be foreseen and dealt with before a default occurred.

Around 1995, credit derivatives were created to provide a new and potentially more effective method of managing credit risk.[13] They allow credit risk to be transferred from the lender to another party. In so doing, they facilitate the separation of interest rate risk from credit risk. Banks can then provide their most important service—lending—knowing that the credit risk can be transferred to another party if so desired. This ability to easily transfer credit risk allows banks to greatly expand their loan business. Given that lending is such a large and vital component of any economy, credit derivatives facilitate economic growth and have expanded to cover, and indeed are primarily focused on, the short-, intermediate-, and long-term bond markets. In fact, credit derivatives are more effective in the bond market, in which terms and conditions are far more standard, than in the bank loan market. Of the four types of credit derivatives, credit default swaps have clearly established themselves as the most widely used instrument. Indeed, in today's markets CDS are nearly the only credit derivative used to any great extent.

In principle, insurance contracts could be written that would allow the transfer of credit risk from one party to another. Credit insurance has existed for many years, but its growth has been constrained by the fact that insurance products are typically more consumer focused than commercially focused. Because it is such an important consumer product, insurance is very heavily regulated. It is very costly for insurance products to expand into new areas with different regulatory authorities. Thus, the ability of a relatively standard product to expand in similar form beyond its regulatory borders is limited. The CDS instrument arose and grew partly in response to this problem. By distinguishing CDS from insurance, the industry was able to effectively

12 These methods include duration-based strategies, gap management, and the use of interest rate derivatives.

13 There is some evidence that the first credit derivative was created by Blythe Masters, a managing director of J.P. Morgan, and was used to manage the potential risk of Exxon defaulting following its oil spill near Valdez, Alaska.

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offer a product that entailed a buyer making a series of promised payments in return for which it received a promise of compensation for losses, a product almost economically identical to insurance but legally distinct.[14]

CDS transactions are executed in the over-the-counter market by phone, instant message, or the Bloomberg message service. Trade information is reported to the **Depository Trust and Clearinghouse Corporation**, which is a US-headquartered entity providing post-trade clearing, settlement, and information services for many kinds of securities in addition to asset custody and asset servicing. New regulations require that almost all CDS be centrally cleared, meaning that parties will send their contracts through clearinghouses that collect and distribute payments and impose margin requirements, as well as mark positions to market. In so doing, a considerable amount of systemic risk is eliminated.

The Bank for International Settlements reported that as of June 2012, the gross notional amount of CDS was about $26.9 trillion with a market value of $1.2 trillion.[15] A rough estimate of the net notional, or promised payments if all possible defaults occur, is about 10% of the gross notional. Single-name CDS are about 60% of the credit derivatives market.

The size of the market today is considerably smaller than it was just a few years ago. For example, in December 2007 CDS gross notional was $57.9 trillion, about twice the size as in December 2011. The decline is accounted for by the fact that the use of CDS fell following the 2008 financial crisis. CDS had been widely used, and indeed overused and mismanaged, by many financial institutions that were ultimately bailed out by governments and central banks. Many of these institutions took credit risk exposures that they thought were diversified or controlled by complex models they had spent millions of dollars and many years developing. Notably, the financial crisis was largely brought about by a real estate crash and the widespread use of subprime mortgages. Credit risk proved to be globally systemic, a possibility not envisioned by risk managers of many well-known institutions, such as AIG. With so many of the large participants in the CDS market effectively out of business, bailed out or taken over, or having to pull back their lending substantially, the use of CDS declined greatly. Nonetheless, the CDS global market is extremely large and well worth our attention.

Until 2010, CDS were essentially unregulated over-the-counter financial instruments. Because of some of the problems discussed earlier, they are now under government regulations or securities and derivatives guidelines in virtually all countries. These regulations require that most CDS transactions be centrally reported and, as noted, most have to be cleared through an authorized clearinghouse.

BASICS OF VALUATION AND PRICING

3

Derivatives are typically valued by constructing a hedge between the derivative and the underlying that produces a risk-free position and merits a return of the risk-free rate. The price of the underlying and certain other variables jointly imply the price of

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14 Probably the most important step in the development of credit default swaps was not calling them "insurance," which would have almost surely triggered a different set of regulations. It is unclear why they are called *swaps*. As presented elsewhere in the curriculum on the subject of derivatives, swaps involve a series of bilateral payments in which parties exchange a series of cash flows. A CDS is clearly a variation of an option and is not at all a swap.

15 By comparison, interest rate swap notional at that same time was about $379 trillion. These figures are obtained from the Bank for International Settlements' semi-annual surveys of derivatives usage.

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the derivative that guarantees a risk-free return on the hedged position. In the context of CDS, pricing means determining the CDS spread or upfront payment given a particular coupon rate for a contract. In turn, this process implies the CDS price.[16]

This principle is fairly easy to apply for conventional derivatives but somewhat more difficult for credit derivatives. For conventional derivatives, the underlying is usually traded in active markets. For example, options on Royal Dutch Shell, futures on a German government bond, and swaps on the yen are relatively easy to value because the underlying instruments trade actively. But the underlying of a CDS is credit, which is a somewhat vague concept. Credit does not "trade" in the traditional sense but exists implicitly within the bond and loan market. The actual valuation of credit, which reveals the price at which credit risk can be sold, is much more difficult to obtain in relation to the valuation of derivatives driven by equities, interest rates, and currencies.

The exact application of these concepts in CDS pricing models is an advanced topic beyond the scope of the CFA Program. It is important, nonetheless, that CFA charterholders have a good grasp of the factors that determine CDS pricing, but the details are not necessary. Thus, we will cover this material at a high level.

3.1 Basic Pricing Concepts

The most important element of CDS pricing is the **probability of default**. With a few exceptions, a loan or bond involves a series of promised payments. Non-payment on any one of these obligations is a default. To illustrate, consider a simple example of a two-year, 5%, $1,000 loan, with one interest payment of $50 due in one year and a final interest and principal payment of $1,050 due in two years. Each of these payments is subject to the possibility of default.

It can be a bit confusing to refer to a general probability of default. There might be a 2% chance of defaulting on the first interest payment but a greater probability of default on the final interest and principal payment because the amount owed is larger and there is a longer period of time until the second payment. The probability of default is normally greater over a longer period of time.[17]

The relevant probability of default is referred to as a concept from statistics called the **hazard rate**. The hazard rate is the probability that an event will occur *given that it has not already occurred*. Once the event occurs, there is no further likelihood of its occurrence. A hazard rate can also be viewed as a conditional probability. It is the probability that something will occur, with the condition that it has not already occurred.

In the life insurance industry, the probability of death clearly meets the concept of a hazard rate. One cannot die if one has already died. Analogously, in the credit industry, default is treated this way.[18] In our example, let the hazard rates be 2% for the first interest payment and 4% for the final interest and principal payment. The 4% rate is the probability that default occurs in Year 2, given that it has not occurred in Year 1. We will assume a 40% recovery rate, which is a common assumption for

16 Recall that we have sometimes distinguished between valuation and pricing for forward, futures, and swaps but not options. Although credit default swaps may be called "swaps," they are really options. Valuation and pricing are, thus, the same concept.

17 The probability of default is typically greater over a longer period of time, because there is more time for the borrower's financial condition to worsen. But there are some exceptions. A borrower could be struggling financially in the short run but might have better prospects in the long run.

18 Technically, a company can default more than once. It can declare bankruptcy, reorganize, continue to operate and even emerge from bankruptcy only to default again, perhaps years later. For example, there are many instances of this occurring in the US airline and auto industries. Credit risk modeling typically does not consider such possibilities because they are fairly uncommon. For our purposes, a CDS terminates with the first credit event, so this event is the principal focus.

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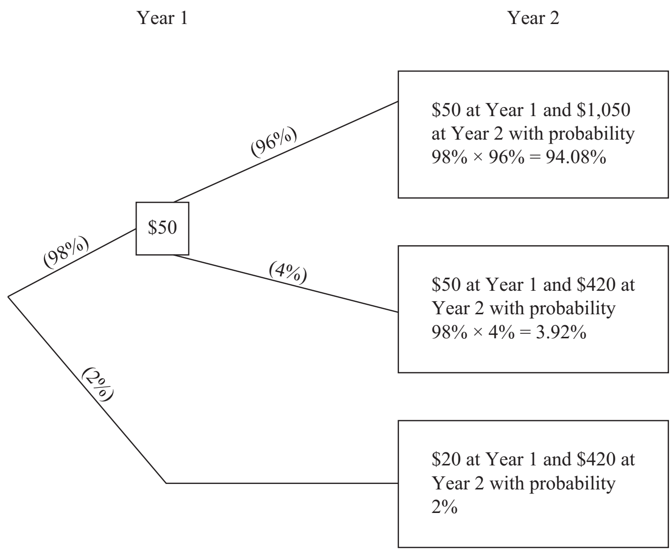
senior unsecured debt. Thus, if default occurs on the \$50 payment, the bondholder will receive \$20 (\$50 × 40%), and if default occurs on the final \$1,050 payment, the bondholder receives \$420 (\$1,050 × 40%). Exhibit 2 shows the possibilities. Note that there are three outcomes: the bondholder receives (1) \$50 at Year 1 and \$1,050 at Year 2 with a probability of 98% × 96% = 94.08%, (2) \$50 at Year 1 and \$420 at Year 2 with probability 98% × 4% = 3.92%, and (3) \$20 at Year 1 and \$420 at Year 2 with probability 2%.[19] These probabilities add up to 100%.

Exhibit 2 Default Possibilities on a Two-Year \$1,000 Loan with Annual Payments at 5% Interest



Now, suppose we ask the question, "what is the probability of default?" There are several possible answers because there are really several questions. The probability of default is 2% on the first payment but 4% on the second. In a more general sense, we might like to know the probability of *any* default occurring or, in a complementary sense, the **probability of survival**. In this problem, the probability of survival is 0.98 multiplied by 0.96, approximately 94.08%. Thus, the probability of default occurring at some time in the life of the loan is 100% − 94.08% = 5.92%.

An important concept in credit analysis is the **loss given default**, which is the amount that will be lost if a default occurs. In the example, that amount cannot be precisely specified because it must refer to a particular default. If the borrower defaults on the first payment, the amount lost is \$50 − \$20 = \$30 on the first payment and \$1,050 − \$420 = \$630 on the second, for a total loss given default of \$660. If the borrower defaults only on the second payment, the loss given default is \$630. From the loss given default, it is possible to calculate the **expected loss**, which is simply the full amount owed minus the expected recovery, or the loss given default, multiplied by the probability of default:

Expected loss = Loss given default × Probability of default

19 Although we say "at Year 1" and "at Year 2," we do not really know when during a year recovery will occur. In the exhibit, we simply assume that the cash flow occurs "at Year 1 (or 2)," but it could occur earlier in the year. Also, for the third outcome, we assume that if default occurs on the first payment, it will also occur on the second but that recovery on the second will be the same as if the first were made in full. This might not be the case in practice, but other estimates can be easily inserted.

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In the example, there is a 2% chance of losing \$660 and a (0.98) × (0.04) = 0.0392, or 3.92%, chance of losing \$630. Thus, unadjusted for time value of money, the expected loss is (0.02) × (\$660) + (0.0392) × (\$630) = \$37.90. This calculation shows that the expected loss is obtained by multiplying the losses given defaults (\$660 and \$630, respectively) by the probabilities of default (2% and 3.92%, respectively).

Now consider another possibility, a 10-year bond with an equivalent hazard rate of 2% each year.[20] Suppose we want to know the probability that the borrower will not default during the entire 10-year period. Of course, if we try to draw a 10-year tree diagram, as in Exhibit 2, it will become very cluttered, but we can still easily answer this question. The probability that a default will occur at some point during the 10 years is one minus the probability of no default in 10 years. The probability of no default in 10 years is (0.98) × (0.98) … (0.98) = $(0.98)^{10}$ = 0.817. Thus, the probability of default is 1 – 0.817 = 0.183, or 18.3%. This somewhat simplified example illustrates how a low probability of default in any one period can turn into a surprisingly high probability of default over a longer period of time.

EXAMPLE 4

Hazard Rate and Probability of Survival

Assume that a company's hazard rate is a constant 8% per year, or 2% per quarter. An investor sells five-year CDS protection on the company with the premiums paid quarterly over the next five years.

1 What is the probability of survival for the first quarter?

2 What is the conditional probability of survival for the second quarter?

3 What is the probability of survival through the second quarter?

Solution to 1:

The probability of survival for the first quarter is 98% (100% minus the 2% hazard rate).

Solution to 2:

The conditional probability of survival for the second quarter is also 98%, because the hazard rate is constant at 2%. In other words, *conditional on the company having survived the first quarter*, there is a 2% probability of default in the second quarter.

Solution to 3:

The probability of survival through the second quarter is 96.04%. The probability of survival through the first quarter is 98%, and the conditional probability of survival through the second quarter is also 98%. The probability of survival through the second quarter is thus 98% × 98% = 96.04%. Alternatively, 1 – 96.04% = 3.96% is the probability of default sometime during the first two quarters.

Understanding the concept of pricing a CDS is facilitated by recognizing that there are essentially two sides, or legs, of a contract. There is the **protection leg**, which is the contingent payment that the credit protection seller may have to make to the credit protection buyer, and the **premium leg**, which is the series of payments the credit protection buyer promises to make to the credit protection seller.

20 The hazard rate is unlikely to be the same each year, but we will use a simple case here to minimize the computations.

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To estimate the value of the protection leg, the probability of each payment, the timing of each payment, and the discount rate must be taken into account.[21] In essence, we need to determine the expected payoff of each promised payment on the reference entity. Having estimated the probability of default for each payment, we find the expected payoff of a given payment on the reference entity by multiplying the payment adjusted for the expected recovery rate by the probability of survival and then discounted at an appropriate rate. The sum of all of these amounts is the expected payoff of the bond or loan, which should, of course, be the price at which the bond is trading in the market. Then, suppose we assume there is no default possibility on the bond. We could then discount all payments at the risk-free rate to obtain the hypothetical value of the bond if it had no credit risk. The difference between these two figures is the value of the credit exposure. In other words, what an investor would pay for the bond, which contains credit risk, minus what the investor would pay if the bond had no credit risk is what it would cost to eliminate the credit risk. This amount is, therefore, the value of the protection leg and is the present value of the contingent obligation of the credit protection seller to the credit protection buyer. Although we could obtain the value of the bond and implicitly the credit premium from the bond's price in the market, we would have to trust that the bond market is properly pricing the credit risk. That may not be the case, as we will discuss later.

Now, we must evaluate the premium leg or present value of the payments made by the protection buyer to the protection seller. With a fixed standardized coupon rate, this calculation would seem simple, but one complication must be considered. For example, for a five-year CDS, the credit protection buyer promises a set of payments over five years, but if the credit event occurs any time during that five-year period, the payments terminate. Hence, the various hazard rates must also be applied to the premium leg to obtain the expected payments promised by the CDS buyer to the seller.

The difference in value of the protection leg and premium leg determines the upfront payment. The party having a claim on the greater present value must make up the cash difference at the initiation date of the contract. Thus, we have

Upfront payment = Present value of protection leg – Present value of premium leg

and if the result is greater (less) than zero, the protection buyer (seller) pays the protection seller (buyer). The actual mechanics of these calculations are somewhat more complex than described here. As noted, for the CFA Program, we take a high-level view of credit default swaps and leave the details to credit derivatives specialists.

3.2 The Credit Curve

The credit spread of a debt instrument is the rate in excess of Libor that investors expect to receive to justify holding the instrument.[22] The credit spread can be expressed roughly as the probability of default multiplied by the loss given default, with the latter in terms of a percentage.[23] The credit spreads for a range of maturities of a

21 There is a technical distinction between the true probability of default and the risk-neutral probability of default. Pricing is done using the risk-neutral probability of default, not the true probability of default. Risk-neutral probability is covered in the Level I readings on derivatives. In this reading, we will not make the distinction explicitly but it should be kept in mind.

22 Libor is not a risk-free rate and contains some credit risk itself. Libor is the rate on loans made from one London bank to another. Given that London banks bear some default risk, Libor is typically higher than the rate on government debt.

23 We previously showed that the expected loss is also the loss given default times the probability of default expressed in currency units. When expressed as a percentage of notional, this relationship is the credit spread. These are all rough approximations because the true relationships are complicated by multiple payments and discounting.

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company's debt make up its **credit curve**. The credit curve is somewhat analogous to the term structure of interest rates, which is the set of rates on default-free debt over a range of maturities, but the credit curve applies to non-government borrowers and incorporates credit risk into each rate.

The CDS market for a given borrower is integrated with the credit curve of that borrower. In fact, given the evolution and high degree of efficiency of the CDS market, the credit curve is essentially determined by the CDS rates. The curve is affected by a number of factors, a key one of which is the set of aforementioned hazard rates. A constant hazard rate will tend to flatten the credit curve.[24] Upward-sloping credit curves imply a greater likelihood of default in later years, whereas downward-sloping credit curves imply a greater probability of default in the earlier years. Downward-sloping curves are less common and often a result of severe near-term stress in the financial markets.

EXAMPLE 5

Change in Credit Curve

A company's 5-year CDS trades at a credit spread of 300 bps, and its 10-year CDS trades at a credit spread of 500 bps.

1 The company's 5-year spread is unchanged, but the 10-year spread widens by 100 bps. Describe the implication of this change in the credit curve.
2 The company's 10-year spread is unchanged, but the 5-year spread widens by 500 bps. Describe the implication of this change in the credit curve.

Solution to 1:

This change implies that although the company is not any riskier in the short term, its longer-term creditworthiness is less attractive. Perhaps the company has adequate liquidity for the time being, but after five years it must begin repaying debt or it will be expected to have cash flow difficulties.

Solution to 2:

This change implies that the company's near-term credit risk is now much greater. In fact, the probability of default will decrease if the company can survive for the next five years. Perhaps the company has run into liquidity issues that must be resolved soon, and if not resolved, the company will default.

3.3 CDS Pricing Conventions

With corporate bonds, we typically refer to their values in terms of prices or spreads. The spread is a somewhat more informative measure than price. People are relatively familiar with a normal range of interest rates, so spreads can be easily compared with interest rates. It is more difficult to compare prices. A high-yield bond can be offered with a coupon equal to its yield and, therefore, a price of par value. At the same time, a low-yield bond with the same maturity can likewise be offered with a coupon equal to its yield, and therefore, its price is at par. These two bonds would have identical prices at the offering date, and their prices might even be close through much of their

24 Because of discounting, the credit curve would not be completely flat even if the hazard rates are constant. For example, for a company issuing 5- and 10-year zero-coupon bonds, there could be equally likely probabilities of default and hence equal expected payoffs. But the present values of the payoffs are not the same and hence the discount rates that equate the present value to the expected payoffs will not be the same. Constant hazard rates tend to flatten a curve but would not flatten it completely unless all rates were zero.

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lives, but they are quite different bonds. Focusing on their prices would, therefore, provide little information. Their spreads are much more informative. With Libor or the risk-free rate as a benchmark, investors can get a sense for the amount of credit risk implied by their prices, maturities, and coupons. The same is true for CDS. Although CDS have their own prices, their spreads are far more informative.

As we briefly described earlier, the convention in the CDS market is for standardized coupons of 1% for investment-grade debt or 5% for high-yield debt. Clearly, the reference entity need not have debt that implies a credit spread of either of these rates. As such, the present value of the promised payments from the credit protection buyer to the credit protection seller can either exceed or be less than the expected payoff. In effect, the payments are either too large or too small for the risk. The present value difference is the upfront premium paid from one party to the other. Hence, the upfront premium is the present value of the credit spread minus the present value of the fixed coupon. Of course, this specification is quite general. A good rough approximation used by the industry is that the upfront premium is the (Credit spread – Fixed coupon) × Duration of the CDS.[25] Moreover, this specification is in terms of rates. The upfront premium must ultimately be converted to a price, which is done by subtracting the percentage premium from 100.

These relationships are summarized as follows:

Present value of credit spread = Upfront premium + Present value of fixed coupon

A good approximation of the present value of a stream of payments can be made by multiplying the payment rate by the duration:

Upfront premium ≈ (Credit spread – Fixed coupon) × Duration

Credit spread ≈ (Upfront premium/Duration) + Fixed coupon

Price of CDS in currency per 100 par = 100 – Upfront premium %

Upfront premium % = 100 – Price of CDS in currency per 100 par

EXAMPLE 6

Premiums and Credit Spreads

1 Assume a high-yield company's 10-year credit spread is 600 bps, and the duration of the CDS is eight years. What is the approximate upfront premium required to buy 10-year CDS protection? Assume high-yield companies have 5% coupons on their CDS.

2 Imagine an investor sold five-year protection on an investment-grade company and had to pay a 2% upfront premium to the buyer of protection. Assume the duration of the CDS to be four years. What are the company's credit spreads and the price of the CDS per 100 par?

25 Recall that duration is a type of cash flow weighted-average maturity for a bond. For a CDS, if default occurs, the payments terminate. Thus, we cannot assume that all payments are made with certainty, and the duration must take this possibility into account for every payment. Normally, one should adjust the duration of a bond for credit losses, but it is not usually done unless the bond pricing model used takes into account the stochastic nature of the credit spread.

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Solution to 1:

To buy 10-year CDS protection, an investor would have to pay a 500 bps coupon plus the present value of the difference between that coupon and the current market spread (600 bps). In this case, the upfront premium would be approximately 100 bps × 8 (duration), or 8% of the notional.

Solution to 2:

The value of the upfront premium is equal to the premium (–2%) divided by the duration (4), or –50 bps. The sign of the upfront premium is negative because the seller is paying the premium rather than receiving it. The credit spread is equal to the fixed coupon (100 bps) plus the running value of the upfront premium (–50 bps), or 50 bps. As a reminder, because the company's credit spread is less than the fixed coupon, the protection seller must pay the upfront premium to the protection buyer. The price in currency would be 100 minus the upfront premium, but the latter is negative, so the price is 100 – (–2) = 102.

3.4 Valuation Changes in CDS during Their Lives

As with any traded financial instrument, a CDS has a value that fluctuates during its lifetime. That value is determined in the competitive marketplace. Market participants constantly assess the current credit quality of the reference entity to determine its current value and (implied) credit spread. Clearly, many factors can change over the life of the CDS. By definition, the duration shortens through time. Likewise, the probability of default, the expected loss given default, and the shape of the credit curve will all change as new information is received. The exact valuation procedure of the CDS is precisely the same as it is when the CDS is first issued and simply incorporates the new inputs. The new market value of the CDS reflects gains and losses to the two parties.

Consider the following example of a five-year CDS with a fixed 1% coupon. The credit spread on the reference entity is 2.5%. In promising to pay 1% coupons to receive coverage on a company whose risk justifies 2.5% coupons, the present value of the protection leg exceeds the present value of the payment leg. The difference is the upfront premium, which will be paid by the CDS buyer to the CDS seller. During the life of the CDS, assume that the credit quality of the reference entity improves, such that the credit spread is now 2.1%. Now, consider a newly created CDS with the same remaining maturity and 1% coupon. The present value of the payment leg would still be less than the present value of the protection leg, but the difference would be less than it was when the original CDS was created because the risk is now less. Logically, it should be apparent that for the original CDS, the seller has gained and the buyer has lost. The difference between the original upfront premium and the new value is the seller's gain and buyer's loss. A rough approximation of the change in value of the CDS for a given change in spread is as follows:[26]

Profit for the buyer of protection ≈ Change in spread in bps × Duration × Notional

Alternatively, we might be interested in the CDS percentage price change, which is obtained as

% Change in CDS price = Change in spread in bps × Duration

26 The relationships expressed in the two equations should be somewhat known to candidates from the fixed-income readings, which illustrate that the percentage change in the price of a bond is approximately the change in yield multiplied by the modified duration. In this case, the change in yield is analogous to the change in spread, measured in basis points. The duration of the CDS is analogous to the duration of the bond on which the CDS is written. The use of the term "modified" with respect to duration is a small adjustment requiring division by one plus the yield.

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EXAMPLE 7

Profit and Loss from Change in Credit Spread

An investor buys \$10 million of five-year CDS protection, and the CDS contract has a duration of four years. The company's credit spread was originally 500 bps and widens to 800 bps.

1. Does the investor (credit protection buyer) benefit or lose from the change in credit spread?
2. Estimate the CDS price change and estimated profit to the investor.

Solution to 1:

The investor owns protection, so he is economically short and benefits from an increase in the company's credit spread. He can sell the protection for a higher premium.

Solution to 2:

The percentage price change is estimated as the change in spread (300 bps) multiplied by the duration (4) or 12%. The profit to the investor is 12% times the notional (\$10 million), or \$1.2 million.

3.5 Monetizing Gains and Losses

As with any financial instrument, changes in the price of a CDS gives rise to opportunities to unwind the position, and either capture a gain or realize a loss. This process is called **monetizing** a gain or loss. Keep in mind that the protection seller is effectively long the reference entity. He has entered into a contract to insure the debt of the reference entity, for which he receives a series of promised payments and possibly an upfront premium. He clearly benefits if the reference entity's credit quality improves because he continues to receive the same compensation but bears less risk. Using the opposite argument, the credit protection buyer benefits from a deterioration of the reference entity's credit quality.[27] Thus, the seller is more or less long the company and the buyer is more or less short the company. As the company's credit quality changes through time, the market value of the CDS changes, giving rise to gains and losses for the CDS counterparties. The counterparties can realize those gains and losses by entering into new offsetting contracts, effectively selling their CDS positions to other parties.

Going back to the example in the previous section, assume that during the life of the CDS, the credit quality of the reference entity improves. The implied upfront premium on a new CDS that matches the terms of the original CDS with adjusted maturity is now the market value of the original CDS. In our example, this new CDS has an upfront premium that would be paid by the buyer to the seller, but that premium is smaller than on the original CDS.

Now, suppose that the buyer of the original CDS wants to unwind his position. He would then enter into this new CDS as a protection seller and receive the newly calculated upfront premium. As we noted, this value is less than what he paid originally. Likewise, the seller could offset his original position by entering into this new CDS as a protection buyer. He would pay an upfront premium that is less than what he originally received. The original protection buyer monetizes a loss and the seller

27 Again, it is important to remember that these statements are limited to the buyer or seller's position in the CDS and not any other instruments held by either party.

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monetizes a gain. The transaction to unwind the CDS does not need to be done with the same original party, although doing so offers some advantages. As clearinghouses begin to be more widely used with CDS, unwind transactions should become even more common and easier to do.

At this point, we have identified two ways of realizing a profit or loss on a CDS. One is to effectively exercise the CDS in response to a default. The other is to unwind the position by entering into a new offsetting CDS in the market. A third, and the least common, method occurs if there is no default. A party can simply hold the position until expiration, at which time the credit protection seller has captured all of the premiums and has not been forced to make any payments, and the seller's obligation for any further payments is terminated. The spread of the CDS will go to zero, in much the same manner as a bond converges toward par as it approaches maturity. The CDS seller clearly gains, having been paid to bear the risk of default that is becoming increasingly unlikely, and the CDS buyer loses.[28]

4 APPLICATIONS OF CDS

Credit default swaps, as demonstrated, facilitate the transfer of credit risk. As simple as that concept seems, there are many different circumstances under which CDS are used. In this section, we consider some applications of this instrument.

Any derivative instrument has two general uses. One is to exploit an expected movement in the underlying. The derivative typically requires less capital and is usually an easier instrument in which to create a short economic exposure as compared with the underlying. The derivatives market can also be more efficient, meaning that it can react to information more rapidly and have more liquidity than the market for the underlying. Thus, information or an expectation of movement in the underlying can often be exploited much better with the derivative than with the underlying directly.

The other trading opportunity facilitated by derivatives is in valuation differences between the derivative and the underlying. If the derivative is mispriced relative to the underlying, one can take the appropriate position in the derivative and an offsetting position in the underlying. If the valuation assessment is correct and other investors come to the same conclusion, the values of the derivative and underlying will converge, and the investor will earn a return that is essentially free of risk because the risk of the underlying has been hedged away by the holding of long and short positions. Whether this happens as planned depends on both the efficiency of the market and the quality of the valuation model. Differences can also exist between the derivative and other derivatives on the same underlying.

These two general types of uses are also the major applications of CDS. We will refer to them as managing credit exposures, meaning the taking on or shedding of credit risk in light of changing expectations and/or valuation disparities. With valuation disparities, the focus is on differences in the pricing of credit risk in the CDS market relative to that of the underlying bonds.

28 Indeed, the buyer loses on the CDS because it paid premiums to receive protection in the event of a default, which did not occur. Although technically a loss, the buyer might well be a creditor of the reference entity, so the buyer's overall position is not a loss. The CDS is, as we have mentioned, somewhat like insurance, so the buyer may not look at it as a loss in the same manner that an individual might not look at an expiring insurance contract on his house as a loss simply because it did not burn down.

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4.1 Managing Credit Exposures

The most basic application of a CDS is to increase or decrease credit exposure. The most obvious such application is for a lender to buy a CDS to reduce its credit exposure to a borrower. For the CDS seller, the trade adds credit exposure. A lender's justification for using a CDS seems obvious. The lender may have assumed too much credit risk but does not want to sell the bond or loan because there can be significant transaction costs, because later it may want the bond or loan back, or because the market for the bond or loan is relatively illiquid. If the risk is temporary, it is almost always easier to temporarily reduce risk by using a CDS. Beyond financial institutions, any organization exposed to credit is potentially a candidate for using CDS.

The justification for selling credit protection is somewhat less obvious. The seller can be a CDS dealer, whose objective is to profit from making markets in CDS. A dealer typically attempts to manage its exposure by either diversifying its credit risks or hedging the risk by entering into a transaction with yet another party, such as by shorting the debt or equity of the reference entity, often accompanied by investment of the funds in a repurchase agreement, or repo. If the dealer manages the risk effectively, the risk assumed in selling the CDS is essentially offset when the payment for assuming the risk exceeds the cost of removing the risk. Achieving this outcome successfully requires sophisticated credit risk modeling, a topic beyond the scope of the CFA Program.

Although dealers make up a large percentage of CDS sellers, not all are dealers. Consider that any bondholder is a buyer of credit and interest rate risk. If the bondholder wants only credit risk, it can obtain it by selling a CDS, which would require far less capital and incur potentially lower overall transaction costs than buying the bond. Moreover, the CDS can easily be more liquid than the bond, so the position can be unwound much more easily.

As noted, it is apparent why a party making a loan might want credit protection. Consider, however, that a party with no exposure to the reference entity might also purchase credit protection. Such a position is called a **naked credit default swap**, and it has resulted in some controversy in regulatory and political circles. In buying a naked CDS, the investor is taking a position that the entity's credit quality will deteriorate, whereas the seller of a naked CDS is taking the position that the entity's credit quality will improve.[29] It is the position of the buyer that has caused some controversy. Some regulators and politicians believe it is inappropriate for a party with no exposure to a borrower to speculate that the borrower's financial condition will deteriorate. This controversy accelerated during the financial crises of 2008–2009 because many investors held these naked CDS and benefited from the crisis.

The counterargument, however, is that elsewhere in the financial markets, such bets are made all of the time in the form of long puts, short futures, and short sales of stocks and bonds. These instruments are generally accepted as a means of protecting oneself against weak if not bad performance in the financial markets. Likewise, a CDS is a means of protecting oneself against terrible economic conditions. Must everyone suffer during a financial crisis? Are there not ways to trade that would reward investors who go against the majority of investors and ultimately are proven correct? Moreover, not having a position in an entity does not mean one does not have exposure. In particular, the default of a sovereign entity or municipality imposes

29 To be clear, a naked CDS does not mean that *both* parties have no exposure to the underlying. Either or both could have no exposure. A naked CDS simply refers to the position of one party. The counterparty may or may not have exposure.

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costs on many citizens and organizations.[30] Other proponents of naked CDS argue that they bring liquidity to the credit market, potentially providing more stability, not less. Nonetheless, naked CDS trading is banned in Europe for sovereign debt, although generally permitted otherwise.

CDS trading strategies, with or without naked exposure, can take several forms. A party can take an outright long or short position, as we have previously discussed. Alternatively, the party can take a long position in one CDS and a short position in another, called a **long/short trade**.[31] One CDS would be on one reference entity, and the other would be on a different entity. This transaction is a bet that the credit position of one entity will improve relative to that of another. The two entities might be related in some way or might produce substitute goods. For example, one might take a position that because of competition and changes in the luxury car industry, the credit quality of Daimler will improve and that of BMW will weaken, so going long a Daimler CDS and shorting a BMW CDS would be appropriate. Similarly, an investor may undertake a long/short trade based on other factors, such as environmental, social, and governance (ESG) considerations. For instance, an investor may be concerned about a company's poor ESG-related practices and policies relative to another company. In this case, the investor could short the CDS of a company with weak ESG practices and policies and go long the CDS of a company with strong ESG practices and policies. Example 8 provides a case study of ESG considerations in a long/short ESG trade.

EXAMPLE 8

Long/Short Trade with ESG Considerations

Overview

An analyst is evaluating two US apparel companies: Atelier and Traxx. Atelier is a large company that focuses on high-end apparel brands. It is profitable despite a high cost structure. Traxx is smaller and less profitable than Atelier. Traxx focuses on less expensive brands and strives to keep costs low. Both companies purchase their merchandise from suppliers all over the world. The analyst recognizes that apparel companies must maintain adequate oversight over their suppliers to control the risks of reputational damage and inventory disruptions. Supplier issues are particularly relevant for Atelier and Traxx following a recent fire that occurred at the factory of Global Textiles, a major supplier to both companies. The fire resulted in multiple casualties and unfavorable news headlines.

The analyst notices a significant difference in the way Atelier and Traxx approach ESG considerations. After the fire at its supplier, Atelier signed an "Accord on Fire and Building Safety," which is a legally binding agreement between global apparel manufacturers, retailers, and trade unions in the country where the fire occurred. After signing the accord, Atelier made a concerted effort to fix and enhance machinery in factories of its suppliers. Its objective was to improve workplace safety—notably, to reduce lost employee time due to factory incidents and the rate of factory accidents and fatalities.

30 Another apparent naked exposure to the reference entity arises from simply having large commercial deposits at a bank, either traditional deposits or collateral for another transaction. If the bank defaults, the funds could be at risk. Technically, this is not naked exposure, but it does not take the form of a traditional loan or bond.

31 In the world of options and futures trading, such a transaction is typically called a spread.

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Investors view Atelier's corporate governance system favorably because management interests and stakeholder interests are strongly aligned. Atelier's board of directors includes a high percentage of independent directors and is notably diverse. In contrast, Traxx's founder is the majority owner of the company and serves as CEO and chairman of the board of directors. Furthermore, Traxx's board is composed mainly of individuals who have minimal industry expertise. As a consequence, Traxx's board was unprepared to adequately respond to the Global Textiles fire. Given the lack of independence and expertise of Traxx's board, investors consider Traxx's corporate governance system to be poor. Because of its emphasis on low costs and reflecting its less experienced board, Traxx chose not to sign the Accord.

Implications for CDS

Single-name CDS on both Atelier and Traxx are actively traded in the market, although Traxx's CDS is less liquid. Before the Global Textiles fire, five-year CDS for Traxx traded at a spread of 250 bps, compared to a spread of 150 bps for the five-year CDS for Atelier. The difference in spreads reflects Traxx's lower trading liquidity, perceived lower creditworthiness (primarily reflecting its smaller size and lower profitability), and hence higher default risk relative to Atelier.

After the Global Textiles fire, spreads on the CDS for all companies in the apparel sector widened considerably. Credit spreads for the five-year CDS on Atelier widened by 60 bps (to 210 bps) and credit spreads for the five-year CDS on Traxx widened by 75 bps (to 325 bps). The analyst believes that over the longer term, the implications of the fire at Global Textiles will be even more adverse for Traxx relative to Atelier. The analyst's view largely reflects Traxx's higher ESG-related risks, especially the perceived weaker safety in its factories and its weaker corporate governance system. In particular, the analyst believes that spreads of Traxx's CDS will remain wider than its pre-fire level of 250 bps, but Atelier's CDS spreads will return to their pre-fire level of 150 bps.

Describe how the analyst can use CDS to exploit the potential opportunity.

Solution

The analyst can try to exploit the potential opportunity by buying protection (shorting) on Traxx five-year CDS and selling protection (going long) on Atelier five-year CDS. This trade would reflect both the anticipated continuing adverse spreads for Traxx relative to the pre-fire level and the return of spreads for Atelier to its lower pre-fire levels. For example, assume Atelier's five-year CDS spread returns to 150 bps from 210 bps, but Traxx's five-year CDS spread narrows to just 300 bps from 325 bps. The difference in spreads between the two companies' CDS would have widened from 115 bps (325 bps − 210 bps) right after the factory fire occurred to 150 bps (300 bps − 150 bps). This 35-bps difference in spread would represent profit (excluding trading costs) to the analyst from the long/short trade.

Similar to a long/short trade involving individual entities (companies), an investor could take a long position in one CDS index and a short position in another. For example, the anticipation of a weakening economy could make one go short a high-yield CDS index and long an investment-grade CDS index. As another example, the expectation of strengthening in the Asian economy relative to the European economy could induce one to go short a European CDS index and long an Asian CDS index.[32]

32 As a reminder, the CDS seller is long credit and the buyer is short credit. Improvements in credit quality benefit (hurt) the CDS seller (buyer).

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Another type of long/short trade, called a **curve trade**, involves buying a CDS of one maturity and selling a CDS on the same reference entity with a different maturity. Consider two CDS maturities, which we will call the short term and the long term to keep things simple. We will assume the more common situation of an upward-sloping credit curve, meaning that long-term CDS rates are higher than short-term rates. If the curve changes shape, it becomes either steeper or flatter. A steeper (flatter) curve means that long-term credit risk increases (decreases) relative to short-term credit risk.[33] An investor who believes that long-term credit risk will increase relative to short-term credit risk (credit curve steepening) can go short a long-term CDS and long a short-term CDS. In the short run, a curve-steepening trade is bullish. It implies that the short-term outlook for the reference entity is better than the long-term outlook. In the short run, a curve-flattening trade is bearish. It implies that the short-run outlook for the reference entity looks worse than the long-run outlook and reflects the expectation of near-term problems for the reference entity.

EXAMPLE 9

Curve Trading

An investor owns some intermediate-term bonds issued by a company and has become concerned about the risk of a near-term default, although he is not very concerned about a default in the long term. The company's two-year duration CDS currently trades at 350 bps, and the four-year duration CDS is at 600 bps.

1. Describe a potential curve trade that the investor could use to hedge the default risk.
2. Explain why an investor may prefer to use a curve trade as a hedge against the company's default risk rather than a straight short position in one CDS.

Solution to 1:

The investor anticipates a flattening curve and can exploit this possibility by positioning himself short (buying protection) in the two-year CDS while going long in the four-year CDS (selling protection).

Solution to 2:

Going short one CDS and long another reduces some of the risk because both positions will react similarly, although not equally, to information about the reference entity's default risk. Moreover, the cost of one position will be partially or more than wholly offset by the premium on the other.

Of course, there can be changes to the credit curve that take the form of simply shifts in the general level of the curve, whereby all rates go up or down by roughly equal amounts. As with long-duration bonds relative to short-duration bonds, the values of longer-term CDS will be more sensitive than those of shorter-term CDS. As an example, a trader who believes that all rates will go up will want to be short CDS but will realize that long-term CDS will move more than short-term CDS. Thus, he might want to be short in long-term CDS and hedge by going long in short-term

33 The considerably less common starting scenario of a downward-sloping credit curve has the opposite interpretation. A steeper curve means that short-term credit risk increases relative to long-term credit risk. Even less common is that of a flat credit curve, in which case a steeper curve can occur either from an increase or decrease in long-term credit risk relative to short-term credit risk.

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CDS. He will balance the sizes of the positions so that the volatility of the position he believes will gain in value will be more than the other position. If more risk is desired, he might choose to trade only one leg, the more volatile one.

4.2 Valuation Differences and Basis Trading

Different investors will have different assessments of the price of credit risk. Such differences of opinion will lead to valuation disparities. Clearly, there can be only one appropriate price at which credit risk can be eliminated, but that price is not easy to determine. The party that has the best estimate of the appropriate price of credit risk can capitalize on its knowledge or ability at the expense of another party. Any such comparative advantage can be captured by trading the CDS against either the reference entity's debt or equity or derivatives on its debt or equity, but such trading is critically dependent on the accuracy of models that isolate the credit risk component of the debt or equity return. As noted, those models are beyond the scope of the CFA Program, but it is important to understand the basic ideas.

The yield on the bond issued by the reference entity to a CDS contains a factor that reflects the credit risk. In principle, the amount of yield attributable to credit risk on the bond should be the same as the credit spread on a CDS. It is, after all, the compensation paid to the party assuming the credit risk, regardless of whether that risk is borne by a bondholder or a CDS seller. But there may be a difference in the credit risk compensation in the bond market and CDS market. This differential pricing can arise from mere differences of opinions, differences in models used by participants in the two markets, differences in liquidity in the two markets, and supply and demand conditions in the repo market, which is a primary source of financing for bond purchases. A difference in the credit spreads in these two markets is the foundation of a strategy known as a **basis trade**.

The general idea behind most basis trades is that any such mispricing is likely to be temporary and the spreads should return to equivalence when the market recognizes the disparity. For example, suppose the bond market implies a 5% credit risk premium whereas the CDS market implies a 4% credit risk premium. The trader does not know which is correct but believes these two rates will eventually converge. From the perspective of the CDS, its premium is too low relative to the bond credit risk premium. From the perspective of the bond, its premium is too high relative to the CDS market, which means its price is too low. So, the CDS market could be pricing in too little credit risk, and/or the bond market could be pricing in too much credit risk. Either market could be correct, but it does not matter. The investor would buy the CDS, thereby purchasing credit protection at what appears to be an unjustifiably low rate, and buy the bond, thereby assuming credit risk and paying an unjustifiably low price for the bond. The risk is balanced because the default potential on the bond is protected by the CDS.[34] If convergence occurs, the trade would capture the 1% differential in the two markets.

To determine the profit potential of such a trade, it is necessary to decompose the bond yield into the risk-free rate plus the funding spread plus the credit spread.[35] The risk-free rate plus the funding spread is essentially Libor. The credit spread is then

34 The bondholder does bear interest rate risk on the bond, but this risk can be hedged with a duration strategy or interest rate derivatives. The general idea is to eliminate all risks and capitalize on the disparity between the price of credit risk in the bond and CDS markets.

35 In practice, this decomposition can be complicated by the existence of embedded options, such as with callable and convertible bonds or when the bond is not selling near par. Those factors would need to be removed in the calculations.

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the excess of the yield over Libor and can be compared with the credit spread in the CDS market. If the spread is higher in the bond (CDS) market than the CDS (bond) market, it is said to be a negative (positive) basis.

EXAMPLE 10

Bonds vs. Credit Default Swaps

An investor wants to be long the credit risk of a given company. The company's bond currently yields 6% and matures in five years. A comparable five-year CDS contract has a credit spread of 3.25%. The investor can borrow in the market at a 2.5% interest rate.

1 Calculate the bond's credit spread.
2 Identify a basis trade that would exploit the current situation.

Solution to 1:

The bond's credit spread is equal to the yield (6%) minus the investor's cost of funding (2.5%). Therefore, the bond's credit spread is currently 3.5%.

Solution to 2:

The bond and CDS markets imply different credit spreads. Credit risk is cheap in the CDS market (3.25%) relative to the bond market (3.5%). The investor should buy protection in the CDS market at 3.25% and go long the bond, thereby earning 3.5% for assuming the credit risk.

Another type of trade using CDS can occur within the instruments issued by a single entity. Credit risk is an element of virtually every unsecured debt instrument or the capital leases issued by a company. Each of these instruments is priced to reflect the appropriate credit risk. Investors can use the CDS market to first determine whether any of these instruments is incorrectly priced relative to the CDS and then buy the cheaper one and sell the more expensive one. Again, there is the assumption that the market will adjust. This type of trading is much more complex, however, because priority of claims means that not all of the instruments pay off equally if default occurs.

EXAMPLE 11

Using CDS to Trade on a Leveraged Buyout

An investor believes that a company will undergo a leveraged buyout (LBO) transaction, whereby it will issue large amounts of debt and use the proceeds to repurchase all of the publicly traded equity, leaving the company owned by management and a few insiders.

1 Why might the CDS spread change?
2 What equity-versus-credit trade might an investor execute in anticipation of such a corporate action?

Solution to 1:

Taking on the additional debt will almost surely increase the probability of default, thereby increasing the CDS spread.

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Solution to 2:

The investor might consider buying the stock and buying CDS protection. Both legs will profit if the LBO occurs because the stock price rises and the CDS price rises as its spread widens to reflect the increased probability of default.

The CDS indexes also permit some opportunities for a type of arbitrage trade. If the cost of the index is not equivalent to the aggregate cost of the index components, the opportunity exists to go long the cheaper instrument and short the more expensive instrument. Again, there is the implicit assumption that convergence will occur. Assuming it does, the investor gains the benefit while basically having neutralized the risk.

A collateralized debt obligation (CDO) is created by assembling a portfolio of debt securities and issuing claims against the portfolio in the form of tranches. These tranches have different priorities of claims, with some tranches responsible for credit losses before others. Yet another type of instrument, called a **synthetic CDO**, is created by combining a portfolio of default-free securities with a combination of credit default swaps undertaken as protection sellers. The default-free securities plus the CDS holdings are, thus, a synthetic CDO because they effectively contain securities subject to default. If an institution can assemble the synthetic CDO at a lower cost than the actual CDO, it can then buy the former and sell the latter, capturing a type of arbitrage profit.

SUMMARY

This reading on credit default swaps provides a basic introduction to these instruments and their markets. The following key points are covered:

- A credit default swap (CDS) is a contract between two parties in which one party purchases protection from another party against losses from the default of a borrower for a defined period of time.
- A CDS is written on the debt of a third party, called the reference entity, whose relevant debt is called the reference obligation, typically a senior unsecured bond.
- A CDS written on a particular reference obligation normally provides coverage for all obligations of the reference entity that have equal or higher seniority.
- The two parties to the CDS are the credit protection buyer, who is said to be short the reference entity's credit, and the credit protection seller, who is said to be long the reference entity's credit. The seller (buyer) is said to be long (short) because the seller is bullish (bearish) on the financial condition of the reference entity.
- The CDS pays off upon occurrence of a credit event, which includes bankruptcy, failure to pay, and, in some countries, restructuring.
- Settlement of a CDS can occur through a cash payment from the credit protection seller to the credit protection buyer as determined by the cheapest-to-deliver obligation of the reference entity, or by physical delivery of the reference obligation from the protection buyer to the protection seller in exchange for the CDS notional.

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- A cash settlement payoff is determined by an auction of the reference entity's debt, which gives the market's assessment of the likely recovery rate. The credit protection buyer must accept the outcome of the auction even though the ultimate recovery rate could differ.
- CDS can be constructed on a single entity or as indexes containing multiple entities.
- The fixed payments made from CDS buyer to CDS seller are customarily set at a fixed annual rate of 1% for investment-grade debt or 5% for high-yield debt.
- Valuation of a CDS is determined by estimating the present value of the protection leg, which is the payment from the protection seller to the protection buyer in event of default, and the present value of the payment leg, which is the series of payments made from the protection buyer to the protection seller. Any difference in the two series results in an upfront payment from the party having the claim on the greater present value to the counterparty.
- An important determinant of the value of the expected payments is the hazard rate, the probability of default given that default has not already occurred.
- CDS prices are often quoted in terms of credit spreads, the implied number of basis points that the credit protection seller receives from the credit protection buyer to justify providing the protection.
- Credit spreads are often expressed in terms of a credit curve, which expresses the relationship between the credit spreads on bonds of different maturities for the same borrower.
- CDS change in value over their lives as the credit quality of the reference entity changes, which leads to gains and losses for the counterparties, even though default may not have occurred or may never occur.
- Either party can monetize an accumulated gain or loss by entering into an offsetting position that matches the terms of the original CDS.
- CDS are used to increase or decrease credit exposures or to capitalize on different assessments of the cost of credit among different instruments tied to the reference entity, such as debt, equity, and derivatives of debt and equity.

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PRACTICE PROBLEMS

The following information relates to Questions 1–6

UNAB Corporation

On 1 January 20X2, Deem Advisors purchased a $10 million six-year senior unsecured bond issued by UNAB Corporation. Six months later (1 July 20X2), concerned about the portfolio's credit exposure to UNAB, Doris Morrison, the chief investment officer at Deem Advisors, purchases a $10 million CDS with a standardized coupon rate of 5%. The reference obligation of the CDS is the UNAB bond owned by Deem Advisors.

On 1 January 20X3, Morrison asks Bill Watt, a derivatives analyst, to assess the current credit quality of UNAB bonds and the value of Deem Advisor's CDS on UNAB debt. Watt gathers the following information on the UNAB's debt issues currently trading in the market:

Bond 1: A two-year senior unsecured bond trading at 40% of par

Bond 2: A six-year senior unsecured bond trading at 50% of par

Bond 3: A six-year subordinated unsecured bond trading at 20% of par

With respect to the credit quality of UNAB, Watt makes the following statement:

> "There is severe near-term stress in the financial markets and UNAB's credit curve clearly reflects the difficult environment."

On 1 July 20X3, UNAB fails to make a scheduled interest payment on the outstanding subordinated unsecured obligation after a grace period; however, the company does not file for bankruptcy. Morrison asks Watt to determine if UNAB experienced a credit event and, if so, to recommend a settlement preference.

Kand Corporation

Morrison is considering purchasing a 10-year CDS on Kand Corporation debt to hedge its current portfolio position. She instructs Watt to determine if an upfront payment would be required and, if so, the amount of the premium. Watt presents the information for the CDS in Exhibit 1.

Exhibit 1	Summary Data for 10-year CDS on Kand Corporation
Credit spread	700 basis points
Duration	7 years
Coupon rate	5%

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Morrison purchases the 10-year CDS on Kand Corporation debt. Two months later the credit spread for Kand Corp. has increased by 200 basis points. Morrison asks Watt to close out the firm's CDS position on Kand Corporation by entering into new offsetting contracts.

Tollunt Corporation

Deem Advisors' chief credit analyst recently reported that Tollunt Corporation's five-year bond is currently yielding 7% and a comparable CDS contract has a credit spread of 4.25%. Since Libor is 2.5%, Watt has recommended executing a basis trade to take advantage of the pricing of the Tollunt's bonds and CDS. The basis trade would consist of purchasing both the bond and the CDS contract.

1 If UNAB experienced a credit event on 1 July, Watt should recommend that Deem Advisors:

A prefer a cash settlement.

B prefer a physical settlement.

C be indifferent between a cash or a physical settlement.

2 According to Watt's statement, the shape of UNAB's credit curve is *most likely*:

A flat.

B upward-sloping.

C downward-sloping.

3 Should Watt conclude that UNAB experienced a credit event?

A Yes.

B No, because UNAB did not file for bankruptcy.

C No, because the failure to pay occurred on a subordinated unsecured bond.

4 Based on Exhibit 1, the upfront premium as a percent of the notional for the CDS protection on Kand Corp. would be *closest* to:

A 2.0%.

B 9.8%.

C 14.0%.

5 If Deem Advisors enters into a new offsetting contract two months after purchasing the CDS protection on Kand Corporation, this action will *most likely* result in:

A a loss on the CDS position.

B a gain on the CDS position.

C neither a loss or a gain on the CDS position.

6 Based on basis trade for Tollunt Corporation, if convergence occurs in the bond and CDS markets, the trade will capture a profit *closest* to:

A 0.25%.

B 1.75%.

C 2.75%.

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The following information relates to Questions 7–15

John Smith, a fixed-income portfolio manager at a €10 billion sovereign wealth fund (SWF), meets with Sofia Chan, a derivatives strategist with Shire Gate Securities (SGS), to discuss investment opportunities for the fund. Chan notes that SGS adheres to ISDA (International Swaps and Derivatives Association) protocols for credit default swap (CDS) transactions and that any contract must conform to ISDA specifications. Before the fund can engage in trading CDS products with SGS, the fund must satisfy compliance requirements.

Smith explains to Chan that fixed-income derivatives strategies are being contemplated for both hedging and trading purposes. Given the size and diversified nature of the fund, Smith asks Chan to recommend a type of CDS that would allow the SWF to simultaneously fully hedge multiple fixed-income exposures.

Next, Smith asks Chan to assess the impact on derivative products of recent events affecting Maxx Corporation, a US company. The SWF holds an unsecured debt instrument issued by Maxx. Chan says she is very familiar with Maxx because many of its unsecured debt obligations are commonly included in broad baskets of bonds used for hedging purposes. SGS recently sold €400 million of protection on the on-the-run CDX high yield (HY) index that includes a Maxx bond; the index contains 100 entities. Chan reports that creditors met with company executives to impose a restructuring on Maxx bonds; as a result, all outstanding principal obligations will be reduced by 30%.

Smith and Chan discuss opportunities to add trading profits to the SWF. Smith asks Chan to determine the probability of default associated with a five-year investment-grade bond issued by Orion Industrial. Selected data on the Orion Industrial bond are presented in Exhibit 1.

Exhibit 1 Selected Data on Orion Industrial Five-Year Bond

Year	Hazard Rate
1	0.22%
2	0.35%
3	0.50%
4	0.65%
5	0.80%

Chan explains that a single-name CDS can also be used to add profit to the fund over time. Chan describes a hypothetical trade in which the fund sells £6 million of five-year CDS protection on Orion, where the CDS contract has a duration of 3.9 years. Chan assumes that the fund closes the position six months later, after Orion's credit spread narrowed from 150 bps to 100 bps.

Chan discusses the mechanics of a long/short trade. In order to structure a number of potential trades, Chan and Smith exchange their respective views on individual companies and global economies. Chan and Smith agree on the following outlooks.

Outlook 1: Italy's economy will weaken.

Outlook 2: The US economy will strengthen relative to that of Canada.

Outlook 3: The credit quality of electric car manufacturers will improve relative to that of traditional car manufacturers.

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Chan believes US macroeconomic data are improving and that the general economy will strengthen in the short term. Chan suggests that a curve trade could be used by the fund to capitalize on her short-term view of a steepening of the US credit curve.

Another short-term trading opportunity that Smith and Chan discuss involves the merger and acquisition market. SGS believes that Delta Corporation may make an unsolicited bid at a premium to the market price for all of the publicly traded shares of Zega, Inc. Zega's market capitalization and capital structure are comparable to Delta's; both firms are highly levered. It is anticipated that Delta will issue new equity along with 5- and 10-year senior unsecured debt to fund the acquisition, which will significantly increase its debt ratio.

7 To satisfy the compliance requirements referenced by Chan, the fund is *most likely* required to:

- **A** set a notional amount.
- **B** post an upfront payment.
- **C** sign an ISDA master agreement.

8 Which type of CDS should Chan recommend to Smith?

- **A** CDS index
- **B** Tranche CDS
- **C** Single-name CDS

9 Following the Maxx restructuring, the CDX HY notional will be *closest* to:

- **A** €396.0 million.
- **B** €398.8 million.
- **C** $400.0 million.

10 Based on Exhibit 1, the probability of Orion defaulting on the bond during the first three years is *closest* to:

- **A** 1.07%.
- **B** 2.50%.
- **C** 3.85%.

11 To close the position on the hypothetical Orion trade, the fund:

- **A** sells protection at a higher premium than it paid at the start of the trade.
- **B** buys protection at a lower premium than it received at the start of the trade.
- **C** buys protection at a higher premium than it received at the start of the trade.

12 The hypothetical Orion trade generated an approximate:

- **A** loss of £117,000.
- **B** gain of £117,000.
- **C** gain of £234,000.

13 Based on the three economic outlook statements, a profitable long/short trade would be to:

- **A** go long a Canadian CDX IG and short a US CDX IG.
- **B** short an iTraxx Crossover and go long an iTraxx Main.
- **C** short electric car CDS and go long traditional car CDS.

14 The curve trade that would *best* capitalize on Chan's view of the US credit curve is to:

- **A** short a 20-year CDX and short a 2-year CDX.
- **B** short a 20-year CDX and go long a 2-year CDX.

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- **C** go long a 20-year CDX and short a 2-year CDX.

15 A profitable equity-versus-credit trade involving Delta and Zega is to:

- **A** short Zega shares and short Delta 10-year CDS.
- **B** go long Zega shares and short Delta 5-year CDS.
- **C** go long Delta shares and go long Delta 5-year CDS.

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SOLUTIONS

1 A is correct. Deem Advisors would prefer a cash settlement. Deem Advisors owns Bond 2 (trading at 50% of par), which is worth more than the cheapest-to-deliver obligation (Bond 1 trading at 40% of par). Deem Advisors can cash settle for \$6 million [= (1 – 40%) × \$10 million] on its CDS contract and sell Bond 2 it owns for \$5 million, for total proceeds of \$11 million. If Deem Advisors were to physically settle the contract, only \$10 million would be received, the face amount of the bonds and they would deliver Bond 2.

   B is incorrect because if Deem Advisors were to physically settle the contract, they would receive only \$10 million, which is less than the \$11 million that could be obtained from a cash settlement. C is incorrect because Deem Advisors would not be indifferent between settlement protocols as the firm would receive \$1 million more with a cash settlement in comparison to a physical settlement.

2 C is correct. A downward-sloping credit curve implies a greater probability of default in the earlier years than in the later years. Downward-sloping curves are less common and often are the result of severe near-term stress in the financial markets.

   A is incorrect because a flat credit curve implies a constant hazard rate (relevant probability of default). B is incorrect because an upward-sloping credit curve implies a greater probability of default in later years.

3 A is correct. UNAB experienced a credit event when it failed to make the scheduled coupon payment on the outstanding subordinated unsecured obligation. Failure to pay, a credit event, occurs when a borrower does not make a scheduled payment of principal or interest on *any* outstanding obligations after a grace period, even without a formal bankruptcy filing.

   B is incorrect because a credit event can occur without filing for bankruptcy. There are three general types of credit events: bankruptcy, failure to pay, and restructuring.

   C is incorrect because a credit event (failure to pay) occurs when a borrower does not make a scheduled payment of principal or interest on *any* outstanding obligations after a grace period, without a formal bankruptcy filing.

4 C is correct. An approximation for the upfront premium is the (Credit spread – Fixed coupon rate) × Duration of the CDS. To buy 10-year CDS protection, Deem Advisors would have to pay an approximate upfront premium of 1400 basis points [(700 – 500) × 7], or 14% of the notional.

   A is incorrect because 200 basis points, or 2%, is derived by taking the simple difference between the credit spread and the fixed coupon rate (700 – 500). B is incorrect because 980 basis points, or 9.8%, is the result of dividing the credit spread by the fixed coupon rate and multiplying by the duration of the CDS [(700/500) × 7].

5 B is correct. Deem Advisors purchased protection, and therefore is economically short and benefits from an increase in the company's spread. Since putting on the protection, the credit spread increased by 200 basis points, and Deem Advisors realizes the gain by entering into a new offsetting contract (sells the protection for a higher premium to another party).

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A is incorrect because a decrease (not increase) in the spread would result in a loss for the credit protection buyer. C is incorrect because Deem Advisors, the credit protection buyer, would profit from an increase in the company's credit spread, not break even.

6 A is correct. A difference in credit spreads in the bond market and CDS market is the foundation of the basis trade strategy. If the spread is higher in the bond market than the CDS market, it is said to be a negative basis. In this case, the bond credit spread is currently 4.50% (bond yield minus Libor) and the comparable CDS contract has a credit spread of 4.25%. The credit risk is cheap in the CDS market relative to the bond market. Since the protection and the bond were both purchased, if convergence occurs, the trade will capture the 0.25% differential in the two markets (4.50% – 4.25%).

B is incorrect because the bond market implies a 4.50% credit risk premium (bond yield minus Libor) and the CDS market implies a 4.25% credit risk premium. Convergence of the bond market credit risk premium and the CDS credit risk premium would result in capturing the differential, 0.25%. The 1.75% is derived by incorrectly subtracting Libor from the credit spread on the CDS (= 4.25% – 2.50%).

C is incorrect because convergence of the bond market credit risk premium and the CDS credit risk premium would result in capturing the differential, 0.25%. The 2.75% is derived incorrectly by subtracting the credit spread on the CDS from the current bond yield (= 7.00% – 4.25%).

7 C is correct. Parties to CDS contracts generally agree that their contracts will conform to ISDA specifications. These terms are specified in the ISDA master agreement, which the parties to a CDS sign before any transactions are made. Therefore, to satisfy the compliance requirements referenced by Chan, the sovereign wealth fund must sign an ISDA master agreement with SGS.

8 A is correct. A CDS index (e.g., CDX and iTraxx) would allow the SWF to simultaneously fully hedge multiple fixed-income exposures.

9 C is correct. When an entity within an index defaults, that entity is removed from the index and settled as a single-name CDS based on its relative proportion in the index. To qualify as a credit event, the restructuring must be involuntary and forced on the borrower by the creditors. Although the Maxx restructuring would be considered a credit event (default) in the eurozone, in the United States, restructuring is not considered a credit event; therefore, the notional amount of $400 million will not change.

10 A is correct. Based on Exhibit 1, the probability of survival for the first year is 99.78% (100% minus the 0.22% hazard rate). Similarly, the probability of survival for the second and third years is 99.65% (100% minus the 0.35% hazard rate) and 99.50% (100% minus the 0.50% hazard rate), respectively. Therefore, the probability of survival of the Orion bond through the first three years is equal to $(0.9978) \times (0.9965) \times (0.9950) = 0.9893$, and the probability of default sometime during the first three years is $1 - 0.9893$, or 1.07%.

11 B is correct. The trade assumes that £6 million of five-year CDS protection on Orion is initially sold, so the fund received the premium. Because the credit spread of the Orion CDS narrowed from 150 bps to 100 bps, the CDS position will realize a financial gain. This financial gain is equal to the difference between the upfront premium received on the original CDS position and the upfront premium to be paid on a new, offsetting CDS position. To close the position and monetize this gain, the fund should unwind the position with a new offsetting CDS, thereby buying protection for a lower premium (relative to the original premium collected) in six months.

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12 B is correct. The gain on the hypothetical Orion trade is £117,000, calculated as follows.

$$\text{Approximate profit} = \text{Change in credit spread (in bps)} \times \text{Duration} \times \text{Notional amount}$$
$$\text{Approximate profit} = (150\text{ bps} - 100\text{ bps}) \times 3.9 \times £6\text{ million}$$
$$\text{Approximate profit} = .005 \times 3.9 \times £6\text{ million} = £117{,}000$$

The SWF gains because they sold protection at a spread of 150 bps and closed out the position by buying protection at a lower spread of 100 bps.

13 B is correct. Based on Outlook 1, Chan and Smith anticipate that Italy's economy will weaken. In order to profit from this forecast, one would go short (buy protection) a high-yield Italian CDS (e.g., iTraxx Crossover) index and go long (sell protection) an investment-grade Italian CDS (e.g., iTraxx Main) index.

14 B is correct. To take advantage of Chan's view of the US credit curve steepening in the short term, a curve trade will entail shorting (buying protection) a long-term (20-year) CDX and going long (selling protection) a short-term (2-year) CDX. A steeper curve means that long-term credit risk increases relative to short-term credit risk.

15 B is correct. If Delta Corporation issues significantly more debt, it raises the probability that it may default, thereby increasing the CDS spread. The shares of Zega will be bought at a premium resulting from the unsolicited bid in the market. An equity-versus-credit trade would be to go long (buy) the Zega shares and short (buy protection) the Delta five-year CDS.

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Derivatives

STUDY SESSIONS

Study Session 14 Derivatives

TOPIC LEVEL LEARNING OUTCOME

The candidate should be able to estimate the value of futures, forwards, options, and swaps and demonstrate how they may be used in various strategies.

Derivatives are used extensively to manage financial risk. Institutions and individuals use derivatives to transfer, modify, or eliminate unwanted interest rate, currency, cash flow, or market exposures. Besides their value in risk management, derivatives can also be effective tools for generating income, enhancing returns, and creating synthetic exposure. Efficiencies in cost, liquidity, ability to short, and limited capital outlay may make derivatives attractive alternatives to their underlying.

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DERIVATIVES

STUDY SESSION 14

Derivatives

This study session introduces key valuation concepts and models for forward commitments (forwards, futures, swaps) and contingent claims (options). Option coverage includes the "Greeks," which measure the effects on value of small changes in underlying asset value, time, volatility, and the risk-free rate.

READING ASSIGNMENTS

Reading 37	Pricing and Valuation of Forward Commitments by Robert E. Brooks, PhD, CFA, and Barbara Valbuzzi, CFA
Reading 38	Valuation of Contingent Claims by Robert E. Brooks, PhD, CFA, and David Maurice Gentle, MEc, BSc, CFA

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READING

37

Pricing and Valuation of Forward Commitments

by Robert E. Brooks, PhD, CFA, and Barbara Valbuzzi, CFA

Robert E. Brooks, PhD, CFA, is at the University of Alabama (USA). Barbara Valbuzzi, CFA (Italy).

LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	**a.** describe and compare how equity, interest rate, fixed-income, and currency forward and futures contracts are priced and valued;
☐	**b.** calculate and interpret the no-arbitrage value of equity, interest rate, fixed-income, and currency forward and futures contracts;
☐	**c.** describe and compare how interest rate, currency, and equity swaps are priced and valued;
☐	**d.** calculate and interpret the no-arbitrage value of interest rate, currency, and equity swaps.

1 INTRODUCTION

Forward commitments cover forwards, futures, and swaps. Pricing and valuation of forward commitments will be introduced here. A forward commitment is a derivative instrument in the form of a contract that provides the ability to lock in a price or rate at which one can buy or sell the underlying instrument at some future date or exchange an agreed-upon amount of money at a series of dates. As many investments can be viewed as a portfolio of forward commitments, this material is important to the practice of investment management.

The reading is organized as follows. Section 2 introduces the principles of the no-arbitrage approach to pricing and valuation of forward commitments. Section 3 presents the pricing and valuation of forwards and futures. Subsections address the cases of equities, interest rates, fixed-income instruments, and currencies as underlyings of forward commitments. Section 4 presents the pricing and valuation of swaps, addressing interest rate, currency, and equity swaps.

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2 PRINCIPLES OF ARBITRAGE-FREE PRICING AND VALUATION OF FORWARD COMMITMENTS

In this section, we examine arbitrage-free pricing and valuation of forward commitments—also known as the no-arbitrage approach to pricing and valuing such instruments. We introduce some guiding principles that heavily influence the activities of arbitrageurs who are price setters in forward commitment markets.

There is a distinction between the pricing and the valuation of forward commitments. Forward commitment pricing involves determining the appropriate forward commitment price or rate when initiating the forward commitment contract. Forward commitment valuation involves determining the appropriate value of the forward commitment, typically after it has been initiated.

Our approach to pricing and valuation is based on the assumption that prices adjust to not allow arbitrage profits. Hence, the material will be covered from an arbitrageur's perspective. Key to understanding this material is to think like an arbitrageur. Specifically, like most people, the arbitrageur would rather have more money today than less. The arbitrageur abides by two fundamental rules:

Rule #1 Do not use your own money.

Rule #2 Do not take any price risk.

The arbitrageur often needs to borrow or lend money to satisfy Rule #1. If we buy the underlying, we borrow the money. If we sell the underlying, we lend the money. These transactions will synthetically create the identical cash flows to a particular forward commitment, but they will be opposite and, therefore, offsetting, which satisfies Rule #2. Note that for Rule #2, the concern is only market price risk related to the underlying and the derivatives used, as explained in detail later. Clearly, if we can generate positive cash flows today and abide by both rules, we have a great business; such is the life of an arbitrageur.

In an effort to demonstrate various pricing and valuation results based on the no-arbitrage approach, we will rely heavily on tables showing cash flows at Times 0 and T. From an arbitrage perspective, if an initial investment requires 100 euros, then we will present it as a –100 euro cash flow. Cash inflows to the arbitrageur have a positive sign, and outflows are negative.

Pricing and valuation tasks based on the no-arbitrage approach imply an inability to create a portfolio with no future liabilities and a positive cash flow today. In other words, if cash and forward markets are priced correctly with respect to each other, we cannot create such a portfolio. That is, we cannot create money today with no risk or future liability. This approach is built on the **law of one price**, which states that if two investments have the same or equivalent future cash flows regardless of what will happen in the future, then these two investments should have the same current price. Alternatively, if the law of one price is violated, someone could buy the cheaper asset and sell the more expensive, resulting in a gain at no risk and with no commitment of capital. The law of one price is built on the value additivity principle, which states that the value of a portfolio is simply the sum of the values of each instrument held in the portfolio.

Throughout this reading, the following key assumptions are made: (1) Replicating instruments are identifiable and investable, (2) market frictions are nil, (3) short selling is allowed with full use of proceeds, and (4) borrowing and lending are available at a known risk-free rate.

Analyses in this reading will rely on the **carry arbitrage model**, a no-arbitrage approach in which the underlying instrument is either bought or sold along with a forward position—hence the term "carry." Carry arbitrage models are also known as

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cost-of-carry arbitrage models or cash-and-carry arbitrage models. Typically, each type of forward commitment will result in a different model, but common elements will be observed. Carry arbitrage models are a great first approximation to explaining observed forward commitment prices in many markets.

The central theme here is that forward commitments are generally priced so as to preclude arbitrage profits. Section 3 demonstrates how to price and value equity, interest rate, fixed-income, and currency forward contracts. We also explain how these results apply to futures contracts.

3 PRICING AND VALUING FORWARD AND FUTURES CONTRACTS

In this section, we examine the pricing of forward and futures contracts based on the no-arbitrage approach. The resulting carry arbitrage models are based on the replication of the forward contract payoff with a position in the underlying that is financed through an external source. Although the margin requirements, mark-to-market features, and centralized clearing in futures markets result in material differences between forward and futures markets in some cases, we focus mainly on cases in which the particular carry arbitrage model can be used in both markets.

We start with a very simple setup to arrive at the primary insight that the current forward or futures price of a non-cash-paying instrument is simply equal to the price of the underlying adjusted upward for the amount that would be earned over the term of the contract by compounding the initial underlying price at the rate that incorporates costs and benefits related to the underlying instrument. Initially, we adopt a simplified approach in which we determine the forward price by compounding the underlying price at the risk-free rate. We then turn to examining the particular nuances of equity, interest rate, fixed-income, and currency forward and futures contracts. Mastery of the simple setup will make understanding the unique nuances in each market easier to comprehend. First, we examine selected introductory material.

3.1 Our Notation

In the following, notations are established for forward and futures contracts that will allow us to express concisely the key pricing and valuation relationships. **Forward price** or **futures price** refers to the price that is negotiated between the parties in the forward or futures contract. The market value of the forward or futures contract, termed **forward value** or **futures value** and sometimes just value, refers to the monetary value of an existing forward or futures contract. When the forward or futures contract is established, the price is negotiated so that the value of the contract on the initiation date is zero. Subsequent to the initiation date, the value can be significantly positive or negative.

Let S_t denote the price of the underlying instrument observed at Time t, where t is the time since the initiation of the forward contract and is expressed as a fraction of years.[1] Consider T as the initial time to expiration, expressed as a fraction of years. S_0 denotes the underlying price observed when the forward contract is initiated, and S_T denotes the underlying price observed when the forward contract expires. Also, let $F_0(T)$ denote the forward price established at the initiation date, 0, and expiring

1 Note that t can be greater than a year—for example t = 1.25. The variable t is expressed in years, not days or months, because interest rates, dividend yields, and most financial returns are expressed as yearly rates.

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at date T, where T represents a period of time later. For example, suppose that on the initiation date (t = 0) a forward contract is negotiated for which $F_0(0.25)$ = €350. Then the forward price for the forward contract is €350, with the contract expiration T = 0.25 years later. Similarly, let $f_0(T)$ denote the futures price for a contract established at the initiation date, 0, that expires at date T. Therefore, uppercase "F" denotes the forward price, whereas lowercase "f" denotes the futures price. Similarly, we let uppercase "V" denote the forward value, whereas lowercase "v" denotes the futures value. Many concepts in this reading apply equally to pricing and valuation of both forwards and futures. When they differ, we will emphasize the distinctions.

A key observation, to which we will return in greater detail, is that as a result of the no-arbitrage approach, when the forward contract is established, the forward price is negotiated so that the market value of the forward contract on the initiation date is zero. Most forward contracts are structured this way and are referred to as **at market**. No money changes hands, meaning that the initial value is zero. The forward contract value when initiated is expressed as $V_0(T) = v_0(T) = 0$. Again, we assume no margin requirements. Subsequent to the initiation date, the forward value can be significantly positive or negative.

At expiration, both the forward contract and the futures contract are equivalent to a spot transaction in the underlying. In fact, forward and futures contracts negotiated at Time T for delivery at Time T are by definition equivalent to a spot transaction at Time T. This property is often called **convergence**, and it implies that at Time T, both the forward price and the futures price are equivalent to the spot price—that is, $F_T(T) = f_T(T) = S_T$.

Let us define $V_t(T)$ as the forward contract value at Time t during the life of the futures contract. At expiration, T,

The market value of a long position in a forward contract value is $V_T(T) = S_T - F_0(T)$.

The market value of a short position in a forward contract value is $V_T(T) = F_0(T) - S_T$.

Let us define $v_t(T)$ as the futures contract value at Time t during the life of the futures contract. Note that as a result of marking to market, the value of a futures contract at expiration is simply the difference in the futures price from the previous day. Our time subscript is expressed in a fraction of a year; hence, we use (t–) to denote the fraction of the year that the previous trading day represents. At expiration, T:

The market value of a long position in a futures contract value before marking to market is $v_t(T) = f_t(T) - f_{t-}(T)$.

The market value of a short position in a futures contract value before marking to market is $v_t(T) = f_{t-}(T) - f_t(T)$.

The futures contract value after daily settlement is $v_t(T) = 0$.

As illustrated later, in this reading we adopt a simplified approach in which the valuation of forward and futures contracts is treated as the same, whereas the forward value and the futures value will be different because of futures contracts being marked to market and forward contracts not being marked to market.[2]

Exhibit 1 shows a forward contract at initiation and expiration. A long position in a forward contract will have a positive value at expiration if the underlying is above the initial forward price, whereas a short position in a forward contract will have a positive value at expiration if the underlying is below the initial forward price.

2 There are specific cases when $f_t(T) \neq F_t(T)$, but they are beyond the scope of this reading.

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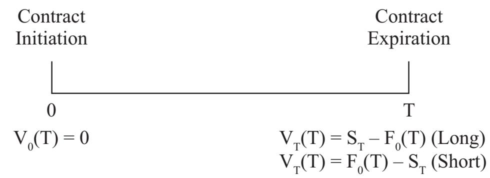
Exhibit 1 Value of a Forward Contract at Initiation and Expiration



We turn now to focus on generic forward contracts.

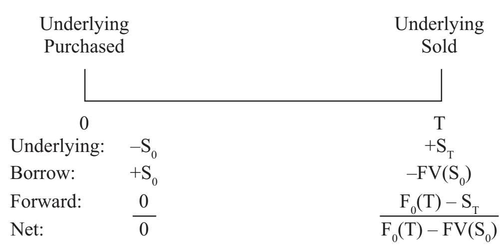
3.2 No-Arbitrage Forward Contracts

We first consider a generic forward contract, meaning that we do not specify the underlying as anything more than just an asset. As we move through this section, we will continue to address specific additional factors to bring each carry arbitrage model closer to real markets. Thus, we will develop several different carry arbitrage models, each one applicable to specific forward commitment contracts.

3.2.1 *Carry Arbitrage Model When There Are No Underlying Cash Flows*

Carry arbitrage models receive their name from the literal interpretation of carrying the underlying over the life of the forward contract. If an arbitrageur enters a forward contract to sell an underlying instrument for delivery at Time T, then to hedge this exposure, one strategy is to buy the underlying instrument at Time 0 with borrowed funds and carry it to the forward expiration date so it can be sold under the terms of the forward contract as illustrated in Exhibit 2.

Exhibit 2 Cash Flows Related to Carrying the Underlying through Calendar Time



For now, we will keep the significant technical issues to a minimum. When possible, we will just use FV and PV to denote the future value and present value, respectively. We are not concerned now about compounding conventions, day count conventions, or even the appropriate risk-free interest rate proxy. We will address these complexities only when necessary.

Carry arbitrage models rest on the no-arbitrage assumptions given earlier. To understand carry arbitrage models, it is helpful to think like an arbitrageur. The arbitrageur seeks to exploit any pricing discrepancy between the futures or forward price and the underlying spot price. The arbitrageur is assumed to prefer more money compared to less money, assuming everything else is the same. We now expand on the two fundamental rules for the arbitrageur.

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Rule #1 Do not use our own money. Specifically, the arbitrageur does not use his or her own money to acquire positions but borrows to purchase the underlying. Also, the arbitrageur does not spend proceeds from short selling transactions but invests them at the risk-free interest rate.

Rule #2 Do not take any price risk. In our discussion, the arbitrageur focuses here only on market price risk related to the underlying and the derivatives used. We do not consider other risks, such as liquidity risk and counterparty credit risk. These topics are covered in more advanced treatments.

Consider the following strategy in which an arbitrageur purchases the underlying instrument with borrowed money in the spot market at price S_0 at Time 0 and later, at Time T, contemporaneously sells the underlying at a price of S_T and repays the loan. The cash flow from this strategy evaluated at Time T is the proceeds from the sales of the underlying, S_T, less $FV_{0,T}(S_0)$ or, more simply, $FV(S_0)$, the price of the underlying purchased at Time 0 grossed up by the finance cost, assumed to be the risk-free interest rate. In other words, the arbitrageur borrows the money to buy the asset, so he will pay back $FV(S_0)$ at Time T, based on the risk-free rate.

Clearly, when S_T is below $FV(S_0)$, this transaction will suffer a loss. Note that breakeven will occur when the underlying value at T exactly equals the future price of the underlying at 0 grossed up by the finance cost or $S_T = FV(S_0)$. If we assume continuous compounding (r_c), then $FV(S_0) = S_0 e^{r_c T}$. If we assume annual compounding (r), then $FV(S_0) = S_0(1 + r)^T$. Note that in practice, observed interest rates are derived from market prices; it is not the other way around. Significant errors can occur if the quoted interest rate is used with the wrong compounding convention.[3] When possible, we just use the generic present value and future value to minimize confusion.

To help clarify, Exhibit 3 shows the cash flows from carrying the underlying, say, stock, assuming $S_0 = 100$, $r = 5\%$, $T = 1$, and $S_T = 90$ or 110.[4] Each step consists of transactions that generate the cash flows shown at times 0 and T. Each row of cash flows in tables such as the one below are termed "steps," and they will involve a wide array of cash flow producing items from market transactions, bank transactions, and other events. The set of transactions is executed simultaneously in practice, not sequentially.

Step 1 Purchase one unit of the underlying at Time 0.

Step 2 Borrow the purchase price. Recall that cash flow is the opposite of investment. An investment of 100 implies a negative cash flow of 100—that is, −100. We assume the interest rate is quoted on an annual compounding basis and time is expressed in fractions of a year.

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3 For many quantitative finance tasks, it is easier to do the analysis with continuous compounding even though the underlying rate quotation conventions are based on another method.

4 Note that S_T can take on any value, but in the table we present just two values, one representing an up move and one representing a down move.

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Exhibit 3 Cash Flows for Financed Position in the Underlying Instrument

Steps	Cash Flows at Time 0	Cash Flows at Time T
1. Purchase underlying at 0 and sell at T	$-S_0 = -100$	$+S_T = 90$ or $+S_T = 110$
2. Borrow funds at 0 and repay with interest at T	$+S_0 = 100$	$-FV(S_0) = -100(1 + 0.05)^1 = -105$
Net cash flow	0	$+S_T - FV(S_0) = 90 - 105 = -15$ or $= 110 - 105 = 5$

Because the two outcomes are not the same, the strategy at this point fails to satisfy the arbitrageur's Rule #2: Do not take any price risk. Thus, to satisfy Rule #2, consider a third transaction that allows one to lock in the value of the underlying at Time T. This result can be achieved by selling, at Time 0, a forward contract on the underlying at price $F_0(T)$, where the underlying will be delivered at Time T.[5] Recall that the value of the forward contract at expiration will simply be the difference between the underlying, S_T, and the initial forward price, $F_0(T)$.

As seen in Exhibit 4, we add two additional steps, again executed simultaneously:

Step 3 Sell a forward contract. As we are seeking to determine the equilibrium forward price, we do not assume that the forward price is initially at market, meaning that the value is zero. Thus, the forward contract value at Time 0, $V_0(T)$, may be non-zero. We illustrate selected numerical values for clarity.

Step 4 Borrow the arbitrage profit in order to capture it today. If the transaction leads to an arbitrage profit at the Time T expiration, you borrow against it. In other words, suppose that in setting up the transaction, you know that it will produce an arbitrage profit of €5. Then you could borrow the present value of €5 and pay it back at expiration with the arbitrage profit. In effect, you are pre-capturing your arbitrage profit by bringing it to the present so as to receive it at Time 0. The amount you borrow will be the forward price minus the future value of the spot price when compounded at the risk-free rate. As we will see shortly, if the forward contract is priced correctly, there will be no arbitrage profit and, hence, no Step 4. Note also that we exclude the case of lending, because it would occur only if you executed a strategy to capture a certain loss, which we presume no one would do.

In this exhibit, the forward price is assumed to be trading at 105.

5 Note that when an arbitrageur needs to sell the underlying, it must be assumed that she does not hold it in inventory and thus must short sell it. When the transaction calls for selling a derivative instrument, such as a forward contract, it is always just selling—technically, not short selling.

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Exhibit 4 Cash Flows for Financed Position in the Underlying Instrument Combined with a Forward Contract

Steps	Cash Flows at Time 0	Cash Flows at Time T
1. Purchase underlying at 0 and sell at T	$-S_0 = -100$	$+S_T = 90$ or $+S_T = 110$
2. Borrow funds at 0 and repay with interest at T	$+S_0 = 100$	$-FV(S_0) = -S_0(1 + r)^T$ $= -100(1 + 0.05)^1 = -105$
3. Sell forward contract at 0 when $F_0(T) = 105$	$+V_0(T)$	$V_T(T) = F_0(T) - S_T = 105 - 90 = 15$ or $V_T(T) = F_0(T) - S_T = 105 - 110 = -5$
4. Borrow arbitrage profit	$+PV[F_0(T)$ $- FV(S_0)]$	$-[F_0(T) - FV(S_0)]$ $= -[105 - 100(1 + 0.05)] = 0$
Net cash flow	$+V_0(T)$ $+ PV[F_0(T)$ $- FV(S_0)]$	$+S_T - FV(S_0) + F_0(T) - S_T$ $- [F_0(T) - FV(S_0)] = 0$ (For every underlying value)

Notice that at expiration the underlying is worth 90 or 110 and the forward contract is worth either 15 or –5. The combination of the underlying and the forward value is 90 + 15 = 105 or 110 – 5 = 105, and that 105 is precisely the amount necessary to pay off the loan. So, there is zero cash flow at expiration under any and all circumstances.

Based on the no-arbitrage approach, a portfolio offering zero cash flow in the future is expected to be valued at zero at Time 0. That is, based on Exhibit 4, the net cash flow at Time 0 can be expressed as $V_0(T) + PV[F_0(T) - FV(S_0)] = 0$. With this perspective, the value of a given short forward contract is, therefore, $V_0(T) = -PV[F_0(T) - FV(S_0)]$, which can be rearranged and denoted $V_0(T) = S_0 - PV[F_0(T)]$. Based on this result, we see that the no-arbitrage forward price is simply the future value of the underlying, or

$$F_0(T) = \text{Future value of underlying} = FV(S_0) \tag{1}$$

In our example, $F_0(T) = FV(S_0) = 105$. In fact, with annual compounding and $T = 1$, we have simply $F_0(1) = S_0(1 + r)^T = 100(1 + 0.05)^1$. The future value refers to the amount of money equal to the spot price invested at the compound risk-free interest rate during the time period. It is not to be confused with or mistaken for the mathematical expectation of the spot price at Time T.

To better understand the arbitrage mechanics, suppose we observe that $F_0(1) = 106$. Based on the prior information, we observe that the forward price is higher than that determined by the carry arbitrage model (recall $F_0(T) = FV(S_0) = 105$). Because the model value is lower than the market forward price, we conclude that the market forward price is too high and should be sold. An arbitrage opportunity exists, and it will involve selling the forward contract at 106. Because of Rule #2—the arbitrageur should not take any market price risk—the second transaction is to purchase the underlying instrument so that gains (or losses) on the underlying will be offset by losses (or gains) on the forward contract. Finally, because of Rule #1—the arbitrageur does not use his or her own money—the third transaction involves borrowing the purchase price of the underlying security. Based on a desire by the arbitrageur to receive future arbitrage profits today, the fourth transaction involves borrowing the known terminal profits. Note that all four transactions are done simultaneously. To summarize, the arbitrage transactions can be represented in the following four steps:

Step 1 Sell the forward contract on the underlying.

Step 2 Purchase the underlying.

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Step 3 Borrow the funds for the underlying purchase.

Step 4 Borrow the arbitrage profit.[6]

Exhibit 5 shows the resulting cash flows from these transactions. This strategy is known as carry arbitrage because we are carrying—that is, we are long—the underlying instrument. Note that if the forward price were 106, the value of the forward contract would be 0.9524 at Time 0. In fact, $V_0(T) = PV[F_0(T) - FV(S_0)] = (106 - 105)/(1 + 0.05) = 0.9524$. But if the counterparty enters a long position in the forward contract at a forward price of 106, valuing it incorrectly, then the forward contract seller has the opportunity to receive the 0.9524 with no liability in the future. In Step 4, the arbitrageur borrows this amount. At Time T, the arbitrage profit of 1 will exactly offset the repayment of this loan. This opportunity represents a portfolio that will be pursued aggressively. It is a clear arbitrage opportunity.

Exhibit 5 Cash Flows with Forward Contract Market Price Too High Relative to Carry Arbitrage Model

Steps	Cash Flows at Time 0	Cash Flows at Time T
1. Sell forward contract on underlying at $F_0(T) = 106$	$V_0(T) = 0$	$V_T(T) = F_0(T) - S_T = 106 - 90 = 16$ or $V_T(T) = F_0(T) - S_T = 106 - 110 = -4$
2. Purchase underlying at 0 and sell at T	$-S_0 = -100$	$+S_T = 90$ or $+S_T = 110$
3. Borrow funds for underlying purchase	$+S_0 = 100$	$-FV(S_0) = -100(1 + 0.05) = -105$
4. Borrow arbitrage profit	$+PV[F_0(T) - FV(S_0)]$ $= (106 - 105)/(1+0.05)$ $= 0.9524$	$-[F_0(T) - FV(S_0)]$ $= -[106 - 100(1+0.05)] = -1$
Net cash flow	0.9524	$16 + 90 - 105 - 1$ or $-4 + 110 - 105 - 1$ $= 0$

Suppose instead we observe a lower forward price of $F_0(T) = 104$. Based on the prior information, we conclude that the forward price is too low when compared to the forward price determined by the carry arbitrage model. In fact, the carry arbitrage model forward price is again $F_0(T) = FV(S_0) = 105$. Thus, Step 1 here is to buy a forward contract, and the value at T is $S_T - F_0(T)$. Because of Rule #2—the arbitrageur not taking any risk—Step 2 is to sell short the underlying instrument. Because of Rule #1—the arbitrageur not using her own money, or technically here spending another entity's money—Step 3 involves lending the short sale proceeds. Finally, to capture the arbitrage profit today, you borrow its present value. Again, to summarize, the arbitrage transactions involve the following four steps:

Step 1 Buy the forward contract on the underlying.

Step 2 Sell the underlying short.

Step 3 Lend the short sale proceeds.

Step 4 Borrow the arbitrage profit.

6 Remember that you are bringing the arbitrage profit from the future, time T, to the present, time 0, by borrowing against it and paying back the loan at T with the arbitrage profit. We exclude the case of lending, because it involves an arbitrage loss and would mean that the arbitrageur invests some of his own money at time 0 and pays out its value at T to cover the arbitrage loss.

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Note that this set of transactions is the exact opposite of the prior case in Exhibit 5. This strategy is known as **reverse carry arbitrage** because we are doing the opposite of carrying the underlying instrument; that is, we are short selling the underlying instrument.

Therefore, unless $F_0(T) = FV(S_0)$, there is an arbitrage opportunity. Notice that if $F_0(T) > FV(S_0)$, then the forward contract is sold and the underlying is purchased. Thus, arbitrageurs drive down the forward price and drive up the underlying price until $F_0(T) = FV(S_0)$ and a risk-free positive cash flow today no longer exists. Further, if $F_0(T) < FV(S_0)$, then the forward contract is purchased and the underlying is sold short. In this case, the forward price is driven up and the underlying price is driven down. Arbitrageurs' market activities will drive forward prices to equal the future value of the underlying, bringing the law of one price into effect once again. Most importantly, if the forward contract is priced at its equilibrium price, there will be no arbitrage profit and thus no Step 4.

EXAMPLE 1

Forward Contract Price

An Australian stock paying no dividends is trading in Australian dollars for A$63.31, and the annual Australian interest rate is 2.75% with annual compounding.

1 Based on the current stock price and the no-arbitrage approach, which of the following values is *closest* to the equilibrium three-month forward price?

A A$63.31

B A$63.74

C A$65.05

2 If the interest rate immediately falls 50 bps to 2.25%, the three-month forward price will:

A decrease.

B increase.

C be unchanged.

Solution to 1:

B is correct. Based on the information given, we know S_0 = A$63.31, r = 2.75% (annual compounding), and T = 0.25. Therefore,

$$F_0(T) = FV_{0,T}(S_0) = 63.31(1 + 0.0275)^{0.25} = \text{A\$}63.7408.$$

Solution to 2:

A is correct, and we know this is true because the forward price is directly related to the interest rate. Specifically,

$$F_0(T) = FV_{0,T}(S_0) = 63.31(1 + 0.0225)^{0.25} = \text{A\$}63.6632.$$

Therefore, we see in this case a fall in interest rates resulted in a decrease in the forward price. This relationship between forward prices and interest rates will generally hold so long as the underlying is not also influenced by interest rates.

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As we see here, remember that one significant implication of this arbitrage activity is that the quoted forward price does not directly reflect expectations of future underlying prices. The only factors that matter are the interest rate and time to expiration. Other factors will be included later as we make the carry arbitrage model more realistic, but we will not be including expectations of future underlying prices. So, in other words, an opinion that the underlying will increase in value, perhaps even substantially, has no bearing on the forward price.

We now turn to the task of understanding the value of an existing forward contract. There are many circumstances in which, once a forward contract has been entered, one wants to know the contract's fair value. The goal is to calculate the position's value at current market prices. It may be due to market-based accounting, in which the accounting statements need to reflect the current fair value of various instruments. Finally, it is simply important to know whether a position in a forward contract is making money or losing money.

The forward value, based on arbitrage, can best be understood by referring to Exhibit 6. Suppose the first transaction involves buying a forward contract with a price of $F_0(T)$ at Time 0 with expiration of Time T. Now consider selling a new forward contract with price $F_t(T)$ at Time t again with expiration of Time T. Exhibit 6 shows the potential cash flows. Remember the equivalence at expiration between the forward price, the futures price, and the underlying price, meaning $F_T(T) = f_T(T) = S_T$. Note that the column labeled "Value at Time t" represents the value of the forward contracts. Note that we are seeking the forward value; hence, this transaction would result in cash flows only if it is actually executed. We need not actually execute the transaction; we just need to see what it would produce if we did. This point is analogous to the fact that if holding a liquid asset, we need not sell it to determine its value; we can simply observe its market price, which gives us an estimate of the price at which we could sell it.

Exhibit 6 Cash Flows for the Valuation of a Long Forward Position

Steps	Cash Flow at Time 0	Value at Time t	Cash Flow at Time T
1. Buy forward contract at 0 at $F_0(T)$	0	$V_t(T)$	$V_T(0, T) = S_T - F_0(T)$
2. Sell forward contract at t at $F_t(T)$	NA	0	$V_T(t, T) = F_t(T) - S_T$
Net cash flows/Value	0	$V_t(T)$	$+F_t(T) - F_0(T)$

There are now three different points in time to consider: Time 0, Time t, and Time T. For clarity, we explicitly state the period for present value, $PV_{t,T}()$ rather than PV(), which means the present value at point t of an amount paid in T – t years, and for future value, $FV_{t,T}()$ rather than FV(), which means the future value in T – t years of an amount paid at point t.

Note that once the offsetting forward is entered, the net position is not subject to market risk in that the cash flow at Time T is not influenced by what happens to the spot price. The position is completely hedged. Therefore, the value observed at Time t of the original forward contract initiated at Time 0 and expiring at Time T is simply the present value of the difference in the forward prices, $PV_{t,T}[F_t(T) - F_0(T)]$. Based on Exhibit 6, the forward value at Time t for a long position in the forward contract entered at Time 0 is the present value of the difference in forward prices, or

$$\begin{aligned} V_t(T) &= \text{Present value of difference in forward prices} \\ &= PV_{t,T}\left[F_t(T) - F_0(T)\right] \end{aligned} \quad (2)$$

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Thus, there is the old forward price, which is the price the participants agreed on when the contract was started, and now there is also the new forward price, which is the price at which any two participants would agree to deliver the underlying at the same date as in the original contract. Of course, now the spot price has changed and some time has elapsed, so the new forward price will likely not equal the old forward price. The value of the contract is simply the present value of the difference in these two prices, with the present value calculated over the remaining life of the contract.

Alternatively, $V_t(T) = S_t - PV_{t,T}[F_0(T)]$.[7] Thus, the long forward contract value can be viewed as the present value, determined using the given interest rate, of the difference in forward prices—the original one and a new one that is priced at the point of valuation. If we know the underlying price at Time t, S_t, then we can estimate the forward price, $F_t(T)= FV_{t,T}(S_t)$. Based on Equation 2, we then solve for the forward value. Note that the short position is simply the negative value of Equation 2.

EXAMPLE 2

Forward Contract Value

Assume that at Time 0 we entered into a one-year forward contract with price $F_0(T) = 105$. Nine months later, at Time $t = 0.75$, the observed price of the stock is $S_{0.75} = 110$ and the interest rate is 5%. The value of the existing forward contract expiring in three months will be *closest* to:

A −6.34.

B 6.27.

C 6.34.

Solution:

B is correct. Note that, based on $F_0(T) = 105$, $S_{0.75} = 110$, r = 5%, and $T - t = 0.25$, the three-month forward price at Time t is equal to $F_t(T) = FV_{t,T}(S_t) = 110(1 + 0.05)^{0.25} = 111.3499$. Therefore, we find that the value of the existing forward entered at Time 0 valued at Time t using the difference method is

$$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)] = (111.3499 - 105)/(1 + 0.05)^{0.25} = 6.2729.$$

Now that we have the basics of forward pricing and forward valuation, we introduce some other realistic carrying costs that influence pricing and valuation.

3.2.2 *Carry Arbitrage Model When Underlying Has Cash Flows*

We have seen that forward pricing and valuation is driven by arbitrageurs seeking to exploit mispricing by either carrying or reverse carrying the underlying instrument. Carry arbitrage requires paying the interest cost, whereas reverse carry arbitrage results in receiving the interest benefit. For many instruments, there are other significant carry costs and benefits. We will now incorporate into forward pricing various costs and benefits related to the underlying instrument. For this reason, we need to introduce some notation.

Let γ (Greek lowercase gamma) denote the **carry benefits** (for example, dividends, foreign interest, and bond coupon payments that would arise from certain underlyings). Let $\gamma_T = FV_{0,T}(\gamma_0)$ denote the future value of underlying carry benefits and

7 From Equation 1 and assuming annual compounding, $F_t(T) = S_t(1 + r)^{(T-t)}$, so $PV_{t,T}[F_t(T)] = PV_{t,T}[S_t(1 + r)^{(T-t)}] = S_t$.

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$\gamma_0 = PV_{0,T}(\gamma_T)$ denote the present value of underlying carry benefits. Let θ (Greek lowercase theta) denote the **carry costs**. For financial instruments, these costs are essentially zero. For commodities, these costs include such factors as waste, storage, and insurance. Let $\theta_T = FV_{0,T}(\theta_0)$ denote the future value of underlying costs and $\theta_0 = PV_{0,T}(\theta_T)$ denote the present value of underlying costs. We do not cover commodities in this reading, but you should be aware of this cost. Moreover, you should note that carry costs are similar to financing costs. Holding a financial asset does not generate direct carry costs, but it does result in the opportunity cost of the interest that could be earned on the money tied up in the asset. Thus, the financing costs that come from the rate of interest and the carry costs that are common to physical assets are equivalent concepts.

The key forward pricing equation, based on these notations, can be expressed as

$$\begin{aligned} F_0(T) &= \text{Future value of underlying adjusted for carry cash flows} \\ &= FV_{0,T}(S_0 + \theta_0 - \gamma_0) \end{aligned} \tag{3}$$

Thus, the forward price is the future value of the underlying adjusted for carry cash flows. Carry costs, like the rate of interest, increase the burden of carrying the underlying instrument through time; hence, these costs are added in the forward pricing equation. Alternatively, carry benefits decrease the burden of carrying the underlying instrument through time; hence, these benefits are subtracted in the forward pricing equation.

In the following discussion, we follow the arbitrage procedure discussed previously, but now we also consider that the underlying pays some form of benefit during the life of the forward contract. Because of the types of instruments considered here, underlying benefits will be our focus. Note, however, that costs are handled in exactly the same way except there is a sign change.

The arbitrageur purchases the underlying with borrowed money at Time 0 and then sells it at Time T. Notice that any benefits from owning the underlying are placed in a risk-free investment. The risk again is that the underlying value (S_T) will decrease between 0 and T, when the position is unwound. Note that breakeven will occur when the underlying value at T exactly equals the future value of the underlying at 0 adjusted for any benefits, or $S_T = FV(S_0) - \gamma_T = FV(S_0 - \gamma_0)$. Thus, based on this breakeven expression, the underlying benefits (γ) have the effect of lowering the cost of carrying the underlying, and therefore, the forward price is lower.

To help clarify, we illustrate in Exhibit 7 the same example as before in which $S_0 = 100$, $r = 5\%$, $T = 1$, and $S_T = 90$ or 110. We now assume the underlying is known to distribute 2.9277 at Time $t = 0.5$: $\gamma_t = 2.9277$. Thus, the time until the distribution of 2.9277 is t, and hence, the present value is $\gamma_0 = 2.9277/(1 + 0.05)^{0.5} = 2.8571$. The time between the distribution and the forward expiration is $T - t = 0.5$, and thus, the future value is $\gamma_T = 2.9277(1 + 0.05)^{0.5} = 3$.

Remember that the steps in these tables simply refer to cash flow producing events and are initiated simultaneously.

Step 1 Purchase the underlying at Time 0, receive the dividend at Time $t = 0.5$, and sell the underlying at Time T.

Step 2 Reinvest the dividend received at Time $t = 0.5$ at the risk-free interest rate until Time T.

Step 3 Borrow the initial cost of the underlying. The strategy again at this point fails to satisfy Rule #2 of the arbitrageur: Do not take any price risk. If the underlying falls in value, then there is price risk.

Step 4 Sell a forward contract. This transaction addresses Rule #2. Specifically, we sell a forward contract at Time 0 and the underlying will be delivered at Time T.

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Step 5 Borrow the arbitrage profit.

Exhibit 7 Cash Flows for Financed Position in the Underlying with Forward

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Purchase underlying at 0, sell at T	$-S_0 = -100$	$+\gamma_t = 2.9277$	$+S_T = 90$ or $+S_T = 110$
2. Reinvest distribution		$-\gamma_t = -2.9277$	$+\gamma_T = 2.9277(1 + 0.05)^{0.5} = 3$
3. Borrow funds	$+S_0 = 100$		$-FV(S_0) = -100(1 + 0.05)^1 = -105$
4. Sell forward contract	$V_0(T)$		$V_T(T)=F_0(T)-S_T = 102 - 90 = 12$ or $= 102 - 110 = -8$
5. Borrow arbitrage profit	$+PV[F_0(T) + \gamma_T - FV(S_0)]$		$-[F_0(T) + \gamma_T - FV(S_0)]$
Net cash flows	$V_0(T) + PV[F_0(T) + \gamma_T - FV(S_0)]$	0	$+S_T + \gamma_T - FV(S_0) + F_0(T) - S_T - [F_0(T) + \gamma_T - FV(S_0)] = 0$

We know in equilibrium the value of the cash flow at Time 0 is zero, or $V_0(T) + PV[F_0(T) + \gamma_T - FV(S_0)] = 0$, and thus $V_0(T) = -PV[F_0(T) + \gamma_T - FV(S_0)]$. If the forward contract has zero value, then the forward price is simply the future value of the underlying less the future value of carry benefits, or

$$\begin{aligned} F_0(T) &= \text{Future value of underlying} - \text{Future value of carry benefits} \\ &= FV(S_0) - \gamma_T \end{aligned}$$

As the carry benefits increase, the forward price decreases. In short, benefits reduce the cost of carrying the asset, and that reduces the forward price. In this example, the equilibrium forward price is $FV_{0,T}(S_0) - \gamma_T = 105 - 3 = 102$. This is the rationale for the carry arbitrage model adjusted for underlying benefits paid, or $F_0(T) = FV_{0,T}(S_0) - \gamma_T$. Note that because $\gamma_T = FV_{0,T}(\gamma_0)$, we can also express the carry benefit adjusted model as $F_0(T) = FV_{0,T}(S_0 - \gamma_0)$. In words, the initial forward price is equal to the future value of the underlying minus the value of any ownership benefits at expiration. Carry benefits lower the carry burden of the arbitrageur. In effect, because the underlying benefits reduce the burden of carrying the underlying, the forward price is lower. We see that the cost of carrying the underlying is now $F_0(T) = 102$, which is lower than the previous example in which $F_0(T) = 105$.

The forward value for a long position when the underlying has carry benefits or carry costs is found in the same way as described previously except that the new forward price, as well as the old, is adjusted to account for these benefits and costs. Specifically,

$$\begin{aligned} V_t(T) &= \text{Present value of difference in forward prices} \\ &= PV_{t,T}\left[F_t(T) - F_0(T)\right] \end{aligned} \quad (4)$$

The forward value is equal to the present value of the difference in forward prices. The benefits and costs are reflected in this valuation equation because they are incorporated in the forward price: $F_t(T) = FV_{t,T}(S_t + \theta_t - \gamma_t)$. Again, the forward value is simply the present value of the difference in forward prices.

Before examining equity, interest rate, fixed-income bond, and currency underlyings, we review an important technical issue related to compounding convention. Assume the underlying is a common stock quoted in euros (€) with an initial price of €100 (S_0 = €100), the European risk-free interest rate is 5% (r = 0.05, annual

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compounding), T = 1 year, and the known dividend payment in t = 0.5 years is γ_t = €2.9277 or in future value terms is γ_T = €3.0. As illustrated previously, the no-arbitrage forward price is €102, which is determined as follows:

$$\begin{aligned} F_0(T) &= FV_{0,T}(S_0 + \theta_0 - \gamma_0) \\ &= [100 + 0 - 2.9277/(1 + 0.05)^{0.5}](1 + 0.05)^1 \\ &= 105 - 3 = €102 \end{aligned}$$

Recall that γ_0 denotes the present value of carry benefits. In this case, the carry benefits are not paid until t = 0.5; hence, discounting is required. Thus, $\gamma_0 = 2.9277/(1 + 0.05)^{0.5} = 2.8571$.

Now let us consider stock indexes, such as the EURO STOXX 50 or the US Russell 3000. With stock indexes, it is difficult to account for the numerous dividend payments paid by underlying stocks that vary in timing and amount. Dividend index point is a measure of the quantity of dividends attributable to a particular index. It is a useful measure of the amount of dividends paid; a very useful number for arbitrage trading. To simplify the problem, a continuous dividend yield is often assumed. What this means is that it is assumed that dividends accrue continuously over the period in question rather than on specific discrete dates, which is not an unreasonable assumption for an index with a large number of component stocks.

Before turning to this carry arbitrage model variation, we will review continuous compounding in general, based on the previous example, because it is a perennial source of confusion. The equivalence between annual compounding and continuous compounding can be expressed as $(1 + r)^T = e^{r_c T}$ or $r_c = \ln[(1 + r)^T]/T = \ln(1 + r)$;[8] "ln" refers to the natural log of the function. Note that in the marketplace, zero coupon bond prices or bank deposit amounts are the underlying instrument and interest rates are derived from prices. Though we often refer to these instruments in terms of quoted rates, ultimately investors are concerned with the resulting cash flows. Therefore, if the quoted interest rate is 5% based on annual compounding as shown in the previous example, then we can solve for the implied interest rate based on continuous compounding, or $r_c = \ln(1 + r) = \ln(1 + 0.05) = 0.0488$, or 4.88%. In most cases, the context makes clear when the rate being used is continuous; hence, we use the subscript c only when clarity is required.

We see that compounding continuously results in a lower quoted rate. What this implies is that a cash flow compounded at 5% annually is equivalent to being compounded at 4.88% continuously. Based on the information in the previous example, the implied dividend yield can be derived. Specifically, the carry arbitrage model with continuous compounding is again the future value of the underlying adjusted for carry and can be expressed as

$$F_0(T) = S_0 e^{(r_c + \theta - \gamma)T} \text{ (Future value of the underlying adjusted for carry)}$$

Note that in this context r_c, θ, and γ are continuously compounded rates.

The carry arbitrage model can also be used when the underlying requires storage costs, needs to be insured, and suffers from spoilage. In these cases, rather than lowering the carrying burden, these costs make it more costly to carry and hence the forward price is higher.

We now apply these results to equity forward and futures contracts.

8 Recall that $\ln(a^x) = x\ln(a)$. Thus, $\ln[(1 + r)^T]/T = \ln(1 + r)$ and time to maturity does not influence this conversion from annual to continuous rates.

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3.3 Equity Forward and Futures Contracts

Although we alluded to equity forward pricing and valuation in the last section, we illustrate with concrete examples the application of carry arbitrage models to equity forward and futures contracts. Remember that here we assume that forward contracts and futures contracts are priced in the same way. It is vital to treat the compounding convention of interest rates appropriately.

If the underlying is a stock, then the carry benefit is the dividend payments as illustrated in the next two examples.

EXAMPLE 3

Equity Futures Contract Price with Continuously Compounded Interest Rates

The continuously compounded dividend yield on the EURO STOXX 50 is 3%, and the current stock index level is 3,500. The continuously compounded annual interest rate is 0.15%. Based on the carry arbitrage model, the three-month futures price will be *closest* to:

A 3,473.85.

B 3,475.15.

C 3,525.03.

Solution:

B is correct. Based on the carry arbitrage model, the forward price is $F_0(T) = S_0 e^{(r_c - \gamma)T}$. The future value of the underlying adjusted for carry, i.e., the dividend payments, over the next year would be $3{,}500e^{(0.0015-0.03)(3/12)} = 3{,}475.15$.

EXAMPLE 4

Equity Forward Pricing and Forward Valuation with Discrete Dividends

Suppose Nestlé common stock is trading for CHF70 and pays a CHF2.20 dividend in one month. Further, assume the Swiss one-month risk-free rate is 1.0%, quoted on an annual compounding basis. Assume that the stock goes ex-dividend the same day the single stock forward contract expires. Thus, the single stock forward contract expires in one month.

The one-month forward price for Nestlé common stock will be *closest* to:

A CHF67.80.

B CHF67.86.

C CHF69.94.

Solution:

B is correct. In this case, we have $S_0 = 70$, $r = 1.0\%$, $T = 1/12$, and $\gamma_T = 2.2$. Therefore, $F_0(T) = FV_{0,T}(S_0 + \theta_0 - \gamma_0) = FV_{0,T}(S_0) + FV_{0,T}(\theta_0) - FV_{0,T}(\gamma_0) = 70(1 + 0.01)^{1/12} + 0 - 2.2 = \text{CHF}67.86$.

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The value of an equity forward contract entered earlier is simply the present value of the difference in the initial forward price and the current forward price as illustrated in the next example.

EXAMPLE 5

Equity Forward Valuation

Suppose we bought a one-year forward contract at 102 and there are now three months to expiration. The underlying is currently trading for 110, and interest rates are 5% on an annual compounding basis.

1 If there are no other carry cash flows, the forward value of the existing contract will be *closest* to:

A –10.00.

B 9.24.

C 10.35.

2 If a dividend payment is announced between the forward's valuation and expiration dates, assuming the news announcement does not change the current underlying price, the forward value will *most likely*:

A decrease.

B increase.

C be the same.

Suppose that instead of buying a forward contract, we buy a one-year *futures* contract at 102 and there are now three months to expiration. Today's futures price is 112.35. There are no other carry cash flows.

3 After marking to market, the futures value of the existing contract will be *closest* to:

A –10.35.

B 0.00.

C 10.35.

4 Compared to the value of a forward contract, the value of a futures contract is *most likely*:

A lower.

B higher.

C the same.

Solution to 1:

B is correct. For this case, we have $F_0(T) = 102$, $S_{0.75} = 110$, $r = 5\%$, and $T - t = 0.25$. Note that the new forward price at t is simply $F_t(T) = FV_{t,T}(S_t) = 110(1 + 0.05)^{0.25} = 111.3499$. Therefore, we have

$$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)] = (111.3499 - 102)/(1 + 0.05)^{0.25} = 9.2365.$$

Thus, we see that the current forward value is greater than the difference between the current underlying value of 110 and the initial forward price of 102 as a result of interest costs resulting in the new forward price being 111.35.

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Solution to 2:

A is correct. The old forward price is fixed. The discounted difference in the new forward price and the old forward price is the value. If we impose a new dividend, it would lower the new forward price and thus lower the value of the old forward contract.

Solution to 3:

B is correct. Futures contracts are marked to market daily, which implies that the market value, resulting in profits and losses, is received or paid at each daily settlement. Hence, the equity futures value is zero each day after settlement has occurred.

Solution to 4:

A is correct. After marking to market, the futures contract value is zero because profits and losses are taken daily. Thus, because we are long the futures or forward contract and the price has risen, the futures value will be lower than the forward value.

We turn now to the widely used interest rate forward and futures contracts.

3.4 Interest Rate Forward and Futures Contracts

Libor, which stands for London Interbank Offered Rate, is a widely used interest rate that serves as the underlying for many derivative instruments. It represents the rate at which London banks can borrow from other London banks. When these loans are in dollars, they are known as Eurodollar time deposits, with the rate referred to as dollar Libor. There are, however, Libor rates for all major non-dollar currencies. Average Libor rates are derived and posted each day at 11:30 a.m. London time. Lenders and participants in the interest rate derivatives market use these posted Libor rates to determine the interest payments on loans and the payoffs of various derivatives.[9] In addition to this London spot market, there are active forward and futures markets for derivatives based on Libor. Our focus will be on forward markets, as represented by forward rate agreements. In order to understand the forward market, however, let us first look at the Libor spot market. Assume the following notation:

$L_i(m)$ = Libor on an m-day deposit observed on day i
NA = notional amount, quantity of funds initially deposited
NTD = number of total days in a year, used for interest calculations (always 360 in the Libor market)
t_m = accrual period, fraction of year for m-day deposit—t_m = m/NTD
TA = terminal amount, quantity of funds repaid when the Libor deposit is withdrawn

For example, suppose day i is designated as Time 0, and we are considering a 90-day Eurodollar deposit (m = 90). Dollar Libor is quoted at 2%; thus, $L_i(m) = L_0(90) = 0.02$. If \$50,000 is initially deposited, then NA = \$50,000. Libor is stated on an actual over 360-day count basis (often denoted ACT/360) with interest paid on an add-on

9 In 2008, financial regulators and many market participants began to suspect that the daily quoted Libor, which was compiled by the British Bankers Association (BBA), was being manipulated by certain banks that submitted their rates to the BBA for use in determining this average. In 2014, the BBA ceded control of the daily Libor reporting process to the Intercontinental Exchange.

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basis.[10] Hence, $t_m = 90/360 = 0.25$. Accordingly, the terminal amount can be expressed as $TA = NA[1 + L_0(m)t_m]$, and the interest paid is thus $TA - NA = NA[L_0(m)t_m]$. In this example, TA = \$50,000[1 + 0.02(90/360)] = \$50,250 and the interest is \$50,250 – \$50,000 = \$250.

Now let us turn to the forward market for Libor. A **forward rate agreement** (FRA) is an over-the-counter (OTC) forward contract in which the underlying is an interest rate on a deposit. An FRA involves two counterparties: the fixed receiver (short) and the floating receiver (long). Thus, being long the FRA means that you gain when Libor rises. The fixed receiver counterparty receives an interest payment based on a fixed rate and makes an interest payment based on a floating rate. The floating receiver counterparty receives an interest payment based on a floating rate and makes an interest payment based on a fixed rate. If we are the fixed receiver, then it is understood without saying that we also are the floating payer, and vice versa. Because there is no initial exchange of cash flows, to eliminate arbitrage opportunities, the FRA price is the fixed interest rate such that the FRA value is zero on the initiation date.

FRAs are identified in the form of "X × Y," where X and Y are months and the multiplication symbol, ×, is read as "by." To grasp this concept and the notion of exactly what is the underlying in an FRA, consider a 3 × 9 FRA, which is pronounced "3 by 9." The 3 indicates that the FRA expires in three months. The underlying is implied by the difference in the 3 and the 9. That is, the payoff of the FRA is determined by six-month Libor when the FRA expires in three months. The notation 3 × 9 is market convention, though it can seem confusing at first. We will see shortly that the rate on the FRA will be determined by the relationship between the spot rate on a nine-month Libor deposit and the spot rate on a three-month deposit when the FRA is initiated. Going short (long) a 3 × 9 FRA effectively replicates going short (long) a nine-month Libor deposit and going long (short) a three-month Libor deposit. And although market convention quotes the time periods as months, the calculations use days based on the assumption of 30 days in a month.

The contract established between the two counterparties settles in cash the difference between a fixed interest payment established on the initiation date and a floating interest payment established on the FRA expiration date. The underlying of an FRA is neither a financial asset nor even a financial instrument; it is just an interest payment. It is also important to understand that the parties to an FRA are not necessarily engaged in a Libor deposit in the spot market. The Libor spot market is simply the benchmark from which the payoff of the FRA is determined. Although a party may use an FRA in conjunction with a Libor deposit, it does not have to do so any more than a party that uses a forward or futures on a stock index has to have a position in the stock index.

In Exhibit 8, we illustrate the key time points in an FRA transaction. The FRA is created and priced at Time 0, the initiation date, and expires h days later. The underlying instrument has m days to maturity as of the FRA expiration date. Thus, the FRA is on m-day Libor. We assume there is a point during the life of the FRA, day g, at which we wish to determine the value of the FRA. So, for example, a 30-day FRA on 90-day Libor would have h = 30, m = 90, and h + m = 120. If we wanted to value the FRA prior to expiration, g could be any day between 0 and 30. The FRA value is the market value on the evaluation date and reflects the fair value of the original position.

10 The add-on basis is one way to quote interest rates and the convention in the Libor market. The idea is that the interest is added on at the end—in contrast, for example, to the discount basis, in which the current price is discounted based on the amount paid at maturity.

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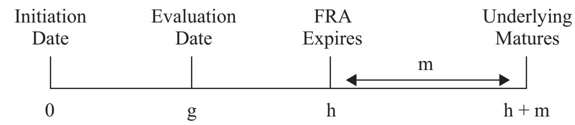
Exhibit 8 Important FRA Dates, Expressed in Days from Initiation



Using the notation in Exhibit 8, let FRA(0,h,m) denote the fixed forward rate set at Time 0 that expires at Time h wherein the underlying Libor deposit has m days to maturity at expiration of the FRA. Thus, the rate set at initiation of a contract expiring in 30 days in which the underlying is 90-day Libor is denoted FRA(0,30,90) and will be a number, such as 1% or 2.5%. Like all standard forward contracts, no money changes hands when an FRA is initiated, so our objective is to price the FRA, meaning to determine the fixed rate [FRA(0,30,90)] such that the value is zero on the initiation date.

When any interest rate derivative expires, there are technically two ways to settle at expiration: "advanced set, settled in arrears" and "advanced set, advanced settled." FRAs are typically settled based on advanced set, advanced settled, whereas swaps and interest rate options are normally based on advanced set, settled in arrears. Let us look at both approaches, because they are both used in the interest rate derivatives markets.

In the earlier example of a Libor deposit of \$50,000 for 90 days at 2%, the rate was set when the money was deposited, interest accrued over the life of the deposit, and the interest was paid and the principal of \$50,250 was repaid at maturity, 90 days later. The term **advanced set** is used because the reference interest rate is set at the time the money is deposited. The advanced set convention is almost always used, because most issuers and buyers of financial instruments want to know the rate on the instrument while they have a position in it.

In an FRA, the term "advanced" refers to the fact that the interest rate is set at Time h, the FRA expiration date, which is the time the underlying deposit starts. The term **settled in arrears** is used when the interest payment is made at Time h + m, the maturity of the underlying instrument. Thus, an FRA with advanced set, settled in arrears works the same way as a typical bank deposit as described in the previous example. At Time h, the interest rate is set, and the interest payment is made at Time h + m. Alternatively, when **advanced settled** is used, the settlement is made at Time h. Thus, in a FRA with the advanced set, advanced settled feature, the FRA expires and settles at the same time. Advanced set, advanced settled is almost always used in FRAs, though we will see advanced set, settled in arrears when we cover interest rate swaps, and it is also used in interest rate options. From this point forward in this reading, all FRAs will be advanced set, advanced settled, as they are in practice.

Mathematically, the settlement amounts for advanced set, advanced settled are determined in the following manner:

Settlement amount at h for receive-floating:

$$\mathrm{NA}\left\{\left[\mathrm{L_h(m)} - \mathrm{FRA(0,h,m)}\right]\mathrm{t_m}\right\}\Big/\left[1 + \mathrm{D_h(m)t_m}\right]$$

Settlement amount at h for receive-fixed:

$$\mathrm{NA}\left\{\left[\mathrm{FRA(0,h,m)} - \mathrm{L_h(m)}\right]\mathrm{t_m}\right\}\Big/\left[1 + \mathrm{D_h(m)t_m}\right]$$

Note the divisor, $1 + D_h(m)t_m$. This term is a discount factor applied to the FRA payoff. It reflects the fact that the rate on which the payoff is determined, $L_h(m)$, is obtained on day h from the Libor spot market, which uses settled in arrears. In the Libor spot market, this rate assumes that a Libor deposit has been made on day h at this rate with interest to be paid on day h + m—that is, settled in arrears. In the FRA market, the payment convention is advanced settle. The discount factor is, therefore,

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appropriately applied to the FRA payment because the payment is received in advance, not in arrears. Often it is assumed that $D_h(m) = L_h(m)$ and we will commonly do so here, but it can be different.[11]

Again, it is important to not be confused by the role played by the Libor spot market in an FRA. In the spot market, Libor deposits are made by various parties that are lending to banks. These rates are used as the benchmark for determining the payoffs of FRAs. The two parties to an FRA do not necessarily engage in any Libor spot transactions. Moreover, Libor spot deposits are settled in arrears, whereas FRA payoffs are settled in advance—hence the discounting.

EXAMPLE 6

Calculating Interest on Libor Spot and FRA Payments

In 30 days, a UK company expects to make a bank deposit of £10,000,000 for a period of 90 days at 90-day Libor set 30 days from today. The company is concerned about a possible decrease in interest rates. Its financial adviser suggests that it negotiate today, at Time 0, a 1 × 4 FRA, an instrument that expires in 30 days and is based on 90-day Libor. The company enters into a £10,000,000 notional amount 1 × 4 receive-fixed FRA that is advanced set, advanced settled. The appropriate discount rate for the FRA settlement cash flows is 0.40%. After 30 days, 90-day Libor in British pounds is 0.55%.

1 The interest actually paid at maturity on the UK company's bank deposit will be *closest* to:

   A £10,000.

   B £13,750.

   C £27,500.

2 If the FRA was initially priced at 0.60%, the payment received to settle it will be *closest* to:

   A –£2,448.75.

   B £1,248.75.

   C £1,250.00.

3 If the FRA was initially priced at 0.50%, the payment received to settle it will be *closest* to:

   A –£1,248.75.

   B £1,248.75.

   C £1,250.00.

Solution to 1:

B is correct. This is a simple Libor deposit of £10,000,000 for 90 days at 0.55%. Therefore, TA = 10,000,000[1 + 0.0055(0.25)] = £10,013,750. So the interest paid at maturity is £13,750.

11 For example, there is a current debate on whether the overnight index swap (OIS) rate is the appropriate discount rate for financial derivatives. Because Libor and the OIS rate are different, we need the capacity to incorporate different rates for the reference rate for settlement and the discount rate for valuation. We do not seek to resolve this debate here. Historically, there have been several candidate discount rates offered, and the popularity of each rate changes over time.

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Solution to 2:

B is correct. In this example, m = 90 (number of days in the deposit), t_m = 90/360 (fraction of year until deposit matures observed at the FRA expiration date), and h = 30 (number of days initially in the FRA). The settlement amount of the 1 × 4 FRA at h for receive-fixed is

$$NA\{[FRA(0,h,m) - L_h(m)]t_m\}/[1 + D_h(m)t_m]$$

$$= [10{,}000{,}000(0.0060 - 0.0055)(0.25)]/[1 + 0.0040(0.25)] = £1{,}248.75.$$

Because the FRA involves paying floating, its value benefited from a decline in rates.

Solution to 3:

A is correct. The data are similar to those in the previous question, but the initial FRA rate was 0.50% and not 0.60%. Thus, the settlement amount of the 1 × 4 FRA at h for receive-fixed is

$$NA\{[FRA(0,h,m) - L_h(m)]t_m\}/[1 + D_h(m)t_m]$$

$$= [10{,}000{,}000(0.0050 - 0.0055)(0.25)]/[1 + 0.0040(0.25)] = -£1{,}248.75$$

The FRA suffered from a rise in rates because it is again paying floating.

With this background, we turn to FRA pricing by illustrating the appropriate FRA(0,h,m) rate that makes the value of the FRA equal to zero on the initiation date. For our purposes, we assume that borrowing and lending can be done at Libor. Also, the notional amount is assumed to be one unit of the designated currency: NA = 1. Finally, we will assume that the discount rate on the FRA settlement is the FRA rate at that point in time.

Consider the following no-arbitrage strategy, depicted in Exhibit 9, in which numerical values are also provided as an aid to understanding the concepts. We illustrate a 3 × 6 FRA for which NA = 1, h = 90, m = 90, t_h = 90/360, $L_0(h) = L_0(90)$ = 1.5%, t_{h+m} = 180/360, $L_0(h + m) = L_0(180)$ = 2.0%, and t_m = 90/360. That is, today 90-day Libor is 1.5% and 180-day Libor is 2%. First, consider the following three arbitrage-related transactions all done at Time 0:

Step 1 Deposit funds for h + m days: At Time 0, deposit an amount equal to $1/[1 + L_0(h)t_h]$, the present value of 1 maturing in h days, in a bank for h + m days at an agreed upon rate of $L_0(h + m)$. After h + m days, withdraw an amount equal to $[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h]$. Based on the data provided, the deposit amount is 1/[1 + 0.015(90/360)] = 0.996264. After h + m days, the withdrawn amount is equal to 0.996264[1 + 0.02(180/360)] = 1.006227. In other words, deposit 0.996264 for 180 days at 2%. One hundred eighty days later, withdraw 1.006227.

Step 2 Borrow funds for h days: At Time 0, borrow 0.996264, corresponding to $\{1/[1 + L_0(h)t_h]\}$, for h days so that the net cash flow at Time 0 is zero. In h days, this borrowing will be worth 1. In other words, borrow 0.996264 for 90 days at 1.5%. In 90 days, pay back 1.

Step 3 At Time h, roll over the maturing loan in Step 2 by borrowing funds for m days at the rate $L_h(m)$. Assume rates rise and $L_h(m)$ = 3.0%. Then at the end of m days, we will owe $[1 + L_h(m)t_m]$ = [1 + 0.03(90/360)] = 1.0075.

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Recall the two rules of the arbitrageur: Rule #1: Do not use our own money. Rule #2: Do not take any price risk. In the transactions above, Rule #1 is satisfied. Unfortunately, Rule #2 is not satisfied because the future value at Time h + m of the borrowed cash flows may be more than the asset cash flows. Note that the risk is that the rate $L_h(m)$ will cause us to roll over the loan in Step 2 at a higher rate that more than offsets the gain from the loan we make in Step 1. This is the case here, because we will owe 1.0075 at the end of period m (Step 3) but will receive only 1.006227 from Step 1 if interest rates go up at Time h to 3%.

This risk can be eliminated by entering a receive-floating FRA on m-day Libor that expires at Time h and has the rate set at FRA(0,h,m). Now assume we roll the FRA payoff forward from h to h + m by investing any gain or borrowing to cover any loss at the rate $L_h(m)$. Let us assume the discount factor in the FRA payoff formula is $1 + L_h(m)t_m$. We see in Exhibit 9 that the following transaction enables us to satisfy Rule #2.

Step 4 Enter a receive-floating FRA and roll the payoff at h to h + m at the rate $L_h(m)$. The payoff at Time h will be $([L_h(m) - \text{FRA}(0,h,m)]t_m)/(1 + L_h(m)t_m)$. There will be no cash flow from this FRA at Time h because this amount will be rolled forward at the rate $L_h(m)t_m$. Therefore, the value realized at Time h + m will be $[L_h(m) - \text{FRA}(0,h,m)]t_m$.

Exhibit 9 Cash Flow Table for Deposit and Lending Strategy with FRA

Steps	Cash Flow at Time 0	Cash Flow at Time h	Cash Flow at Time h + m
1. Make deposit for h + m days	$-1/[1 + L_0(h)t_h]$ $= -0.996264$	0	$+[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h]$ $= 1.006227$
2. Borrow funds for h days	$+1/[1 + L_0(h)t_h]$ $= +0.996264$	–1	
3. Borrow funds for m days initiated at h		+1	$-[1 + L_h(m)t_m] = -1.0075$
4. Receive-floating FRA and roll payoff at $L_h(m)$ rate from h to h + m	0	0	$+[L_h(m) - \text{FRA}(0, h, m)]t_m$ $= [0.03 - \text{FRA}(0,h,m)](90/360)$
Net cash flows	0	0	$+[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h] -$ $[1 + L_h(m)t_m]$ $+[L_h(m) - \text{FRA}(0,h,m)]t_m$

Recall that the goal is to identify the appropriate FRA(0,h,m) rate that makes the value of the FRA equal to zero on the initiation date. The terminal cash flows as expressed in the table can be used to solve for the FRA fixed rate. Because the transaction starts off with no initial investment or receipt of cash, the net cash flows at Time h + m should equal zero; thus,

$$+\left[1 + L_0(h + m)t_{h+m}\right]/\left[1 + L_0(h)t_h\right] - \left[1 + L_h(m)t_m\right] + \left[L_h(m) - \text{FRA}(0,h,m)\right]t_m = 0$$

Solving for the FRA fixed rate, we have

$$\text{FRA}(0,h,m) = \left\{\left[1 + L_0(h + m)t_{h+m}\right]/\left[1 + L_0(h)t_h\right] - 1\right\}/t_m \tag{5}$$

This equation looks complex, but it is really quite simple. In fact, it may well be quite familiar. It is essentially the compound value of $1 invested at the longer-term Libor for h + m days divided by the compound value of $1 invested at the shorter-term Libor for h days minus 1 and then annualized. The result is simply *the forward rate in the Libor term structure*. Recall that with simple interest, a one-period forward rate

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is found by solving the expression $[1 + y(1)][1 + f(1)] = [1 + y(2)]^2$, where y denotes the one- and two-period yield to maturity and f denotes the forward rate in the next period. The equation above is similar but simply addresses the unique features of add-on interest rate calculations. Based on the numbers used in the previous two tables, we note

$$\begin{aligned} FRA(0,90,90) &= \{[1 + L_0(180)t_{180}]/[1 + L_0(90)t_{90}] - 1\}/t_{90} \\ &= \{[1 + 0.02(180/360)]/[1 + 0.015(90/360)] - 1\}/(90/360) \\ &= 0.024907 \text{ or } 2.49\%.^{12} \end{aligned}$$

EXAMPLE 7

FRA Fixed Rate

Based on market quotes on Canadian dollar (C$) Libor, the six-month C$ Libor and the nine-month C$ Libor are presently at 1.5% and 1.75%, respectively. Assume a 30/360-day count convention. The 6 × 9 FRA fixed rate will be *closest* to:

A 2.00%.

B 2.23%.

C 2.25%.

Solution:

B is correct. Based on the information given, we know L(180) = 1.5% and L(270) = 1.75%. The 6 × 9 FRA rate is thus

$$FRA(0,h,m) = \{[1 + L_0(h + m)t_{h+m}]/[1 + L_0(h)t_h] - 1\}/t_m$$

$$FRA(0,180,90) = \{[1 + 0.0175(270/360)]/[1 + 0.015(180/360)] - 1\}/(90/360)$$

$$FRA(0,180,90) = [(1.013125/1.0075) - 1]4 = 0.0223325, \text{ or } 2.23\%$$

We can now value an existing FRA using the same general approach as we did with the forward contracts previously covered; specifically, we can enter into an offsetting transaction at the new rate that would be set on an FRA that expires at the same time as our original FRA. By taking the opposite position, the new FRA offsets the old one. That is, if we are long the old FRA, we will receive the rate $L_h(m)$ at h. We will go short a new FRA that will force us to pay $L_h(m)$ at h. Consider the following strategy, illustrated in Exhibit 10, in which we again assume that NA = 1. Let us assume that we initiate an FRA that expires in 90 days and is based on 90-day Libor. The fixed rate at initiation is 2.49%. Thus, t_m = 90/360, and FRA(0,h,m) = FRA(0,90,90) = 2.49%. When the FRA expires and makes its payoff, assume that we do not collect or pay the payoff; instead, we roll it forward by lending it (if a gain) or borrowing it (if a loss) from period h to period h + m at the rate $L_h(m)$. We then collect or pay the rolled forward value at h + m. Thus, there is no cash realized at Time h.

Now having entered into the long FRA with the intention of rolling the payoff forward, let us now position ourselves 30 days later, at Time g, at which there are 60 days remaining in the life of the FRA. Assume that at this point, the rate on an

12 The result given in this example can be compared with the result from a simple approximation technique. Notice that for this FRA, 90 is half of 180. Thus, we can use the simple arithmetic average equation—here, (1/2)1.5% + (1/2)X = 2.0%—and solve for the missing variable X: X = 2.5%. Knowing this approximation will always be biased slightly high, we know we are looking for an answer that is a little less than 2.5%. This is a nice way to check your final answer.

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FRA based on 90-day Libor that expires in 60 days is 2.59%. Thus, FRA(g,h – g,m) = FRA(30,60,90) = 2.59%. We go short this FRA, and as with the long FRA, we roll forward its payoff from Time h to h + m. Therefore, there is no cash realized from this FRA at Time h. This strategy is illustrated in Exhibit 10.

Exhibit 10 Cash Flows for FRA Valuation

Steps	Cash Flow at Time g	Cash Flow at Time h	Cash Flow at Time h + m
1. Receive-floating FRA (settled in arrears) at Time 0; roll forward at Rate $L_h(m)$ from h to h + m		0	$+\{[L_h(m) - FRA(0,h,m)]t_m\}$ $= +(L_h(m) - 0.0249)(90/360)$
2. Receive-fixed FRA (settled in arrears) at Time g; roll forward at Rate $L_h(m)$ from h to h + m	0	0	$+[FRA(g,h-g,m) - L_h(m)]t_m$ $= +[0.0259 - L_h(m)](90/360)$
Net cash flows	0	0	$+[FRA(g,h-g,m) - FRA(0,h,m)]t_m$ $= +(0.0259 - 0.0249)(90/360)$ $= 0.00025$

To recap, the original FRA that we wish to value had its fixed rate set at 2.49% when it was initiated. Now, 30 days later, a new offsetting FRA can be created at 2.59%. The value of the offset position is 10 bps (2.59% – 2.49%) times 90/360 paid at Time h + m, assuming we roll the FRA payoffs forward. We will receive this amount at h + m, so it must be discounted back to Time g in order to obtain the value.

Because the cash flows at h + m are now known with certainty at g, this offsetting transaction at Time g has completely eliminated all of the risk at Time h + m. Our task, however, is to determine the fair value of the original FRA at Time g. Therefore, we need the present value of this Time h + m cash flow at Time g. That is, the value of the old FRA is the present value of the difference in the new FRA rate and the old FRA rate. Specifically, we let $V_g(0,h,m)$ be the value of the FRA at Time g that was initiated at Time 0, expires at Time h, and is based on m-day Libor. Note that discounting will be over the period h + m – g. With $D_g(h + m - g)$ as the discount rate, the value is

$$V_g(0,h,m) = \left\{\left[FRA(g,h-g,m) - FRA(0,h,m)\right]t_m\right\} / \left[1 + D_g(h+m-g)t_{h+m-g}\right] \quad (6)$$

where the new FRA rate is the formula we previously learned, simply applied to this new offsetting transaction:

$$FRA(g,h-g,m) = \{[1 + L_g(h+m-g)t_{h+m-g}]/[1 + L_g(h-g)t_{h-g}] - 1\}/t_m$$

Thus, the date g value of the receive-floating FRA initiated at date 0 is merely the present value of the difference in FRA rates, one entered on date g and one entered on date 0. Traditionally, it is assumed that the discount rate, $D_g(h + m - g)$, is equal to the underlying floating rate, $L_g(h + m - g)$, but that is not necessary.[13] Let us assume a 150-day rate of 3% on day g. Thus, $L_g(h + m - g) = L_{30}(150) = 3\%$. Then the value of the FRA would be

$$V_g(0,h,m) = V_{30}(0,90,90) = 0.00025/[1 + 0.03(150/360)] = 0.000247.$$

13 Again, there is a current debate on whether the OIS rate should be used for discounting; hence, we may have a different discount rate, but in any case, that rate would be known at time g.

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And of course, this amount is per notional of 1. Thus, the answer found here must be multiplied by the actual notional amount as demonstrated in the following example.

EXAMPLE 8

FRA Valuation

Suppose we entered a receive-floating 6 × 9 FRA at a rate of 0.86%, with notional amount of C$10,000,000 at Time 0. The six-month spot Canadian dollar (C$) Libor was 0.628%, and the nine-month C$ Libor was 0.712%. Also, assume the 6 × 9 FRA rate is quoted in the market at 0.86%. After 90 days have passed, the three-month C$ Libor is 1.25% and the six-month C$ Libor is 1.35%, which we will use as the discount rate to determine the value at g. We have h = 180 and m = 90.

Assuming the appropriate discount rate is C$ Libor, the value of the original receive-floating 6 × 9 FRA will be *closest* to:

A C$14,500.

B C$14,625.

C C$14,651.

Solution:

C is correct. Initially, we have $L_0(h) = L_0(180) = 0.628\%$, $L_0(h + m) = L_0(270) = 0.712\%$, and FRA(0,180,90) = 0.86%. After 90 days (g = 90), we have $L_g(h - g) = L_{90}(90) = 1.25\%$ and $L_g(h + m - g) = L_{90}(180) = 1.35\%$. Interest rates rose during this period; hence, the FRA likely has gained value because the position is receive-floating. First, we compute the new FRA rate at Time g and then estimate the fair FRA value as the discounted difference in the new and old FRA rates. The new FRA rate at Time g, denoted FRA(g,h − g,m) = FRA(90,90,90), is the rate on day 90 of an FRA to expire in 90 days in which the underlying is 90-day Libor. That rate is found as

$$\begin{aligned} \text{FRA}(g,h-g,m) &= \text{FRA}(90,90,90) \\ &= \{[1 + L_g(h+m-g)t_{h+m-g}]/[1 + L_g(h-g)t_{h-g}] - 1\}/t_m, \end{aligned}$$

and based on the information in this example, we have

$$\begin{aligned} \text{FRA}(90,90,90) = \{&[1 + L_{90}(180 + 90 - 90)(180/360)]/[1 + L_{90}(180 - 90) \\ &(90/360)] - 1\}/(90/360). \end{aligned}$$

Substituting the values given in this problem, we find

$$\begin{aligned} \text{FRA}(90,90,90) = \{&[1 + 0.0135(180/360)]/[1 + 0.0125(90/360)] - 1\}/ \\ &(90/360) = [(1.00675/1.003125) - 1]4 = 0.0145\text{, or } 1.45\%. \end{aligned}$$

Therefore,

$$\begin{aligned} V_g(0,h,m) &= V_{90}(0,180,90) \\ &= 10{,}000{,}000[(0.0145 - 0.0086)(90/360)]/[1 + 0.0135(180/360)] \\ &= 14{,}651. \end{aligned}$$

Again, floating rates rose during this period; hence, the FRA enjoyed a gain. Notice that the FRA rate rose by roughly 59 bps (= 145 − 86), and 1 bp for 90-day money and a 1,000,000 notional amount is 25. Thus, we can also estimate

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the terminal value as 10 × 25 × 59 = 14,750. As with all fixed-income strategies, understanding the value of a basis point is often helpful when estimating profits and losses and managing the risks of FRAs.

We now turn to the specific features of various forward and futures markets. The same general principles will apply, but the specifics will be different.

3.5 Fixed-Income Forward and Futures Contracts

Fixed-income forward and futures contracts have several unique issues that influence the specifics of the carry arbitrage model. First, in some countries the prices of fixed-income securities (termed "bonds" here) are quoted without the interest that has accrued since the last coupon date. The quoted price is sometimes known as the clean price. Naturally, when buying a bond, one must pay the full price, which is sometimes called the dirty price, so the accrued interest is included. Nonetheless, it is necessary to understand how the quoted bond price and accrued interest compose the true bond price and the effect this convention has on derivative pricing. The quote convention for futures contracts, whether based on clean or dirty prices, usually corresponds to the quote convention in the respective bond market. In this section, we will largely treat forwards and futures the same, except in certain places where noted.

In general, accrued interest is computed based on the following linear interpolation formula:

Accrued interest = Accrual period × Periodic coupon amount, or

AI = (NAD/NTD) × (C/n)

where NAD denotes the number of accrued days since the last coupon payment, NTD denotes the number of total days during the coupon payment period, n denotes the number of coupon payments per year, and C is the stated annual coupon amount. For example, after two months (60 days), a 3% semi-annual coupon bond with par of 1,000 would have accrued interest of AI = (60/180) × (30/2) = 5. Note that accrued interest is expressed in currency (not percent) and the number of total days (NTD) depends on the coupon payment frequency (semi-annual on 30/360 day count convention would be 180).

Second, fixed-income futures contracts often have more than one bond that can be delivered by the seller. Because bonds trade at different prices based on maturity and stated coupon, an adjustment known as the conversion factor is used in an effort to make all deliverable bonds roughly equal in price.

Third, when multiple bonds can be delivered for a particular maturity of a futures contract, a cheapest-to-deliver bond typically emerges after adjusting for the conversion factor. The conversion factor is a mathematical adjustment to the amount required when settling a futures contract that is supposed to make all eligible bonds equal the same amount. For example, the conversion factor may seek to adjust each bond to an equivalent 6% coupon bond. The conversion factor adjustment, however, is not precise. Thus, the seller will deliver the bond that is least expensive.

For bond markets in which the quoted price includes the accrued interest and in which futures or forward prices assume accrued interest is in the bond price quote, the futures or forward price simply conforms to the general formula we have previously discussed. Recall that the futures or forward price is simply the future value of the underlying in which finance costs, carry costs, and carry benefits are all incorporated or

$$\begin{aligned} F_0(T) &= \text{Future value of underlying adjusted for carry cash flows} \\ &= FV_{0,T}(S_0 + \theta_0 - \gamma_0) \end{aligned}$$

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Again, Time 0 is the forward contract trade initiation date, and Time T is the contract expiration date. For the fixed-income bond, let T + Y denote the underlying instrument's current time to maturity. Therefore, Y is the time to maturity of the underlying bond at Time T, when the contract expires. Let $B_0(T + Y)$ denote the quoted price observed at Time 0 of a fixed-rate bond that matures at Time T + Y and pays a fixed coupon rate. For bonds quoted without accrued interest, let AI_0 denote the accrued interest at Time 0. The carry benefits are the bond's fixed coupon payments, $\gamma_0 = PVCI_{0,T}$, meaning the present value of all coupon interest paid over the forward contract horizon from Time 0 to Time T. The corresponding future value of these coupons is $\gamma_T = FVCI_{0,T}$. Finally, there are no carry costs, and thus $\theta_0 = 0$. To be consistent with prior notation, we have

$$S_0 = \text{Quoted bond price} + \text{Accrued interest} = B_0(T + Y) + AI_0$$

We could just insert this price into the previous equation, letting $\gamma_0 = PVCI_{0,T}$, and thereby obtain the futures price the simple and traditional way. But fixed-income futures contracts often permit delivery of more than one bond and use a conversion factor system to provide this flexibility. In these markets, the futures price, $F_0(T)$, is defined here as the quoted futures price, $QF_0(T)$, times the conversion factor, CF(T). In fact, the futures contract settles against the quoted bond price without accrued interest. Thus, the total profit or loss on a long futures position is $B_T(T + Y) – F_0(T)$. Based on our notation above, we can represent this profit or loss as $(S_T – AI_T) – F_0(T)$. Therefore, the fixed-income forward or futures price including the conversion factor, termed the "adjusted price," can be expressed as[14]

$$\begin{aligned} F_0(T) &= QF_0(T)CF(T) \\ &= \text{Future value of underlying adjusted for carry cash flows} \\ &= FV_{0,T}\left[S_0 - PVCI_{0,T}\right] = FV_{0,T}\left[B_0(T+Y) + AI_0 - PVCI_{0,T}\right] \end{aligned} \tag{7}$$

In other words, the actual futures price is $F_0(T)$, but in the market, the availability of multiple deliverable bonds gives rise to the adjustment factor. Hence, the price you would see quoted is QF_0.

Recall that the bracketed term $B_0(T + Y) + AI_0 – PVCI_{0,T}$ is just the full spot price minus the present value of the coupons over the life of the forward or futures contract. The fixed-income forward or futures price is thus the future value of the quoted bond price plus accrued interest less any coupon payments made during the life of the contract. Again, the quoted bond price plus the accrued interest is the spot price: It is in fact the price you would have to pay to buy the bond. Market conventions in some countries just happen to break this price out into the quoted price plus the accrued interest.

Now let us explore carry arbitrage in the bond market, assuming that accrued interest is broken out and that multiple bonds are deliverable, thereby requiring the use of the conversion factor. Consider the following transactions:

Step 1 Buy the underlying bond, requiring S_0 cash flow.

Step 2 Borrow an amount equivalent to the cost of the underlying bond, S_0.

Step 3 Sell the futures contract at $F_0(T)$.

Step 4 Borrow the arbitrage profit.

Exhibit 11 shows the cash flow consequences for this portfolio in which the futures price is not in equilibrium. Note that $FVCI_{0,T}$ denotes the future value as of Time T of any coupons paid during the life of the futures contract. Again, for illustration

14 In this section, we will use the letter F to denote either the forward price or the futures price times the conversion factor.

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purposes, we provide a numerical example: Suppose T = 0.25, CF(T) = 0.8, $B_0(T + Y)$ = 107 (the quoted price), $FVCI_{0,T}$ = 0.0 (meaning no coupon payments over the life of the contract), AI_0 = 0.07 (the accrued interest at Time 0), AI_T = 0.20 (the accrued interest at Time T), $QF_0(T)$ = 135 (the quoted futures price), and r = 0.2%. Thus, $S_0 = B_0(T + Y) + AI_0 = 107 + 0.07 = 107.07$ (the full or spot price), and $F_0(T) = CF(T) QF_0(T) = 0.8(135) = 108$ (the adjusted price). At Time T, suppose $B_T(T + Y) = 110$ and thus $S_T = B_T(T + Y) + AI_T = 110 + 0.20 = 110.20$. Because $FVCI_{0,T} = 0.0$, there are no coupons paid over the life of the futures. Note that the adjusted price, $F_0(T)$, is 108 whereas the future value adjusted for carry cash flows (Equation 7) is $(107 + 0.07)(1.002)^{0.25} = 107.12$. Adding the accrued interest at expiration (AI_T = 0.20) to the adjusted futures price gives 108.20. The difference between 108.20 and 107.12 is 1.08, which means that the futures contract is overpriced by 1.08. Thus, the arbitrage will involve borrowing the arbitrage profit, which is the present value of 1.08, or 1.0795—that is, $108(1.002)^{-0.25}$.

Exhibit 11 Cash Flows for Fixed Rate Coupon Bond Futures Pricing

Steps	Cash Flow at Time 0	Cash Flow at Time T
1. Buy bond	$-S_0 = -[B_0(T + Y) + AI_0]$ $= -[107 + 0.07]$ $= -107.07$	$S_T + FVCI_{0,T}$ $= 110.20 + 0.0$ $= 110.20$
2. Borrow	$+S_0 = 107.07$	$-FV_{0,T}(S_0)$ $= -(1+0.002)^{0.25}(107.07)$ $= -107.12$
3. Sell futures	0	$F_0(T) - B_T(T + Y)$ $= 108 - 110$ $= -2$
4. Borrow arbitrage profit	$+PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $= (1 + 0.002)^{-0.25}[108 - 107.12 + 0.20 + 0.0]$ $= 1.0795$	$-[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $= -[108 - 107.12 + 0.20 + 0.0]$ $= -1.08$
Net cash flows	$+PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}]$ $=1.0795$	0

Thus, the value of the Time 0 cash flows should be zero or else there is an arbitrage opportunity. The numerical example provided shows a 1.0795 cash flow at Time 0 per bond. If the value in the Time 0 column for net cash flows is positive, then conduct the carry arbitrage of buy bond, borrow, and sell futures (again, termed *carry arbitrage* because the underlying is "carried"). If the Time 0 column is negative, then conduct the reverse carry arbitrage of short sell bond, lend, and buy futures (termed *reverse carry arbitrage* because the underlying is not carried but is sold short).

Thus, in equilibrium, to eliminate an arbitrage opportunity, we expect

$$PV_{0,T}[F_0(T) - FV_{0,T}(S_0) + AI_T + FVCI_{0,T}] = 0$$

or

$$F_0(T) = FV_{0,T}(S_0) - AI_T - FVCI_{0,T}$$

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For clarity, substituting for $F_0(T)$ and S_0 and solving for the quoted futures price, we have

$$\begin{aligned} QF_0(T) &= \text{Conversion factor adjusted future} \\ &\quad \text{value of underlying adjusted for carry} \\ &= \left[1/CF(T)\right]\left\{FV_{0,T}\left[B_0(T+Y) + AI_0\right] - AI_T - FVCI_{0,T}\right\} \end{aligned} \tag{8}$$

In the example above, we have

$$\begin{aligned} QF_0(T) &= [1/CF(T)]\{FV_{0,T}[B_0(T + Y) + AI_0] - AI_T - FVCI_{0,T}\} \\ &= (1/0.8)[(1 + 0.002)^{0.25}(107 + 0.07) - 0.20 - 0.0] = 133.65 \end{aligned}$$

Note that the futures price of 135 used for calculations in Exhibit 11 was higher than the equilibrium futures price of 133.65; hence, the arbitrage transaction of selling the futures contract resulted in a riskless positive cash flow.

EXAMPLE 9

Estimating the Euro-Bund Futures Price

Euro-bund futures have a contract value of €100,000, and the underlying consists of long-term German debt instruments with 8.5 to 10.5 years to maturity. They are traded on the Eurex. Suppose the underlying 2% German bund is quoted at €108 and has accrued interest of €0.083 (one-half of a month since last coupon). The euro-bund futures contract matures in one month. At contract expiration, the underlying bund will have accrued interest of €0.25, there are no coupon payments due until after the futures contract expires, and the current one-month risk-free rate is 0.1%. The conversion factor is 0.729535. In this case, we have $T = 1/12$, $CF(T) = 0.729535$, $B_0(T + Y) = 108$, $FVCI_{0,T} = 0$, $AI_0 = 0.5(2/12) =$ €0.083, $AI_T = 1.5(2/12) = 0.25$, and $r = 0.1\%$. The equilibrium euro-bund futures price based on the carry arbitrage model will be *closest* to:

A €147.57.

B €147.82.

C €148.15.

Solution:

B is correct. The carry arbitrage model for forwards and futures is simply the future value of the underlying with adjustments for unique carry features. With bond futures, the unique features include the conversion factor, accrued interest, and any coupon payments. Thus, the equilibrium euro-bund futures price can be found using the carry arbitrage model in which

$$F_0(T) = FV_{0,T}(S_0) - AI_T - FVCI_{0,T}$$

or

$$QF_0(T) = [1/CF(T)]\{FV_{0,T}[B_0(T + Y) + AI_0] - AI_T - FVCI_{0,T}\}$$

Thus, we have

$$QF_0(T) = [1/0.729535][(1 + 0.001)^{1/12}(108 + 0.083) - 0.25 - 0] = 147.82$$

In equilibrium, the euro-bund futures price should be approximately €147.82 based on the carry arbitrage model.

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Because of the mark-to-market settlement procedure, the value of a bond futures is essentially the price change since the previous day's settlement. That value is captured at the settlement at the end of the day, at which time the value of a bond futures contract, like other futures contracts, is zero.

We now turn to the task of estimating the fair value of the bond forward contract at a point in time during its life. Forwards are not settled daily, so the value is not formally realized until expiration. Suppose the first transaction is buying an at-market bond forward contract at Time 0 with expiration of Time T. Now consider selling a new bond forward contract at Time t again with expiration of Time T. Exhibit 12 shows the potential cash flows. Because this is a bond forward contract, we assume either no conversion factor or effectively a conversion factor of 1. Suppose now $B_T(T + Y) = 108$, $F_0(T) = 107.12$, and $F_t(T) = 107.92$.

Exhibit 12 Cash Flows for Offsetting a Long Forward Position

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Buy bond forward contract at 0	0	$V_t(T)$	$V_T(0,T) = B_T(T + Y) - F_0(T)$ $= 108 - 107.12 = 0.88$
2. Sell bond forward contract at t	NA	0	$V_T(t,T) = F_t(T) - B_T(T + Y)$ $= 107.92 - 108 = -0.08$
Net cash flows	0	$V_t(T)$	$F_t(T) - F_0(T)$ $= 107.92 - 107.12 = 0.8$

Note that the net position from these bond forward transactions is risk free. It is independent of the underlying bond value, $B_T(T + Y)$. Therefore, the forward value observed at Time t of a Time T maturity bond forward contract is simply the present value—denoted $PV_{t,T}()$—of the difference in forward prices. That is,

$$V_t(T) = \text{Present value of difference in forward prices} = PV_{t,T}[F_t(T) - F_0(T)]$$

Based on our example in the table and assuming $T - t = 0.1$ and $r = 0.15\%$, we have $V_t(T) = (107.92 - 107.12)/(1 + 0.0015)^{0.1} = 0.79988$. Note that this is the same result as the generic case with a simple conversion factor adjustment. Recall that the conversion factor is an adjustment to make all bonds roughly equal in value.

EXAMPLE 10

Estimating the Value of a Euro-Bund Forward Position

Suppose that one month ago, we purchased five euro-bund forward contracts with two months to expiration and a contract notional of €100,000 each at a price of 145 (quoted as a percentage of par). The euro-bund forward contract now has one month to expiration. Again, assume the underlying is a 2% German bund quoted at 108 and has accrued interest of 0.0833 (one-half of a month since last coupon). At the contract expiration, the underlying bund will have accrued interest of 0.25, there are no coupon payments due until after the forward contract expires, and the current annualized one-month risk-free rate is 0.1%.

Based on the current forward price of 148, the value of the euro-bund forward position will be *closest* to:

A €2,190.

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B €14,998.

C €15,000.

Solution:

B is correct. Because we are given both forward prices, the solution is simply

$$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)] = (148 - 145)/(1 + 0.001)^{1/12} = 2.9997$$

which is 2.9997 per €100 par value because this forward price was quoted as a percentage of par. Because five contracts each with €100,000 par were entered, we have 0.029997(€100,000)5 = €14,998.50. Note that when interest rates are so low and the forward contract has a short maturity, then the present value effect is minimal.

3.6 Currency Forward and Futures Contracts

Currency derivative contracts require careful attention to the unit of value. For example, if we are discussing bond futures, then the underlying is perceived in currency per unit of par value. If we are trading gold futures, then the quotation will be in currency per troy ounce. If trading a common stock, then it will be in currency per share. When trading currency itself, great care must be taken to know which currency is the base currency. When quoting an exchange rate, we will say that the foreign currency is trading for a certain number of units of domestic currency. For example, we could say, "The euro is trading for $1.30," meaning that €1 is worth $1.30. We use the shorthand notation of DC/FC to refer to the price of one unit of foreign currency expressed in terms of domestic currency units when embedded in an equation.[15] With currency, perspective makes a significant difference. Thus, when pricing and valuing currency forwards and futures contracts, a clear perspective requires considerable care. The carry arbitrage model with foreign exchange presented here is also known as **covered interest rate parity** and sometimes just **interest rate parity**.

Recall that currency forward contracts are agreements to exchange one currency for another on a future date at an exchange rate the counterparties agree on today. One approach to pricing is based on a forward exchange rate satisfying an arbitrage relationship that equates the investment return on two alternative but equivalent investment strategies. We illustrate these two strategies assuming the domestic currency is British pounds (£) and the foreign currency is the euro (€).

Strategy #1:

We simply invest one currency unit in a domestic risk-free bond. Thus, at Time T, we have the original investment grossed up at the domestic interest rate or the future value of 1DC, denoted FV(1DC). For example, the future value at Time T of this strategy can be expressed as $FV_{£,T}(1)$, given British pounds as the domestic currency.

Strategy #2:

We engage in three simultaneous transactions termed *steps* here. In Step 1, the domestic currency is converted at the current spot exchange rate, S_0(FC/DC), into the foreign currency (FC). At this point, 1 domestic currency unit is being converted to the foreign currency; hence, we use S_0(FC/DC) generically or S_0(€/£) in our example. Note that the final answer will express the spot exchange rate as the reciprocal $1/S_0$(FC/DC) = S_0(DC/FC). In Step 2, FC is

15 Some practitioners prefer to express the discussion here as FC/DC, contradicting normal mathematics as well as contradicting standard market quotations, such as $ per bushel of wheat or $ per ounce of gold.

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invested at the foreign risk-free rate until Time T. For example, the future value at Time T of this strategy can be expressed as $FV_{€,T}(1)$, given that the euro is the foreign currency. In Step 3, a forward foreign exchange contract is entered to sell the foreign currency at Time T in exchange for domestic currency with the forward rate denoted $F_0(DC/FC,T)$. So, for example, $F_0(£/€,T)$ is the rate on a forward commitment at Time 0 to sell one euro for British pounds at Time T. This transaction can be looked at as being short the euro in pound terms or being long the pound in euro terms for delivery at Time T.

We are examining two ways to invest British pounds at Time 0, and both strategies should result in the same value in domestic currency units at Time T. If not, then an arbitrage opportunity exists. Remember that the current spot exchange rate, $S_0(£/€)$, is the number of British pounds for one euro. Again, in our example, $FV_{£,T}(1)$ denotes the future value of one British pound and $FV_{€,T}(1)$ denotes the future value of one euro.[16] Based on the two strategies, the value at Time T follows:

Strategy 1. Future value at Time T of investing £1: $FV_{£,T}(1)$

Strategy 2. Future value at Time T of investing £1: $F_0(£/€,T)FV_{€,T}(1)S_0(€/£)$

Assuming both strategies lead to the same number of British pounds at Time T, we have $FV_{£,T}(1) = F_0(£/€)FV_{€,T}(1)S_0(€/£)$. Note that $S_0(£/€) = 1/S_0(€/£)$, simply reflecting the reciprocal of the exchange rate. Thus, solving for the forward foreign exchange rate, the forward rate can be expressed as

$$\begin{aligned} F_0(£/€,T) &= \text{Future value of spot exchange rate adjusted for foreign rate} \\ &= FV_{£,T}(1)\Big/\left[FV_{€,T}(1)S_0(€/£)\right] = S_0(£/€)FV_{£,T}(1)\Big/FV_{€,T}(1) \end{aligned} \tag{9}$$

The carry adjustment, though it looks different, is similar to what we did in other carry models. In the numerator, we have simply the future value of the spot exchange rate. Rather than subtracting the carry benefit of foreign interest—the euro here—we divide by the future value of one euro, based on the euro interest rate. The effect is similar: The higher the foreign interest rate, the greater the benefit, and hence, the lower the forward or futures price will be.

If the two strategies result in different values at Time T, then the arbitrageur would buy the strategy offering the higher value at Time T and sell the strategy offering the lower value at Time T. This arbitrage activity would result in no cash flow today and positive cash flow at expiration. As with previous examples, we could borrow the arbitrage profit today and pay the loan back when the profit is captured at T.

EXAMPLE 11

Pricing Forward Foreign Exchange Contracts

Suppose the current spot exchange rate, $S_0(£/€)$, is £0.792 (what 1€ is trading for in £). Further assume that the annual compounded annualized risk-free rates are 1% for the British pound and 0.3% for the euro.

1 The arbitrage-free one-year foreign exchange forward rate, $F_0(£/€,T)$ (expressed as the number of £ per 1€), will be *closest* to:

A 0.792.

B 0.794.

C 0.798.

16 Note that the interest could be compounded annually, continuously, or by any other method at this point; hence, we use the generic future value specification.

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2 Now suppose the foreign exchange forward rate, $F_0(£/€,T)$, is observed to be below the foreign exchange spot rate, $S_0(£/€)$. Based on the carry arbitrage model, compared to British interest rates, the eurozone interest rate will *most likely* be:

A lower.

B higher.

C the same.

Solution to 1:

C is correct. Based on the information given, we have $S_0(£/€) = 0.792$, T = 1 year, $r_£ = 1.0\%$, and $r_€ = 0.3\%$ (both with annual compounding). Therefore,

$$F_0(£/€,1) = S_0(£/€)FV_{£,1}(1)/FV_{€,1}(1) = 0.792(1 + 0.01)^1/(1 + 0.003)^1 = 0.798,$$

or £0.798/€.

Solution to 2:

B is correct. Note that if we observe that $F_0(£/€,T)$ is smaller than $S_0(£/€)$, then the carry arbitrage model provides a simple explanation: The British interest rate is lower than the eurozone interest rate. Based on the carry arbitrage model, foreign exchange forward rates solely reflect interest-related carry costs. Specifically, $F_0(£/€,T) < S_0(£/€)$ if and only if $r_£ < r_€$.

Note that the future value expressions in Equation 9 are in the same pattern as the spot exchange rate. If the spot exchange rate is expressed as 1€ is trading for £—denoted $S_0(£/€)$ and $F_0(£/€,T)$—then the future value ratio is $FV_{£,T}(1)/FV_{€,T}(1)$. If we assume annual compounding and denote the risk-free rates $r_£$ and $r_€$, respectively, we have

$$F_0(£/€,T) = S_0(£/€)(1 + r_£)^T/(1 + r_€)^T \text{ (Annually compounded version)}$$

If we assume continuous compounding and denote these risk-free rates in domestic (UK) and eurozone as $r_{£,c}$ and $r_{€,c}$, respectively, we have

$$F_0(£/€,T) = S_0(£/€)\, e^{(r_{£,c} - r_{€,c})T} \text{ (Continuously compounded version)}$$

To summarize, we identify several ways we get tripped up in understanding currency forward and futures contracts. First, if we let DC denote generically domestic currency and FC denote generically foreign currency, then there are two representations of the carry arbitrage model based on $S_0(FC/DC) = 1/S_0(DC/FC)$ and $F_0(FC/DC) = 1/F_0(DC/FC)$. If we assume annual compounding, we have either

$$F_0(DC/FC,T) = S_0(DC/FC)\frac{(1 + r_{DC})^T}{(1 + r_{FC})^T} \text{ or } F_0(FC/DC,T) = S_0(FC/DC)\frac{(1 + r_{FC})^T}{(1 + r_{DC})^T}$$

A good way to remember this relationship is that the interest rate in the numerator should be the rate for the country whose currency is specified in the spot rate quote. Thus, if the spot rate quote is in euros, the numerator should be the euro interest rate. Then the interest rate in the denominator is the rate in the other country.

Second, interest rates can be quoted in a wide variety of ways, including annual compounding (previous equation) and continuous compounding (following equation).

$$F_0(DC/FC,T) = S_0(DC/FC)e^{(r_{DC,c} - r_{FC,c})T} \text{ or } F_0(FC/DC,T) = S_0(FC/DC)e^{(r_{FC,c} - r_{DC,c})T}$$

Here, likewise, the currency quote should match the first interest rate. Thus, if the spot rate is quoted in euros, then the first interest rate in the exponential will be the euro rate.

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In equilibrium, $F_0(£/€,T) = S_0(£/€)FV_£(1)/FV_€(1)$; otherwise, positive future cash flow can be generated with no initial investment, which is an arbitrage profit.

We now turn to the task of estimating the fair value of the foreign exchange forward contract. The forward value, based on arbitrage, can best be understood by referring to Exhibit 13. Suppose the first transaction is buying a foreign exchange forward contract at Time 0 with expiration of Time T. Now consider selling a new foreign exchange forward contract at Time t also with expiration of Time T. Exhibit 13 shows the potential cash flows again using British pounds (£) as the domestic currency and euros (€) as the foreign currency. Suppose T = 1, T – t = 0.5, $F_0(£/€,T) = 0.804$, $F_t(£/€,T) = 0.901$, $S_T(£/€) = 1.2$, and $r_{£,t} = 1.2\%$. In other words, six months ago we bought a forward contract at 0.804, and the new forward price is 0.901.

Exhibit 13 Cash Flows for Offsetting a Long Forward Position

Steps	Cash Flow at Time 0	Cash Flow at Time t	Cash Flow at Time T
1. Buy forward contract at 0	0	$V_t(T)$	$V_T(0,T) = S_T(£/€) - F_0(£/€,T)$ $= 1.2 - 0.804 = 0.396$
2. Sell forward contract at t	NA	0	$V_T(t,T) = F_t(£/€,T) - S_T(£/€)$ $= 0.901 - 1.2 = -0.299$
Net cash flows	0	$V_t(T)$	$+F_t(£/€,T) - F_0(£/€,T)$ $= 0.901 - 0.804 = 0.097$

Note that the net position is again risk free. Therefore, the forward value observed at t of a T maturity forward contract is simply the present value of the difference in foreign exchange forward prices. That is,

$$\begin{aligned} V_t(T) &= \text{Present value of the difference in forward prices} \\ &= PV_{£,t,T}\left[F_t(£/€,T) - F_0(£/€,T)\right] \end{aligned} \tag{10}$$

Based on our numerical example, we have $V_t(T) = (0.901 - 0.804)/(1 + 0.012)^{0.5} =$ £0.0964/€.

EXAMPLE 12

Computing the Foreign Exchange Forward Contract Value

A corporation sold €10,000,000 against a British pound forward at a forward rate of £0.8000 for €1 at Time 0. The current spot market at Time t is such that €1 is worth £0.7500, and the annually compounded risk-free rates are 0.80% for the British pound and 0.40% for the euro. Assume at Time t there are three months until the forward contract expiration.

1 The forward price $F_t(£/€,T)$ at Time t will be *closest* to:

A 0.72.

B 0.74.

C 0.75.

2 The value of the foreign exchange forward contract at Time t will be *closest* to:

A £492,000.

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B £495,000.

C £500,000.

Solution to 1:

C is correct. Note that the forward price at Time t is

$$\begin{aligned} F_t(£/€,T) &= S_t(£/€)FV_{£,t,T}(1)/FV_{€,t,T}(1) \\ &= 0.75(1 + 0.008)^{0.25}/(1 + 0.004)^{0.25} \\ &= 0.7507. \end{aligned}$$

Solution to 2:

A is correct. The value per euro to the seller of the foreign exchange futures contract at Time t is simply the present value of the difference between the initial forward price and the £/€ forward price at Time t or

$$\begin{aligned} V_t(T) &= PV_{£,t,T}[F_0(£/€,T) - F_t(£/€,T)] \\ &= (0.8000 - 0.7507)/(1 + 0.008)^{0.25} \\ &= £0.0492 \text{ per euro.} \end{aligned}$$

Note that the corporation has an initial short position, so the short position of a €10,000,000 notional amount has a positive value of €10,000,000(£0.0492/€) = £492,000 for the corporation because the forward rate fell between Time 0 and Time t.

We conclude this section with observations on the similarities and differences between forward and futures contracts.

3.7 Comparing Forward and Futures Contracts

For every market considered here, the carry arbitrage model provides an approach for both pricing and valuing forward contracts. Recall the two generic expressions:

$F_0(T) = FV_{0,T}(S_0 + \theta_0 - \gamma_0)$ (Forward pricing)

$V_t(T) = PV_{t,T}[F_t(T) - F_0(T)]$ (Forward valuation)

Carry costs (θ_0) increase the forward price, and carry benefits (γ_0) decrease the forward price. The arbitrageur is carrying the underlying, and costs increase the burden whereas benefits decrease the burden. The forward value can be expressed as either the present value of the difference in forward prices or as a function of the current underlying price adjusted for carry cash flows and the present value of the initial forward price.

Futures prices are generally found using the same model, but futures values are different because of the daily marking to market. Recall that the futures values are zero at the end of each day because profits and losses are taken daily.

In summary, the carry arbitrage model provides a compelling way to price and value forward and futures contracts. In short, the forward or futures price is simply the future value of the underlying adjusted for any carry cash flows. The forward value is simply the present value of the difference in forward prices at an intermediate time in the contract. The futures value is zero after marking to market. We turn now to pricing and valuing swaps.

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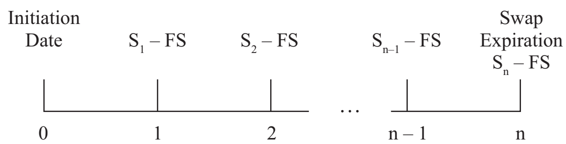
PRICING AND VALUING SWAP CONTRACTS

4

Based on the foundational materials in the last section on using the carry arbitrage model for pricing and valuing forward and futures contracts, we now apply this approach to pricing and valuing swap contracts. Swap contracts can be synthetically created by either a portfolio of underlying instruments or a portfolio of forward contracts. We focus here solely on the portfolio of underlying instruments approach.

We consider a receive-floating and pay-fixed interest rate swap. The swap will involve a series of n future cash flows at points in time represented simply here as 1, 2, ..., n. Let S_i denote the generic floating interest rate cash flow based on some underlying, and let FS denote the cash flow based on some fixed interest rate. We assume that the last cash flow occurs at the swap expiration. Exhibit 14 shows the cash flows of a generic swap. Later we will let S_i denote the floating cash flows tied to currency movements or equity movements.

Exhibit 14 Generic Swap Cash Flows: Receive-Floating, Pay-Fixed



We again will rely on the arbitrage approach for determining the pricing of a swap. This procedure involves finding the fixed swap rate such that the value of the swap at initiation is zero. Recall that the goal of the arbitrageur is to generate positive cash flows with no risk and no investment of one's own capital. Thus, it is helpful to be able to synthetically create a swap with a portfolio of other instruments. A receive-floating, pay-fixed swap is equivalent to being long a floating-rate bond and short a fixed-rate bond. Assuming both bonds were purchased at par, the initial cash flows are zero and the par payments at the end offset each other. Thus, the fixed bond payment should be equivalent to the fixed swap payment. Exhibit 15 shows the view of a swap as a pair of bonds. Note that the coupon dates on the bonds match the settlement dates on the swap and the maturity date matches the expiration date of the swap.[17]

17 As with all derivative instruments, there are numerous technical details that have been simplified here. We will explore some of these details shortly.

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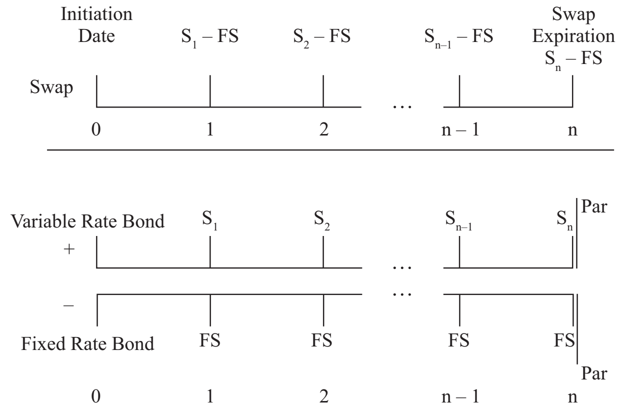
Exhibit 15 Receive-Floating, Pay-Fixed as a Portfolio of Bonds



As futures contracts can be viewed as marketable forward contracts, swaps can also be viewed as a portfolio of futures contracts.[18] In addition, because a single forward contract can be viewed as a portfolio of a call and a put option, a swap can also be viewed as a portfolio of options.[19]

Market participants often use swaps to transform one series of cash flows into another. For example, suppose that because of the relative ease of issuance, REB, Inc. sells a fixed-rate bond to investors. Based on careful analysis of the interest rate sensitivity of the company's assets, REB's leadership deems a Libor-based variable rate bond to be more appropriate. By entering a receive-fixed, pay-floating interest rate swap, REB can create a synthetic floating-rate bond, as illustrated in Exhibit 16. REB issues fixed-rate bonds and thus must make periodic fixed-rate-based payments, denoted FIX. REB then enters a receive-fixed (FIX) and pay-floating (FLT) interest rate swap. The two fixed rate payments cancel, leaving on net the floating-rate payments. Thus, we say that REB has created a synthetic floating-rate loan.

Exhibit 16 REB's Synthetic Floating-Rate Bond Based on Fixed-Rate Bond Issuance with Receive-Fixed Swap

The example in Exhibit 16 is for a swap in which the underlying is an interest rate. There are also currency swaps and equity swaps. Currency swaps can be used in a similar fashion, but the risks being addressed are both interest rate and currency exposures. Equity swaps can also be used in a similar fashion, but the risk being addressed is equity exposure.

Swaps have several technical nuances that can have a significant influence on pricing and valuation. Differences in payment frequency and day count methods often have a material impact on pricing and valuation. Another difficult issue is identifying the

18 In practice, futures have standardized characteristics, so there is rarely a set of futures contracts that can perfectly replicate a swap.

19 For example, a long forward contract is equivalent to a long call and a short put with the strike price equal to the forward price.

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appropriate discount rate to apply to the future cash flows. We turn now to examining three types of swap contracts—interest rate, currency, and equity—with a focus on pricing and valuation.

4.1 Interest Rate Swap Contracts

One approach to pricing and valuing interest rate swaps is based on a pair of bonds. We first need to introduce some basic notation and typical structures. It is important to understand that because they are OTC products in which the characteristics are agreed upon by the counterparties, swaps can be designed with an infinite number of variations. For example, a plain vanilla Libor-based interest rate swap can involve different frequencies of cash flow settlements and day count conventions. In fact, a swap can have both semi-annual payments and quarterly payments, as well as actual day counts and day counts based on 30 days per month. Also, the notional amount can vary across the maturities, such as would occur when aligning a swap with an amortizing loan. Thus, it is important to build in our models the flexibility to handle these variations and issues. Unless stated otherwise, we will assume the notional amounts are all equal to one (NA = 1); hence, we do not consider amortizing swaps here. Swap values per 1 notional amount can be simply multiplied by the actual notional amount to arrive at the swap's fair market value.

Interest rate swaps have two legs, typically a floating leg (FLT) and a fixed leg (FIX). The floating leg cash flow (denoted S_i to be consistent with other underlying instruments) can be expressed as

$$S_i = CF_{FLT,i} = AP_{FLT,i} r_{FLT,i} = \left(\frac{NAD_{FLT,i}}{NTD_{FLT,i}}\right) r_{FLT,i}$$

and the fixed leg cash flow (denoted FS) can be expressed as

$$FS = CF_{FIX,i} = AP_{FIX,i} r_{FIX} = \left(\frac{NAD_{FIX,i}}{NTD_{FIX,i}}\right) r_{FIX}$$

where CF_i simply reminds us that our focus is on cash flows, AP_i denotes the accrual period, $r_{FLT,i}$ denotes the observed floating rate appropriate for Time i, NAD_i denotes the number of accrued days during the payment period, NTD_i denotes the total number of days during the year applicable to cash flow i, and r_{FIX} denotes the fixed swap rate. The accrual period accounts for the payment frequency and day count methods. The two most popular day count methods are known as 30/360 and ACT/ACT. As the name suggests, 30/360 treats each month as having 30 days, and thus a year has 360 days. ACT/ACT treats the accrual period as having the actual number of days divided by the actual number of days in the year (365 or 366). Finally, the convention in the swap market is that the floating interest rate is assumed to be advanced set and settled in arrears; thus, $r_{FLT,i}$ is set at the beginning of period i and paid at the end.[20] If we assume constant accrual periods, the receive-fixed, pay-floating net cash flow can be expressed as

$$FS - S_i = AP\left(r_{FIX} - r_{FLT,i}\right)$$

20 Often, interest rate swaps are used to convert floating-rate loans to synthetic fixed rate loans. These floating-rate loans are advanced set, settled in arrears. Otherwise, while interest is accruing, we have no idea what rate is being applied until the end. Thus, with advanced set, settled in arrears, the interest begins accruing at a known rate and then the interest is paid at the end of the period, whereupon the interest rate is reset once again.

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and the receive-floating, pay-fixed net cash flow can be expressed as

$$S_i - FS = AP\left(r_{FLT,i} - r_{FIX}\right)$$

As a simple example, if the fixed rate is 5%, the floating rate is 5.2%, and the accrual period is 30 days based on a 360 day year, the payment of a receive-fixed, pay-floating swap is calculated as (30/360)(0.05 – 0.052) = –0.000167 per notional of 1. Because the floating rate exceeds the fixed rate, the party that pays floating (and receives fixed) would pay this amount to the party that receives floating (and pays fixed). In other words, there is only a single payment made from one party to the other.

We now turn to swap pricing. Exhibit 17 shows the cash flows for an interest rate swap along with a pair of bonds each with the same par amount.[21] Suppose the arbitrageur enters a receive-fixed, pay-floating interest rate swap with some initial value V. Because we are exploring the equilibrium fixed swap rate, we do not first assume the swap value is in fact zero or in equilibrium. Because this swap will lose value when floating rates rise, the arbitrageur purchases a variable rate bond whose value is denoted VB—satisfying Rule #2 of not taking any risk. Note that the terms of the variable rate bond are selected to match exactly the floating payments of the swap. To satisfy Rule #1 of not spending money, a fixed-rate bond is sold short—equivalent to borrowing funds—with terms to match exactly the fixed payments of the swap.

Exhibit 17 Cash Flows for Receive-Fixed Swap Hedge with Bonds

Steps	Time 0	Time 1	Time 2	...	Time n
1. Receive fixed swap	–V	$+FS - S_1$	$+FS - S_2$	...	$+FS - S_n$
2. Buy floating-rate bond	–VB	$+S_1$	$+S_2$	...	$+S_n$ + Par
3. Short sell fixed-rate bond	+FB	–FS	–FS	...	–(FS + Par)
Net cash flows	–V – VB + FB	0	0	0	0

Thus, the fixed coupon such that the floating-rate bond price equals the fixed-rate bond price is the equilibrium fixed swap rate. That is, in equilibrium we must have –V – VB + FB = 0 or else there is an arbitrage opportunity. For a receiver of a fixed rate and payer of a floating rate, the value of the swap is

$$V = \text{Value of fixed bond} - \text{Value of floating bond} = FB - VB \tag{11}$$

The value of a receive-fixed, pay-floating interest rate swap is simply the value of buying a fixed-rate bond and issuing a floating-rate bond.[22] If we further stipulate that pricing the swap means to determine the fixed rate such that the value of the swap at initiation is zero, then the value of the fixed bond must equal the value of the floating bond.

21 The underlying bonds have a designated par value on which their interest payments are based, whereas swaps are based on a notional amount that is never paid. The notional amount determines the size of the swap interest payments. Thus, a swap is like an offsetting pair of bonds with interest payments but no principal payments. In general, the notional amount of the swap will equal the par value of the underlying bonds.
22 In Exhibit 17, the trades illustrated in Steps 2 and 3 are synthetically creating an offsetting position; hence, the floating bond is purchased and the fixed bond is short sold.

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The value of a floating-rate bond, assuming we are on a reset date and the interest payment matches the discount rate, is par, assumed to be 1 here. The value of a fixed bond is as follows:

$$\text{Fixed bond rate: } FB = C\sum_{i=1}^{n} PV_{0,t_i}(1) + PV_{0,t_n}(1) \quad (12)$$

where C denotes the coupon amount for the fixed-rate bond and $PV_{0,t_i}(1)$ is the appropriate present value factor for the i^{th} fixed cash flow.

Based on the value of these bonds and noting that the fixed coupon amount is equivalent to the fixed swap rate, r_{FIX}, we obtain the swap pricing equation:

$$\text{Swap pricing equation: } r_{FIX} = \frac{1 - PV_{0,t_n}(1)}{\sum_{i=1}^{n} PV_{0,t_i}(1)} \quad (13)$$

The fixed swap rate is simply one minus the final present value term divided by the sum of present values. Therefore, one interpretation of the fixed swap rate is that it should be equal to the fixed rate on a par bond, which is the ratio one minus the present value of the final cash flow all divided by an annuity.[23]

The fixed swap leg cash flow for a unit of notional amount is simply the fixed swap rate adjusted for the accrual period, or $FS_i = AP_{FIX,i} r_{FIX}$. Alternatively, the annualized fixed swap rate is equal to the fixed swap leg cash flow divided by the fixed rate accrual period, or $r_{FIX,i} = FS/AP_{FIX,i}$. Note that if the accrual period varies across the swap payments, then the fixed swap payment will also vary. Thus, when relevant, a subscript i will be used. Often the fixed leg accrual period is constant; hence, the subscript can be safely omitted.

EXAMPLE 13

Solving for the Fixed Swap Rate Based on Present Value Factors

Suppose we are pricing a five-year Libor-based interest rate swap with annual resets (30/360 day count). The estimated present value factors, $PV_{0,t_i}(1)$, are given in the following table.

Maturity (years)	Present Value Factors
1	0.990099
2	0.977876
3	0.965136
4	0.951529
5	0.937467

The fixed rate of the swap will be *closest* to:

A 1.0%.

B 1.3%.

C 1.6%.

23 The denominator of Equation 13 is simply the sum of the present values of receiving one currency unit on each payment date or an annuity.

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Solution:

B is correct. Note that the sum of present values is

$$\sum_{i=1}^{n} PV_{0,t_i}(1) = 0.990099 + 0.977876 + 0.965136 + 0.951529 + 0.937467$$
$$= 4.822107$$

Therefore, the solution for the fixed swap rate is

$$r_{FIX} = \frac{1 - PV_{0,t_n}(1)}{\sum_{i=1}^{n} PV_{0,t_i}(1)} = \frac{1 - 0.937467}{4.822107} = 0.012968\text{, or } 1.2968\%$$

We now turn to interest rate swap valuation. Following a similar pattern as forward contracts, Exhibit 18 shows the cash flows for a receive-fixed interest rate swap initiated at Time 0 but that needs to be valued at Time t expressed per unit of the underlying currency. We achieve this valuation through entering an offsetting swap—receive-floating, pay-fixed. The floating sides offset, leaving only the difference in the fixed rates. We assume n′ remaining cash flows. At Time t, the swap value is represented as the funds need to generate the appropriate future cash flows.

Exhibit 18 Cash Flows for Receive-fixed Swap Valued at Time t

Steps	Time t	Time 1	Time 2	…	Time n′
1. Receive fixed swap (Time 0)	$-V$	$+FS_0 - S_1$	$+FS_0 - S_2$	…	$+FS_0 - S_{n'}$
2. Receive floating swap (Time t)	0	$S_1 - FS_t$	$S_2 - FS_t$	…	$S_{n'} - FS_t$
Net cash flows	$-V$	$FS_0 - FS_t$	$FS_0 - FS_t$	…	$FS_0 - FS_t$

Thus, the value of a fixed rate swap at some future point in Time t is simply the sum of the present value of the difference in fixed swap rates times the stated notional amount (denoted NA), or

$$V = NA(FS_0 - FS_t)\sum_{i=1}^{n'} PV_{t,t_i} \tag{14}$$

It is important to be clear on which side this value applies. The rate FS_0 is the fixed rate established at the start of the swap and goes to the party receiving fixed. Thus, when Equation 14 with FS_0 having a positive sign is used, it provides the value to the party receiving fixed. The negative of this amount is the value to the fixed rate payer.

The examples illustrated here show swap valuation only on a payment date. If a swap is being valued between payment dates, some adjustments are necessary. We do not pursue this topic here.

EXAMPLE 14

Solving for the Swap Value Based on Present Value Factors

Suppose two years ago we entered a €100,000,000 seven-year receive-fixed Libor-based interest rate swap with annual resets (30/360 day count). The fixed rate in the swap contract entered two years ago was 2%. Again, the estimated present value factors, $PV_{0,t_i}(1)$, are repeated from the previous example.

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Maturity (years)	Present Value Factors
1	0.990099
2	0.977876
3	0.965136
4	0.951529
5	0.937467

From the previous example, we know the current equilibrium fixed swap rate is 1.3% (two years after the swap was originally entered).

1 The value (in thousands) for the party receiving the fixed rate will be *closest* to:

A −€5,000.

B €3,375.

C €4,822.

2 The value (in thousands) for the party in the swap receiving the floating rate will be *closest* to:

A −€4,822.

B −€3,375.

C €5,000.

Solution to 1:

B is correct. Recall the sum of present values is 4.822107. Thus, the swap value per dollar notional is

$$\begin{aligned} V &= \left(FS_0 - FS_t\right)\sum_{i=1}^{n'} PV_{t,t_i} \\ &= \left(0.02 - 0.013\right)4.822107 \\ &= 0.03375 \end{aligned}$$

Thus, the swap value is €3,375,000.

Solution to 2:

B is correct. The equivalent receive-floating swap value is simply the negative of the receive-fixed swap value.

4.2 Currency Swap Contracts

A currency swap is a contract in which two counterparties agree to exchange future interest payments in different currencies. These interest payments can be based on either a fixed interest rate or a floating interest rate. Thus, with the addition of day count options and payment frequencies, there are many different ways to set up a currency swap. There are four major types of currency swaps: fixed-for-fixed, floating-for-fixed, fixed-for-floating, and floating-for-floating.

Currency swaps come in a wide array of types and structures. We review a few key features. First, currency swaps often but not always involve an exchange of notional amounts at both the initiation of the swap and at the expiration of the swap. Second, the payment on each leg of the swap is in a different currency unit, such as euros and Japanese yen, and the payments are not netted. Third, each leg of the swap can be either fixed or floating. To understand the pricing and valuation of currency swaps,

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we need a general approach that is flexible enough to handle each of these situations. We first focus on the fixed-for-fixed currency swaps with a very simple structure and only then consider other variations.

Currency swap pricing has three key variables: two fixed interest rates and one notional amount. Pricing a currency swap involves solving for the appropriate notional amount in one currency, given the notional amount in the other currency, as well as two fixed interest rates such that the currency swap value is zero at initiation. Because one notional amount is given, there are three swap pricing variables.

Because we are focused on fixed-for-fixed currency swaps, we need notation that reflects the different generic currency units. Thus, we let k = a and b to reflect two different currency units, such as euros and yen. Letters are used rather than numbers to avoid confusion with calendar time. The value of a fixed-rate bond in Currency k can be expressed generically as

$$FB_k = C_k \sum_{i=1}^{n} PV_{0,t_i,k}(1) + PV_{0,t_n,k}(Par_k)$$

where k = a or b, C_k denotes the periodic fixed coupon amount in Currency k, $\sum_{i=1}^{n} PV_{0,t_i,k}(1)$ denotes the present value from Time 0 to Time t_i discounting at the Currency k risk-free rate, and Par_k denotes the k currency unit par value. We do not assume par equals 1 because the notional amounts are typically different in each currency within the currency swap.

Exhibit 19 shows the cash flows for a fixed-for-fixed currency swap along with an offsetting pair of fixed-rate bonds. In this case, notice that the two bonds are in different currencies.[24] We assume the arbitrage cash flows will be evaluated in currency unit a. Therefore, all cash flows are converted to Currency a in the cash flow table based on the exchange rate denoted S_i—expressed as the number of units of Currency a for one unit of Currency b at Time i. We again ignore the technical nuances and assume the same accrual periods on both legs of the swap. Note that all the future cash flows, expressed in Currency a, are zero because the coupon rates on the fixed-rate bonds were selected to equal the fixed swap rates. Because we are demonstrating swap pricing, we do not assume the currency swap is initially valued correctly; hence, V can be either positive or negative. We initially use a negative sign, because an investment usually involves negative cash flows. We assume the par value of each bond is the same as the notional amount of each leg of the swap. From the arbitrageur's perspective, whether there is an exchange of notional amounts on the initiation date is not relevant because this exchange will be done at the current foreign exchange rate, and hence, it will have a fair value of zero. It is important, however, that this exchange of notional amounts is done at expiration. Because the swap notional amounts differ between the two currencies, it would be confusing to express these results per unit of Currency a. Therefore, each leg of the swap is assumed to have different notional amounts, but $Par_a = NA_a$ and $Par_b = NA_b$ in order to achieve zero cash flow at Time n.

Exhibit 19 Cash Flows for Currency Swap Hedged with Bonds

Steps	Time 0	Time 1	Time 2	...	Time n
1. Enter currency swap	$-V_a$	$+FS_a - S_1FS_b$	$+FS_a - S_2FS_b$	...	$+FS_a + NA_a$ $- S_n(FS_b + NA_b)$
2. Short sell bond in Currency a	$+FB_a(C_a = FS_a)$	$-FS_a$	$-FS_a$	...	$-(FS_a + Par_a)$

24 Technically, we build these portfolios such that the initial value in each currency is par.

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Exhibit 19 (Continued)

Steps	Time 0	Time 1	Time 2	...	Time n
3. Buy bond in Currency b	$-S_0FB_b(C_b = FS_b)$	$+S_1FS_b$	$+S_2FS_b$	...	$+S_n(FS_b + Par_b)$
Net cash flows	$-V_a + FB_a - S_0FB_b$	0	0	0	0

Based on this table, in equilibrium we must have

$$-V_a + FB_a - S_0FB_b = 0$$

and the fixed-for-fixed currency swap value is

$$V_a = FB_a - S_0FB_b$$

or else there is an arbitrage opportunity. Notice that the two-bond approach allows the arbitrageur to avoid having to convert one currency into another in the future. This approach mitigates all future currency exposure and basically identifies the current exchange rate that makes the value of the two bonds equal. Remember that the exchange rate S_0 is the number of Currency a units for one unit of Currency b at Time 0; thus, S_0FB_b is expressed in Currency a units.

Exhibit 20 provides a simple illustration of an at-market 10-year receive-fixed US$ and pay-fixed € swap, for which the annual reset coupon amount in US dollars is US$10 with par of US$1,300 and the annual reset coupon amount in euros is €9 with par of €1,000. Both bonds are assumed to be trading at par and have a 10-year maturity. This exhibit assumes a current spot exchange rate (S_0) at which €1 trades for US$1.3, and selected future spot exchange rates are S_1 = \$1.5, S_2 = \$1.1, and S_{10} = \$1.2. These future spot exchange rates are used to illustrate the conversion of future euro cash flows into US dollars, but notice that the cash flows are all zero regardless of the future spot exchange rates. In other words, we could have used any numbers for S_1, S_2, and S_{10}.

Exhibit 20 Numerical Example of Currency Swap Hedged with Bonds

Steps	Time 0	Time 1	Time 2	...	Time 10
1. Enter currency swap	0	+\$10 – (\$1.5/€)€9 = –\$3.5	+\$10 – (\$1.1/€)€9 = \$0.1	...	+\$10 + \$1,300 –(\$1.2/€)(€9 + €1,000) = \$99.2
2. Short sell US dollar bond	+\$1,300	–\$10	–\$10	...	–(\$10 + \$1,300)
3. Buy euro bond	–(\$1.3/€)€1,000	+(\$1.5/€)€9	+(\$1.1/€)€9	...	+(\$1.2/€)(€9 + €1,000)
Net cash flows	0	0	0	0	0

Clearly, if the initial swap value is not at market or zero, then there are arbitrage opportunities. If the initial swap value is positive, then this set of transactions would be implemented. If the initial swap value is negative, then the opposite set of transactions would be implemented. Specifically, enter a pay-US dollar, receive-euro swap, buy Currency a bonds, and short sell Currency b bonds. As before, the swap value after initiation is a simple variation of the expression above—specifically,

$$V_a = FB_a - S_0FB_b = 1{,}300 - 1.3(1{,}000) = 0$$

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Note further that $C_a = FS_a$ and $C_b = FS_b$ are fixed swap payment amounts stipulated in the currency swap. One way to find the equilibrium currency swap price (that is, the two fixed rates) is to identify the initial coupon rates ($C_{0,a}$ and $C_{0,b}$) such that the two bonds trade at par—specifically,

$$FB_a(C_{0,a},Par_a) = Par_a$$

and

$$FB_b(C_{0,b},Par_b) = Par_b$$

In equilibrium, the notional amounts of the two legs of the currency swap are $NA_b = Par_b$ and $NA_a = Par_a = S_0Par_b$. That is, one first decides the par value desired in one currency and then solves for the implied notional amount in the other currency.

The goal is to determine the fixed rates of the swap such that the current swap value is zero; then we have

$$FB_a(C_{0,a},Par_a) = S_0FB_b(C_{0,b},Par_b)$$

Because the fixed swap rate does not depend on the notional amounts, the fixed swap rates are found in exactly the same manner as the fixed interest rate swap rate. For emphasis, we repeat the equilibrium fixed swap rate equations for each currency:

$$r_{FIX,a} = \frac{1 - PV_{0,t_n,a}(1)}{\sum_{i=1}^{n} PV_{0,t_i,a}(1)}$$

and

$$r_{FIX,b} = \frac{1 - PV_{0,t_n,b}(1)}{\sum_{i=1}^{n} PV_{0,t_i,b}(1)} \quad (15)$$

Again, the fixed swap rate in each currency is simply one minus the final present value term divided by the sum of present values. We need to be sure that the present value terms are expressed on the basis of the appropriate currency.

We illustrate currency swap pricing with spot rates by way of an example.

EXAMPLE 15

Currency Swap Pricing with Spot Rates

A US company needs to borrow 100 million Australian dollars (A$) for one year for its Australian subsidiary. The company decides to issue US-denominated bonds in an amount equivalent to A$100 million. Then the company enters into a one-year currency swap with quarterly reset (30/360 day count) and the exchange of notional amounts at initiation and at maturity. At the swap's initiation, the US company receives the notional amount in Australian dollars and pays to the counterparty, a swap dealer, the notional amount in US dollars. At the swap's expiration, the US company pays the notional amount in Australian dollars and receives from the counterparty the notional amount in US dollars. Based on interbank rates, we observe the following spot rates today, at Time 0:

Days to Maturity	A$ Spot Interest Rates (%)	US$ Spot Interest Rates (%)
90	2.50	0.10
180	2.60	0.15

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Days to Maturity	A$ Spot Interest Rates (%)	US$ Spot Interest Rates (%)
270	2.70	0.20
360	2.80	0.25

Assume that the counterparties in the currency swap agree to an A$/US$ spot exchange rate of 1.140 (expressed as number of Australian dollars for US$1).

1 The annual fixed swap rates for Australian dollars and US dollars, respectively, will be *closest* to:

A 2.80% and 0.10%.

B 2.77% and 0.25%.

C 2.65% and 0.175%.

2 The notional amount (in US$ millions) will be *closest* to:

A 88.

B 100.

C 114.

3 The fixed swap quarterly payments in the currency swap will be *closest* to:

A A$692,000 and US$55,000.

B A$220,000 and US$173,000.

C A$720,000 and US$220,000.

Solution to 1:

B is correct. We first find the PV factors and then solve for the fixed swap rates. The present value expression based on spot rates (not forward rates) is $PV_{0,t_i}(1) = \dfrac{1}{1 + r_{Spot_i}\left(\dfrac{NAD_i}{NTD}\right)}$. Spot rates cover the entire period from 0 to t_i, unlike forward rates, which cover incremental periods. Based on the data given, we construct the following present value data table. The calculations are shown to the sixth decimal place in an effort to minimize rounding error. Rounding differences may occur in the solutions.

Days to Maturity	A$ Spot Interest Rates (%)	Present Value (A$1)	US$ Spot Interest Rates (%)	Present Value (US$1)
90	2.50	0.993789[a]	0.10	0.999750
180	2.60	0.987167	0.15	0.999251[b]
270	2.70	0.980152	0.20	0.998502
360	2.80	0.972763	0.25	0.997506
	Sum:	*3.933870*	*Sum:*	*3.995009*

[a] A$0.993789 = 1/[1 + 0.0250(90/360)].
[b] US$0.999251 = 1/[1 + 0.00150(180/360)].

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Therefore, the Australian dollar periodic rate is

$$r_{FIX.AUD} = \frac{1 - PV_{0,t_4,AUD}(1)}{\sum_{i=1}^{4} PV_{0,t_i,AUD}(1)} = \frac{1 - 0.972763}{3.933870}$$

$$= 0.00692381 \text{ or } 0.692381\%$$

and the US dollar periodic rate is

$$r_{FIX.USD} = \frac{1 - PV_{0,t_4,USD}(1)}{\sum_{i=1}^{4} PV_{0,t_i,USD}(1)} = \frac{1 - 0.997506}{3.995009}$$

$$= 0.00062422 \text{ or } 0.062422\%$$

The annualized rate is simply (360/90) times the period results: 2.7695% for Australian dollars and 0.2497% for US dollars.

Solution to 2:

A is correct. The US dollar notional amount is calculated as A\$100 million divided by the current spot exchange rate at which US\$1 dollar trades for A\$1.1400. This exchange is equal to US\$87,719,298 (= A\$100,000,000/1.14).

Solution to 3:

A is correct. The fixed swap payments in currency units equal the periodic swap rate times the appropriate notional amounts. From the answers to 1 and 2, we have

$$\begin{aligned} FS_{A\$} &= NA_{A\$}(AP)r_{FIX,A\$} \\ &= A\$100{,}000{,}000(90/360)(0.027695) \\ &= A\$692{,}375 \end{aligned}$$

and

$$\begin{aligned} FS_{US\$} &= NA_{US\$}(AP)r_{FIX,US\$} \\ &= US\$87{,}719{,}298(90/360)(0.002497) \\ &= US\$54{,}759. \end{aligned}$$

Therefore, one approach to pricing currency swaps is to view the swap as a pair of fixed-rate bonds. The main advantage of this approach is that all foreign exchange considerations are moved to the initial exchange rate. We do not need to address future foreign currency transactions. Also, note that a fixed-for-floating currency swap is simply a fixed-for-fixed currency swap paired with a floating-for-fixed interest rate swap. Also, we do not technically "price" a floating-rate swap, because we do not designate a single coupon rate, and the value of such a swap is par on any reset date. Thus, we have the capacity to price any variation of currency swaps.

We now turn to currency swap valuation. Recall that with currency swaps, there are two main sources of risk: interest rates and exchange rates. Exhibit 21 shows the cash flows from three transactions. Note this exhibit is similar to the currency swap pricing exhibit, but the currency swap was initiated at Time 0 and here we are evaluating it at Time t. Step 1 shows the cash flows for a fixed-for-fixed currency swap expressed in units of Currency a. Step 2 is borrowing or short selling a bond in Currency a to generate sufficient funds to exactly offset the currency swap cash flows that are in units of Currency a. Step 3 is lending or buying a bond in Currency b to generate sufficient funds to exactly offset the currency swap cash flows that are in units of Currency b. The net cash flows at each future point in time are zero. Recall that S_i denotes the

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spot exchange rate in units of Currency a for each unit of Currency b at Time t_i. Thus, $S_tFS_{b,0}$ is the value of the Currency b fixed cash flow expressed in Currency a at Time t. From a value perspective, $FS_{b,0}$ is equivalent in value in Currency b to $S_tFS_{b,0}$ in Currency a. Hence, the future net cash flows are all zero.

Exhibit 21 Cash Flows for Currency Swap Hedged with Bonds

Steps	Time t	Time 1	Time 2	...	Time n′
1. Currency swap	$-V_a$	$+FS_{a,0} - S_1FS_{b,0}$	$+FS_{a,0} - S_2FS_{b,0}$	...	$+FS_{a,0} + NA_{a,0} - S_{n'}(FS_{b,0} + NA_{b,0})$
2. Short sell bond (a)	$+FB_a$	$-FS_{a,0}$	$-FS_{a,0}$	...	$-(FS_{a,0} + NA_{a,0})$
3. Buy bond (b)	$-S_tFB_b$	$+S_1FS_{b,0}$	$+S_2FS_{b,0}$	...	$+S_{n'}(FS_{b,0} + NA_{b,0})$
Net cash flows	0	0	0		0

The value of a fixed-for-fixed currency swap at some future point in time, Time t, is simply the difference in a pair of fixed-rate bonds, one expressed in Currency a and one expressed in Currency b. To express the bonds in the same currency units, we convert the Currency b bond into units of Currency a through a spot foreign exchange transaction. Hence, we have

$$\begin{aligned} V_a &= FB_a - S_0FB_b \\ &= FS_{a,0}\sum_{i=1}^{n'} PV_{t,t_i,a} + NA_{a,0}PV_{t,t_{n'},a} - S_t\left(FS_{b,0}\sum_{i=1}^{n'} PV_{t,t_i,b} + NA_{b,0}PV_{t,t_{n'},b}\right) \end{aligned}$$

Note that the fixed swap amount (FS) is the per-period fixed swap rate times the notional amount. Therefore, the currency swap valuation equation can be expressed as

$$\begin{aligned} V_a = {} & NA_{a,0}\left(r_{FIX,a,0}\sum_{i=1}^{n'} PV_{t,t_i,a} + PV_{t,t_{n'},a}\right) - \\ & S_tNA_{b,0}\left(r_{FIX,b,0}\sum_{i=1}^{n'} PV_{t,t_i,b} + PV_{t,t_{n'},b}\right) \end{aligned} \qquad (16)$$

As mentioned, the terms in Equation 16 represent the difference of two fixed-rate bonds. The first term in braces is the value of a long position in a bond with face value of 1 unit of Currency a, which is then multiplied by the notional amount of the swap, in Currency a (i.e., represented by $NA_{a,0}$). This product represents the value of the cash inflows to the counterparty receiving interest payments in Currency a in the swap. The second term (after the minus sign) are outflows, and represents the value of a short bond position with face value of 1 unit of Currency b, which is multiplied by the product of the swap notional amount in Currency b (NA_b) and the current exchange rate, S_t (stated in units of Currency a per unit of Currency b). That gives the value of the payments, in currency a terms, made by the party receiving interest in Currency a and paying interest in Currency b in the swap. V_a is then the value of the swap to the party receiving Currency a while the value of the swap to the party receiving Currency b in the swap is $-V_a$.

Example 16 examines the case of a company using a currency swap to effectively convert a bond issued in US dollars to a bond issued in Australian dollars. In solving the problem, take care to identify Currency a (implied by the how the exchange rate, S_t, is given) and the party receiving interest payments in Currency a in the swap.

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EXAMPLE 16

Currency Swap Valuation with Spot Rates

This example builds on the previous example addressing currency swap pricing. Recall that a US company needed to borrow 100 million Australian dollars (A$) for one year for its Australian subsidiary. The company decided to borrow in US dollars (US$) an amount equivalent to A$100 million by issuing US-denominated bonds. The company entered into a one-year currency swap with a swap dealer. The swap uses quarterly reset (30/360 day count) and exchange of notional amounts at initiation and at maturity. At the swap's expiration, the US company pays the notional amount in Australian dollars and receives from the dealer the notional amount in US dollars. The fixed rates were 2.7695% for Australian dollars and 0.2497% for US dollars. Initially, the notional amount in US dollars was US$87,719,298 with a spot exchange rate of A$1.14 for US$1.

Assume 60 days have passed and we observe the following market information:

Days to Maturity	A$ Spot Interest Rates (%)	Present Value (A$1)	US$ Spot Interest Rates (%)	Present Value (US$1)
30	2.00	0.998336	0.50	0.999584
120	1.90	0.993707	0.40	0.998668
210	1.80	0.989609	0.30	0.998253
300	1.70	0.986031	0.20	0.998336
	Sum:	3.967683	*Sum*:	3.994841

The currency spot exchange rate is now A$1.13 for US$1.

1 The current value to the swap dealer in A$ of the currency swap entered into 60 days ago will be closest to:

A –A$13,557,000.

B A$637,620.

C A$2,145,200.

2 The current value in USD to the US firm of the currency swap entered into 60 days ago will be *closest* to:

A –$2,673,705.

B –$1,898,400.

C $334,730.

Solution to 1:

C is correct. The US firm issues a $87.7 bond and enters a swap with the swap dealer. The initial exchange rate is given as A$1.14 for US$1, so Currency a is $A. The swap dealer is receiving quarterly interest payments in Currency a (A$). The swap may be diagrammed as shown below:

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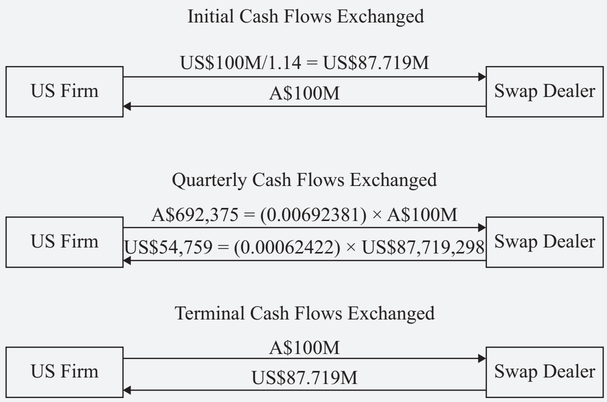

After 60 days the new exchange rate is A$1.13 per US$1, and the term structure has changed in both markets. Equation 16 gives the value of the swap at time t, V_a. This the value of the swap to the party receiving interest payments in Australian dollars, which is the swap dealer. Thus Equation 16, the value to the firm receiving A$ is:

$$
\begin{aligned}
V_a &= NA_{a,0}\left(r_{FIX,a,0}\sum_{i=1}^{n'} PV_{t,t_i,a} + PV_{t,t_{n'},a}\right) - S_0 NA_{b,0}\left(r_{FIX,b,0}\sum_{i=1}^{n'} PV_{t,t_i,b} + PV_{t,t_{n'},b}\right) \\
&= 100{,}000{,}000[0.00692381(3.967683) + 0.986031] - 1.13(87{,}719{,}298) \\
&\quad [0.00062422(3.994841) + 0.998336] \\
&= \text{A\$}2{,}145{,}203
\end{aligned}
$$

The first term in Equation 16 represents the PV of the dealer's incoming cash flows in A$, effectively a long position in A$ bond. Remember, the dealer is receiving quarterly interest payments in A$ and will receive the A$100M terminal payment. To compute the PV of the AUD cashflows, the notional amount is multiplied by a term inside the braces which represents the periodic interest rate multiplied by the sum of the PV factors for the four payments plus the PV factor for the terminal cash flow. The second term is the PV of the USD outflows (effectively a short bond in Currency b, here USD). The PV of the quarterly interest payments and terminal payment are calculated using the new term structure and converted into AUD. Thus, we have the value of the long AUD bond minus the value of short USD bond (stated in AUD terms). This gives V_a the value of the swap to the party receiving Currency a, that is the value from the perspective of the dealer.

Solution to 2:

B is correct. In terms of Solution 1 above, the value to the US firm is $-V_a$. This represents the value to the firm making interest payments in Currency a (A$).

$-V_a$ = –A$2,145,203, which converted to USD is
$-V_a$ = –A$2,145,203 × (1$/A$1.13) = –$1,898,410

Note that the US company issues (short) a bond in USD in their home market and uses a swap to effectively convert to AUD bond issue. Understanding the swap as two bonds, the US firm is long a USD bond (USD is Currency b in this example) and short a bond in A$ (Currency a). The swap offsets the US firm's USD bond issue (short). The swap allows the US firm to make A$ interest payments to the swap dealer, or to effectively issue a bond in A$ (Currency a).

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The swap value is negative to the US firm due to changes in the term structure and exchange rate. The A$ has strengthened against the US$, so now the US firm must pay periodic interest and principal cash flows in A$ at a rate of $1.13A$/1US$. That is, the firm gets fewer A$ for their US$. The new term structure of Example 16 offers lower rates to A$ borrowers; this also contributes to the negative swap value for the US firm. The firm had agreed to pay higher periodic A$ rates in the swap; now the present value of those outflows has increased.

4.3 Equity Swap Contracts

Drawing on our prior definition of a swap, we define an equity swap in the following manner: An **equity swap** is an OTC derivative contract in which two parties agree to exchange a series of cash flows whereby one party pays a variable series that will be determined by an equity and the other party pays either (1) a variable series determined by a different equity or rate or (2) a fixed series. An equity swap is used to convert the returns from an equity investment into another series of returns, which, as noted, either can be derived from another equity series or can be a fixed rate. Equity swaps are widely used in equity portfolio investment management to modify returns and risks.

We examine three types of equity swaps: receive-equity return, pay-fixed; receive-equity return, pay-floating; and receive-equity return, pay-another equity return. Like interest rate swaps and currency swaps, there are several unique nuances for equity swaps. We highlight just a few. First, the underlying reference instrument for the equity leg of an equity swap can be an individual stock, a published stock index, or a custom portfolio. Second, the equity leg cash flow can be with or without dividends. Third, all the interest rate swap nuances exist with equity swaps that have a fixed or floating interest rate leg.

We focus here on viewing an equity swap as a portfolio of an equity position and a bond. The equity swap cash flows can be expressed as follows:

NA(Equity return – Fixed rate) (for receive-equity, pay-fixed),

NA(Equity return – Floating rate) (for receive-equity, pay-floating), and

NA(Equity return$_a$ – Equity return$_b$) (for receive-equity, pay-equity),

where a and b denote different equities. Note that an equity-for-equity swap can be viewed simply as a receive-equity a, pay-fixed swap combined with a pay-equity b, receive-fixed swap. The fixed payments cancel out, and we have synthetically created an equity-for-equity swap.

EXAMPLE 17

Equity Swap Cash Flows

Suppose we entered into a receive-equity index and pay-fixed swap. It is quarterly reset, 30/360 day count, €5,000,000 notional amount, pay-fixed (1.6% annualized, quarterly pay, or 0.4% per quarter).

1 If the equity index return was 4.0% for the quarter (not annualized), the equity swap cash flow will be *closest* to:

   A –€220,000.

   B –€180,000.

   C €180,000.

2 If the equity index return was –6.0% for the quarter (not annualized), the equity swap cash flow will be closest to:

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A –€320,000.

B –€180,000.

C €180,000.

Solution to 1:

C is correct. Note that the equity index return is reported on a quarterly basis. It is not an annualized number. The fixed leg is often reported on an annual basis. Thus, one must carefully interpret the different return conventions. In this case, receive-equity index counterparty cash flows are as follows:

€5,000,000(0.04 – 0.004) = €180,000 (Receive 4%, pay 0.4% for the quarter)

Solution to 2:

A is correct. Similar to 1, we have

€5,000,000(–0.06 – 0.004) = –€320,000 (Receive –6%, pay 0.4% for the quarter)

When the equity leg of the swap is negative, then the receive-equity counterparty must pay both the equity return as well as the fixed rate (or whatever the payment terms are). Note, also, that equity swaps may cause liquidity problems. As seen here, if the equity return is negative, then the receive-equity return, pay-floating or pay-fixed swap may result in a large negative cash flow.

The cash flows for the equity leg of an equity swap can be expressed as

$$S_i = NA_E R_{E_i}$$

where R_{E_i} denotes the periodic return of the equity either with or without dividends as specified in the swap contract and NA_E denotes the notional amount. The cash flows for the fixed interest rate leg of the equity swap are the same as those of an interest rate swap, or

$$FS = NA_E AP_{FIX} r_{FIX}$$

where AP_{FIX} denotes the accrual period for the fixed leg for which we assume the accrual period is constant and r_{FIX} here denotes the fixed rate on the equity swap.

For equity swaps, the equity position could be a wide variety of claims, including the return on a stock index with or without dividends and the return on an individual stock with or without dividends. For our objectives here, we ignore the influence of dividends by assuming the equity swap leg assumes all dividends are reinvested in the equity position.[25] The equity leg of the swap is produced by selling the equity position on a reset date and reinvesting the original equity notional amount, leaving a remaining balance that is the cash flow required of the equity swap leg.[26] Exhibit 22 shows the cash flows from an equity swap arbitrage transaction.

---

25 The arbitrage transactions for an equity swap when dividends are not included are extremely complex and beyond our objectives.

26 Technically, we just sell off any equity value in excess of NA_E or purchase additional shares to return the equity value to NA_E, effectively generating S_i.

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Exhibit 22 Cash Flows for Receive-Fixed Equity Swap Hedged with Equity and Bond

Steps	Time 0	Time 1	Time 2		Time n
1. Enter equity swap	–V	$+FS - S_1$	$+FS - S_2$		$+FS - S_n$
2. Buy NA_E equity	$-NA_E$	$+S_1$	$+S_2$		$+S_n + NA_E$
3. Short sell fixed-rate bond	+FB(C = FS)	–FS	–FS		–(FS + Par)
4. Borrow arbitrage profit	$-PV(Par - NA_E)$				$Par - NA_E$
Net cash flows	$-V - NA_E + FB - PV(Par - NA_E)$	0	0	0	0

Let us examine the Time 1 cash flow. The equity swap is receive-fixed, pay-equity. For Step 1, if the equity-related cash flow S_1 is less than the fixed-leg cash flow, then the swap generates a positive cash flow to this counterparty. For Step 2, the cash flow is simply the cash flow related to the equity movement and dividends, if applicable. Essentially, if the position value is greater than NA_E, then the excess value is sold off, but if the position value is less than NA_E, then an additional equity position is acquired. For Step 3, the short bond position requires the payment of coupons. Note that these coupons, by construction, equal the fixed leg cash flows. The sum of these three transactions is always zero.

Note the final cash flow for the long position in the equity includes the final sale of the underlying equity position. The final periodic return on the equity plus the original equity value will equal the proceeds from the final sale of the underlying equity position. Note that for the terminal cash flows to equal zero, we must either set the bond par value to equal the initial equity position or finance this difference. In this case, the bond par value could be different from the notional amount of equity. Therefore, in equilibrium, we have $-V - NA_E + FB - PV(Par - NA_E) = 0$, and hence, the equity swap value is $V = -NA_E + FB - PV(Par - NA_E)$.

The fixed swap rate can be expressed as the r_{FIX} rate such that $FB_0 = NA_E + PV(Par - NA_E)$. Note that assuming $NA_E = Par = 1$,

$$r_{FIX} = \frac{1 - PV_{0,t_n}(1)}{\sum_{i=1}^{n} PV_{0,t_i}(1)}$$

You should recognize that the pricing of an equity swap is identical to the pricing of a comparable interest rate swap even though the future cash flows are dramatically different. If the swap required a floating payment, there would be no need to price the swap, as the floating side effectively prices itself at par automatically at the start. If the swap involves paying one equity return against another, there would also be no need to price it. You could effectively view this arrangement as paying equity a and receiving a fixed rate as specified above and receiving equity b and paying the same fixed rate. The fixed rates would cancel.

Valuing an equity swap after the swap is initiated (V_t) is similar to valuing an interest rate swap except that rather than adjust the floating-rate bond for the last floating rate observed (remember, advanced set), we adjust the value of the notional amount of equity, or

$$V_t = FB_t(C_0) - (S_t/S_{t-})NA_E - PV(Par - NA_E) \tag{17}$$

where $FB_t(C_0)$ denotes the Time t value of a fixed-rate bond initiated with coupon C_0 at Time 0, S_t denotes the current equity price, S_{t-} denotes the equity price observed at the last reset date, and PV() denotes the present value function from Time t to the swap maturity time.

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EXAMPLE 18

Equity Swap Pricing

In Examples 13 and 14 related to interest rate swaps, we considered a five-year, annual reset, 30/360 day count, Libor-based swap. The following table provides the present values per €1.

Maturity (years)	Present Value Factors
1	0.990099
2	0.977876
3	0.965136
4	0.951529
5	0.937467

Assume an annual reset Libor floating-rate bond trading at par. The fixed rate was previously found to be 1.2968%. Given these same data, the fixed interest rate in the EURO STOXX 50 equity swap is *closest* to:

A 0.0%.

B 1.1%.

C 1.3%.

Solution:

C is correct. The fixed rate on an equity swap is the same as that on an interest rate swap or 1.2968% as in Example 13. That is, the fixed rate on an equity swap is simply the fixed rate on a comparable interest rate swap.

EXAMPLE 19

Equity Swap Valuation

Suppose six months ago we entered a receive-fixed, pay-equity five-year annual reset swap in which the fixed leg is based on a 30/360 day count. At the time the swap was entered, the fixed swap rate was 1.5%, the equity was trading at 100, and the notional amount was 10,000,000. Now all spot interest rates have fallen to 1.2% (a flat term structure), and the equity is trading for 105.

1 The fair value of this equity swap is *closest* to:

   A –€300,000.

   B –€500,000.

   C €500,000.

2 The value of the equity swap will be *closest* to zero if the stock price is:

   A 100.

   B 102.

   C 105.

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Solution to 1:

A is correct. Because we have not yet passed the first reset date, there are five remaining cash flows for this equity swap. The fair value of this swap is found by solving for the fair value of the implied fixed-rate bond. We then adjust for the equity value. The fixed rate of 1.5% results in fixed cash flows of 150,000 at each settlement. Applying the respective present value factors, which are based on the new spot rates of 1.2%, gives us the following:

Date (in years)	Present Value Factors (PV)	Fixed Cash Flow	PV(Fixed Cash Flow)*
0.5	0.994036	150,000	149,105
1.5	0.982318	150,000	147,348
2.5	0.970874	150,000	145,631
3.5	0.959693	150,000	143,954
4.5	0.948767	10,150,000	9,629,981
		Total:	10,216,019

* Answers may differ due to rounding.

Therefore, the fair value of this equity swap is 10,216,019 less 10,500,000 [= (105/100)10,000,000], or a loss of 283,981.

Solution to 2:

B is correct. The stock price at which this equity swap's fair value is zero would require (Par = NA_E in this case)

$$V_t = FB_t(C_0) - (S_t/S_{t-})NA_E$$

The value of the fixed leg is now approximately 102% of par; a stock price of 102 will result in a value of zero,

$$V_t = 102 - (S_t/100)100 = 0$$

where S_t is 102.

SUMMARY

This reading on forward commitment pricing and valuation provides a foundation for understanding how forwards, futures, and swaps are both priced and valued.

Key points include the following:

- The arbitrageur would rather have more money than less and abides by two fundamental rules: Do not use your own money, and do not take any price risk.
- The no-arbitrage approach is used for the pricing and valuation of forward commitments and is built on the key concept of the law of one price, which states that if two investments have the same future cash flows, regardless of what happens in the future, these two investments should have the same current price.
- Throughout this reading, the following key assumptions are made:
  - Replicating instruments are identifiable and investable.

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- Market frictions are nil.
- Short selling is allowed with full use of proceeds.
- Borrowing and lending is available at a known risk-free rate.

- Carry arbitrage models used for forward commitment pricing and valuation are based on the no-arbitrage approach.
- With forward commitments, there is a distinct difference between pricing and valuation; pricing involves the determination of the appropriate fixed price or rate, and valuation involves the determination of the contract's current value expressed in currency units.
- Forward commitment pricing results in determining a price or rate such that the forward contract value is equal to zero.
- The price of a forward commitment is a function of the price of the underlying instrument, financing costs, and other carry costs and benefits.
- With equities, currencies, and fixed-income securities, the forward price is determined such that the initial forward value is zero.
- With forward rate agreements, the fixed interest rate is determined such that the initial value of the FRA is zero.
- Futures contract pricing here can essentially be treated the same as forward contract pricing.
- Because of daily marking to market, futures contract values are zero after each daily settlement.
- The general approach to pricing and valuing swaps as covered here is using a replicating or hedge portfolio of comparable instruments.
- With a basic understanding of pricing and valuing a simple interest rate swap, it is a straightforward extension to pricing and valuing currency swaps and equity swaps.
- With interest rate swaps and some equity swaps, pricing involves solving for the fixed interest rate.
- With currency swaps, pricing involves solving for the two fixed rates as well as the notional amounts in each currency.

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PRACTICE PROBLEMS

The following information relates to Questions 1–7

Donald Troubadour is a derivatives trader for Southern Shores Investments. The firm seeks arbitrage opportunities in the forward and futures markets using the carry arbitrage model.

Troubadour identifies an arbitrage opportunity relating to a fixed-income futures contract and its underlying bond. Current data on the futures contract and underlying bond are presented in Exhibit 1. The current annual compounded risk-free rate is 0.30%.

Exhibit 1 Current Data for Futures and Underlying Bond

Futures Contract		Underlying Bond	
Quoted futures price	125.00	Quoted bond price	112.00
Conversion factor	0.90	Accrued interest since last coupon payment	0.08
Time remaining to contract expiration	Three months	Accrued interest at futures contract expiration	0.20
Accrued interest over life of futures contract	0.00		

Troubadour next gathers information on three existing positions.

Position 1 (Nikkei 225 Futures Contract):

Troubadour holds a long position in a Nikkei 225 futures contract that has a remaining maturity of three months. The continuously compounded dividend yield on the Nikkei 225 Stock Index is 1.1%, and the current stock index level is 16,080. The continuously compounded annual interest rate is 0.2996%.

Position 2 (Euro/Yen Forward Contract):

One month ago, Troubadour purchased euro/yen forward contracts with three months to expiration at a quoted price of 100.20 (quoted as a percentage of par). The contract notional amount is ¥100,000,000. The current forward price is 100.05, and the current annualized risk-free rate is 0.30%.

Position 3 (JPY/USD Currency Forward Contract):

Troubadour holds a short position in a yen/US dollar forward contract with a notional value of $1,000,000. At contract initiation, the forward rate was ¥112.10 per $1. The forward contract expires in three months. The current

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spot exchange rate is ¥112.00 per $1, and the annually compounded risk-free rates are –0.20% for the yen and 0.30% for the US dollar. The current quoted price of the forward contract is equal to the no-arbitrage price.

Troubadour next considers an equity forward contract for Texas Steel, Inc. (TSI). Information regarding TSI common shares and a TSI equity forward contract is presented in Exhibit 2.

Exhibit 2 Selected Information for TSI

- The price per share of TSI's common shares is $250.
- The forward price per share for a nine-month TSI equity forward contract is $250.562289.
- Assume annual compounding.

Troubadour takes a short position in the TSI equity forward contract. His supervisor asks, "Under which scenario would our position experience a loss?"

Three months after contract initiation, Troubadour gathers information on TSI and the risk-free rate, which is presented in Exhibit 3.

Exhibit 3 Selected Data on TSI and the Risk-Free Rate

- The price per share of TSI's common shares is $245.
- The risk-free rate is 0.325% (quoted on an annual compounding basis).
- TSI recently announced its regular semiannual dividend of $1.50 per share that will be paid exactly three months before contract expiration.
- The market price of the TSI equity forward contract is equal to the no-arbitrage forward price.

1 Based on Exhibit 1 and assuming annual compounding, the arbitrage profit on the bond futures contract is *closest* to:
   A 0.4158.
   B 0.5356.
   C 0.6195.
2 The current no-arbitrage futures price of the Nikkei 225 futures contract (Position 1) is *closest* to:
   A 15,951.81.
   B 16,047.86.
   C 16,112.21.
3 The value of Position 2 is *closest* to:
   A –¥149,925.
   B –¥150,000.
   C –¥150,075.
4 The value of Position 3 is *closest* to:
   A –¥40,020.

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B ¥139,913.

C ¥239,963.

5 Based on Exhibit 2, Troubadour should find that an arbitrage opportunity relating to TSI shares is

A not available.

B available based on carry arbitrage.

C available based on reverse carry arbitrage.

6 The *most appropriate* response to Troubadour's supervisor's question regarding the TSI forward contract is:

A a decrease in TSI's share price, all else equal.

B an increase in the risk-free rate, all else equal

C a decrease in the market price of the forward contract, all else equal.

7 Based on Exhibits 2 and 3, and assuming annual compounding, the per share value of Troubadour's short position in the TSI forward contract three months after contract initiation is *closest* to:

A $1.6549.

B $5.1561.

C $6.6549.

The following information relates to Questions 8–16

Sonal Johnson is a risk manager for a bank. She manages the bank's risks using a combination of swaps and forward rate agreements (FRAs).

Johnson prices a three-year Libor-based interest rate swap with annual resets using the present value factors presented in Exhibit 1.

Exhibit 1 Present Value Factors

Maturity (years)	Present Value Factors
1	0.990099
2	0.977876
3	0.965136

Johnson also uses the present value factors in Exhibit 1 to value an interest rate swap that the bank entered into one year ago as the receive-floating party. Selected data for the swap are presented in Exhibit 2. Johnson notes that the current equilibrium two-year fixed swap rate is 1.12%.

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Exhibit 2	Selected Data on Fixed for Floating Interest Rate Swap
Swap notional amount	$50,000,000
Original swap term	Three years, with annual resets
Fixed swap rate (since initiation)	3.00%

One of the bank's investments is exposed to movements in the Japanese yen, and Johnson desires to hedge the currency exposure. She prices a one-year fixed-for-fixed currency swap involving yen and US dollars, with a quarterly reset. Johnson uses the interest rate data presented in Exhibit 3 to price the currency swap.

Exhibit 3 Selected Japanese and US Interest Rate Data

Days to Maturity	Yen Spot Interest Rates	US Dollar Spot Interest Rates
90	0.05%	0.20%
180	0.10%	0.40%
270	0.15%	0.55%
360	0.25%	0.70%

Johnson next reviews an equity swap with an annual reset that the bank entered into six months ago as the receive-fixed, pay-equity party. Selected data regarding the equity swap, which is linked to an equity index, are presented in Exhibit 4. At the time of initiation, the underlying equity index was trading at 100.00.

Exhibit 4	Selected Data on Equity Swap
Swap notional amount	$20,000,000
Original swap term	Five years, with annual resets
Fixed swap rate	2.00%

The equity index is currently trading at 103.00, and relevant US spot rates, along with their associated present value factors, are presented in Exhibit 5.

Exhibit 5 Selected US Spot Rates and Present Value Factors

Maturity (years)	Spot Rate	Present Value Factors
0.5	0.40%	0.998004
1.5	1.00%	0.985222
2.5	1.20%	0.970874
3.5	2.00%	0.934579
4.5	2.60%	0.895255

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Johnson reviews a 6 × 9 FRA that the bank entered into 90 days ago as the pay-fixed/receive-floating party. Selected data for the FRA are presented in Exhibit 6, and current Libor data are presented in Exhibit 7. Based on her interest rate forecast, Johnson also considers whether the bank should enter into new positions in 1 × 4 and 2 × 5 FRAs.

Exhibit 6 6 × 9 FRA Data

FRA term	6 × 9
FRA rate	0.70%
FRA notional amount	US$20,000,000
FRA settlement terms	Advanced set, advanced settle

Exhibit 7 Current Libor

30-day Libor	0.75%
60-day Libor	0.82%
90-day Libor	0.90%
120-day Libor	0.92%
150-day Libor	0.94%
180-day Libor	0.95%
210-day Libor	0.97%
270-day Libor	1.00%

Three months later, the 6 × 9 FRA in Exhibit 6 reaches expiration, at which time the three-month US dollar Libor is 1.10% and the six-month US dollar Libor is 1.20%. Johnson determines that the appropriate discount rate for the FRA settlement cash flows is 1.10%.

8 Based on Exhibit 1, Johnson should price the three-year Libor-based interest rate swap at a fixed rate *closest* to:

A 0.34%.

B 1.16%.

C 1.19%.

9 From the bank's perspective, using data from Exhibit 1, the current value of the swap described in Exhibit 2 is *closest* to:

A –$2,951,963.

B –$1,849,897.

C –$1,943,000.

10 Based on Exhibit 3, Johnson should determine that the annualized equilibrium fixed swap rate for Japanese yen is *closest* to:

A 0.0624%.

B 0.1375%.

C 0.2496%.

11 From the bank's perspective, using data from Exhibits 4 and 5, the fair value of the equity swap is *closest* to:

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A –$1,139,425.

B –$781,323.

C –$181,323.

12 Based on Exhibit 5, the current value of the equity swap described in Exhibit 4 would be zero if the equity index was currently trading the *closest* to:

A 97.30.

B 99.09.

C 100.00.

13 From the bank's perspective, based on Exhibits 6 and 7, the value of the 6 × 9 FRA 90 days after inception is *closest* to:

A $14,817.

B $19,647.

C $29,635.

14 Based on Exhibit 7, the no-arbitrage fixed rate on a new 1 × 4 FRA is *closest* to:

A 0.65%.

B 0.73%.

C 0.98%.

15 Based on Exhibit 7, the fixed rate on a new 2 × 5 FRA is *closest* to:

A 0.61%.

B 1.02%.

C 1.71%.

16 Based on Exhibit 6 and the three-month US dollar Libor at expiration, the payment amount that the bank will receive to settle the 6 × 9 FRA is *closest* to:

A $19,945.

B $24,925.

C $39,781.

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SOLUTIONS

1 B is correct.

The no-arbitrage futures price is equal to the following:

$$F_0(T) = \text{FV}_{0,T}(T)[B_0(T+Y) + \text{AI}_0 - \text{PVCI}_{0,T}]$$
$$F_0(T) = (1 + 0.003)^{0.25}(112.00 + 0.08 - 0)$$
$$F_0(T) = (1 + 0.003)^{0.25}(112.08) = 112.1640$$

The adjusted price of the futures contract is equal to the conversion factor multiplied by the quoted futures price:

$$F_0(T) = \text{CF}(T)\text{QF}_0(T)$$
$$F_0(T) = (0.90)(125) = 112.50$$

Adding the accrued interest of 0.20 in three months (futures contract expiration) to the adjusted price of the futures contract gives a total price of 112.70.

This difference means that the futures contract is overpriced by 112.70 – 112.1640 = 0.5360. The available arbitrage profit is the present value of this difference: $0.5360/(1.003)^{0.25} = 0.5356$.

2 B is correct. The no-arbitrage futures price is

$$F_0(T) = S_0 e^{(r_c - \gamma)T}$$
$$F_0(T) = 16{,}080e^{(0.002996 - 0.011)(3/12)} = 16{,}047.68$$

3 A is correct. The value of Troubadour's euro/yen forward position is calculated as

$$V_t(T) = \text{PV}_{t,T}[F_t(T) - F_0(T)]$$
$$V_t(T) = (100.05 - 100.20)/(1 + 0.0030)^{2/12} = -0.149925 \text{ (per ¥100 par value)}$$

Therefore, the value of the Troubadour's forward position is

$$V_t(T) = -\frac{0.149925}{100}(¥100{,}000{,}000) = -¥149{,}925$$

4 C is correct. The current no-arbitrage price of the forward contract is

$$F_t(¥/\$,T) = S_t(¥/\$)\text{FV}_{¥,t,T}(1)/\text{FV}_{\$,t,T}(1)$$
$$F_t(¥/\$,T) = ¥112.00(1 - 0.002)^{0.25}/(1 + 0.003)^{0.25} = ¥111.8602$$

Therefore, the value of Troubadour's position in the ¥/$ forward contract, on a per dollar basis, is

$$V_t(T) = \text{PV}_{¥,t,T}[F_0(¥/\$,T) - F_t(¥/\$,T)]$$
$$= (112.10 - 111.8602)/(1 - 0.002)^{0.25} = ¥0.239963 \text{ per } \$1$$

Troubadour's position is a short position of $1,000,000, so the short position has a positive value of (¥0.239963/$) × $1,000,000 = ¥239,963 because the forward rate has fallen since the contract initiation.

5 A is correct. The carry arbitrage model price of the forward contract is

$$\text{FV}(S_0) = S_0(1 + r)^T = \$250(1 + 0.003)^{0.75} = \$250.562289$$

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The market price of the TSI forward contract is \$250.562289. A carry or reverse carry arbitrage opportunity does not exist because the market price of the forward contract is equal to the carry arbitrage model price.

6 B is correct. From the perspective of the long position, the forward value is equal to the present value of the difference in forward prices:

$$V_t(T) = \mathrm{PV}_{t,T}[F_t(T) - F_0(T)],$$

where $F_t(T) = \mathrm{FV}_{t,T}(S_t + \theta_t - \gamma_t)$.

All else equal, an increase in the risk-free rate before contract expiration would cause the forward price, $F_t(T)$, to increase. This increase in the forward price would cause the value of the TSI forward contract, from the perspective of the short, to decrease. Therefore, an increase in the risk-free rate would lead to a loss on the short position in the TSI forward contract.

7 C is correct. The no-arbitrage price of the forward contract, three months after contract initiation, is

$$F_{0.25}(T) = \mathrm{FV}_{0.25,T}(S_{0.25} + \theta_{0.25} - \gamma_{0.25})$$
$$F_{0.25(T)} = [\$245 + 0 - \$1.50/(1 + 0.00325)^{(0.5 - 0.25)}](1 + 0.00325)^{(0.75 - 0.25)} = \$243.8966$$

Therefore, from the perspective of the long, the value of the TSI forward contract is

$$V_{0.25}(T) = \mathrm{PV}_{0.25,T}[F_{0.25}(T) - F_0(T)]$$
$$V_{0.25}(T) = (\$243.8966 - \$250.562289)/(1 + 0.00325)^{0.75 - 0.25} = -\$6.6549$$

Because Troubadour is short the TSI forward contract, the value of his position is a gain of \$6.6549.

8 C is correct. The swap pricing equation is

$$r_{FIX} = \frac{1 - \mathrm{PV}_{0,t_n}(1)}{\sum_{i=1}^{n} \mathrm{PV}_{0,t_i}(1)}$$

That is, the fixed swap rate is equal to 1 minus the final present value factor (in this case, Year 3) divided by the sum of the present values (in this case, the sum of Years 1, 2, and 3). The sum of present values for Years 1, 2, and 3 is calculated as

$$\sum_{i=1}^{n} \mathrm{PV}_{0,t_i}(1) = 0.990099 + 0.977876 + 0.965136 = 2.933111$$

Thus, the fixed-swap rate is calculated as

$$r_{FIX} = \frac{1 - 0.965136}{2.933111} = 0.01189 \text{ or } 1.19\%$$

9 B is correct. The value of a swap from the perspective of the receive-fixed party is calculated as

$$V = NA(FS_0 - FS_t)\sum_{i=1}^{n'} \mathrm{PV}_{t,t_i}$$

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The swap has two years remaining until expiration. The sum of the present values for Years 1 and 2 is

$$\sum_{i=1}^{n'} \mathrm{PV}_{t,t_i} = 0.990099 + 0.977876 = 1.967975$$

Given the current equilibrium two-year swap rate of 1.00% and the fixed swap rate at initiation of 3.00%, the swap value per dollar notional is calculated as

$$V = (0.03 - 0.0112)1.967975 = 0.036998$$

The current value of the swap, from the perspective of the receive-fixed party, is \$50,000,000 × 0.036998 = \$1,849,897.

From the perspective of the bank, as the receive-floating party, the value of the swap is –\$1,849,897.

10 C is correct. The equilibrium swap fixed rate for yen is calculated as

$$\hat{r}_{FIX,JPY} = \frac{1 - \mathrm{PV}_{0,t_4,JPY}(1)}{\sum_{i=1}^{4} PV_{0,t_4,JPY}(1)}$$

The yen present value factors are calculated as

$$\mathrm{PV}_{0,t_i}(1) = \frac{1}{1 + r_{Spot_i}\left(\frac{NAD_i}{NTD}\right)}$$

90-day PV factor = 1/[1 + 0.0005(90/360)] = 0.999875.
180-day PV factor = 1/[1 + 0.0010(180/360)] = 0.999500.
270-day PV factor = 1/[1 + 0.0015(270/360)] = 0.998876.
360-day PV factor = 1/[1 + 0.0025(360/360)] = 0.997506.

Sum of present value factors = 3.995757.

Therefore, the yen periodic rate is calculated as

$$\hat{r}_{FIX,JPY} = \frac{1 - 0.997506}{3.995757} = 0.000624 \text{ or } 0.0624\%$$

The annualized rate is (360/90) times the periodic rate of 0.0624%, or 0.2496%.

11 B is correct. The value of an equity swap is calculated as

$$V_t = \mathrm{FB}_t(C_0) - \left(\frac{S_t}{S_{t-}}\right)\mathrm{NA}_E$$

The swap was initiated six months ago, so the first reset has not yet passed; thus, there are five remaining cash flows for this equity swap. The fair value of the swap is determined by comparing the present value of the implied fixed-rate bond with the return on the equity index. The fixed swap rate of 2.00%, the swap notional amount of \$20,000,000, and the present value factors in Exhibit 5 result in a present value of the implied fixed-rate bond's cash flows of \$19,818,677:

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Date (in years)	PV Factors	Fixed Cash Flow	PV (fixed cash flow)
0.5	0.998004 or 1/[1 + 0.0040(180/360)]	\$400,000	\$399,202
1.5	0.985222 or 1/[1 + 0.0100(540/360)]	\$400,000	\$394,089
2.5	0.970874 or 1/[1 + 0.0120(900/360)]	\$400,000	\$388,350
3.5	0.934579 or 1/[1 + 0.0200(1,260/360)]	\$400,000	\$373,832
4.5	0.895255 or 1/[1 + 0.0260(1,620/360)]	\$20,400,000	\$18,263,205
Total			\$19,818,677

The value of the equity leg of the swap is calculated as (103/100)(\$20,000,000) = \$20,600,000.

Therefore, the fair value of the equity swap, from the perspective of the bank (receive-fixed, pay-equity party) is calculated as

$$V_t = \$19{,}818{,}677 - \$20{,}600{,}000 = -781{,}323$$

12 B is correct. The equity index level at which the swap's fair value would be zero can be calculated by setting the swap valuation formula equal to zero and solving for S_t:

$$0 = \mathrm{FB}_t\left(C_0\right) - \left(\frac{S_t}{S_{t-}}\right)\mathrm{NA}_E$$

The value of the fixed leg of the swap has a present value of \$19,818,677, or 99.0934% of par value:

Date (years)	PV Factors	Fixed Cash Flow	PV (fixed cash flow)
0.5	0.998004	\$400,000	\$399,202
1.5	0.985222	\$400,000	\$394,089
2.5	0.970874	\$400,000	\$388,350
3.5	0.934579	\$400,000	\$373,832
4.5	0.895255	\$20,400,000	\$18,263,205
Total			\$19,818,677

Treating the swap notional value as par value and substituting the present value of the fixed leg and S_0 into the equation yields

$$0 = 99.0934 - \left(\frac{S_t}{100}\right)100$$

Solving for S_t yields

$$S_t = 99.0934$$

13 A is correct. The current value of the 6 × 9 FRA is calculated as

$$V_g(0,h,m) = \{[\mathrm{FRA}(g,h-g,m) - \mathrm{FRA}(0,h,m)]t_m\}/[1 + D_g(h+m-g)t_{h+m-g}]$$

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The 6 × 9 FRA expires six months after initiation. The bank entered into the FRA 90 days ago; thus, the FRA will expire in 90 days. To value the FRA, the first step is to compute the new FRA rate, which is the rate on Day 90 of an FRA that expires in 90 days in which the underlying is the 90-day Libor, or FRA(90,90,90):

$$\text{FRA}(g,h-g,m) = \{[1 + L_g(h-g+m)t_{h-g+m}]/[1 + L_0(h-g)t_{h-g}] - 1\}/t_m$$
$$\text{FRA}(90,90,90) = \{[1 + L_{90}(180-90+90)(180/360)]/[1 + L_{90}(180-90)(90/360)] - 1\}/(90/360)$$
$$\text{FRA}(90,90,90) = \{[1 + L_{90}(180)(180/360)]/[1 + L_{90}(90)(90/360)] - 1\}/(90/360)$$

Exhibit 7 indicates that $L_{90}(180) = 0.95\%$ and $L_{90}(90) = 0.90\%$, so

$$\text{FRA}(90,90,90) = \{[1 + 0.0095(180/360)]/[1 + 0.0090(90/360)] - 1\}/(90/360)$$
$$\text{FRA}(90,90,90) = [(1.00475/1.00225) - 1](4) = 0.009978, \text{ or } 0.9978\%$$

Therefore, given the FRA rate at initiation of 0.70% and notional principal of $20 million from Exhibit 1, the current value of the forward contract is calculated as

$$V_g(0,h,m) = V_{90}(0,180,90)$$
$$V_{90}(0,180,90) = \$20{,}000{,}000[(0.009978 - 0.0070)(90/360)]/[1 + 0.0095(180/360)].$$
$$V_{90}(0,180,90) = \$14{,}887.75/1.00475 = \$14{,}817.37.$$

14 C is correct. The no-arbitrage fixed rate on the 1 × 4 FRA is calculated as

$$\text{FRA}(0,h,m) = \{[1 + L_0(h+m)t_{h+m}]/[1 + L_0(h)t_h] - 1\}/t_m$$

For a 1 × 4 FRA, the two rates needed to compute the no-arbitrage FRA fixed rate are $L(30) = 0.75\%$ and $L(120) = 0.92\%$. Therefore, the no-arbitrage fixed rate on the 1 × 4 FRA rate is calculated as

$$\text{FRA}(0,30,90) = \{[1 + 0.0092(120/360)]/[1 + 0.0075(30/360)] - 1\}/(90/360).$$
$$\text{FRA}(0,30,90) = [(1.003066/1.000625) - 1]4 = 0.009761, \text{ or } 0.98\% \text{ rounded}$$

15 B is correct. The fixed rate on the 2 × 5 FRA is calculated as

$$\text{FRA}(0,h,m) = \{[1 + L_0(h+m)t_{h+m}]/[1 + L_0(h)t_h] - 1\}/t_m$$

For a 2 × 5 FRA, the two rates needed to compute the no-arbitrage FRA fixed rate are $L(60) = 0.82\%$ and $L(150) = 0.94\%$. Therefore, the no-arbitrage fixed rate on the 2 × 5 FRA rate is calculated as

$$\text{FRA}(0,60,90) = \{[1 + 0.0094(150/360)]/[1 + 0.0082(60/360)] - 1\}/(90/360)$$
$$\text{FRA}(0,60,90) = [(1.003917/1.001367) - 1]4 = 0.010186, \text{ or } 1.02\% \text{ rounded}$$

16 A is correct. Given a three-month US dollar Libor of 1.10% at expiration, the settlement amount for the bank as the receive-floating party is calculated as

$$\text{Settlement amount (receive floating)} = \text{NA}\{[L_h(m) - \text{FRA}(0,h,m)]t_m\}/[1 + D_h(m)t_m]$$
$$\text{Settlement amount (receive floating)} = \$20{,}000{,}000[(0.011 - 0.0070)(90/360)]/[1 + 0.011(90/360)]$$
$$\text{Settlement amount (receive floating)} = \$20{,}000/1.00275 = \$19{,}945.15$$

Therefore, the bank will receive $19,945 (rounded) as the receive-floating party.

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READING

38

Valuation of Contingent Claims

by Robert E. Brooks, PhD, CFA, and David Maurice Gentle, MEc, BSc, CFA

Robert E. Brooks, PhD, CFA, is at the University of Alabama (USA). David Maurice Gentle, MEc, BSc, CFA, is at Omega Risk Consulting (Australia).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** describe and interpret the binomial option valuation model and its component terms;
☐	**b.** calculate the no-arbitrage values of European and American options using a two-period binomial model;
☐	**c.** identify an arbitrage opportunity involving options and describe the related arbitrage;
☐	**d.** calculate and interpret the value of an interest rate option using a two-period binomial model;
☐	**e.** describe how the value of a European option can be analyzed as the present value of the option's expected payoff at expiration;
☐	**f.** identify assumptions of the Black–Scholes–Merton option valuation model;
☐	**g.** interpret the components of the Black–Scholes–Merton model as applied to call options in terms of a leveraged position in the underlying;
☐	**h.** describe how the Black–Scholes–Merton model is used to value European options on equities and currencies;
☐	**i.** describe how the Black model is used to value European options on futures;
☐	**j.** describe how the Black model is used to value European interest rate options and European swaptions;
☐	**k.** interpret each of the option Greeks;
☐	**l.** describe how a delta hedge is executed;
☐	**m.** describe the role of gamma risk in options trading;
☐	**n.** define implied volatility and explain how it is used in options trading.

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1 INTRODUCTION

A contingent claim is a derivative instrument that provides its owner a right but not an obligation to a payoff determined by an underlying asset, rate, or other derivative. Contingent claims include options, the valuation of which is the objective of this reading. Because many investments contain embedded options, understanding this material is vital for investment management.

Our primary purpose is to understand how the values of options are determined. Option values, as with the values of all financial instruments, are typically obtained using valuation models. Any financial valuation model takes certain inputs and turns them into an output that tells us the fair value or price. Option valuation models, like their counterparts in the forward, futures, and swaps markets, are based on the principle of no arbitrage, meaning that the appropriate price of an option is the one that makes it impossible for any party to earn an arbitrage profit at the expense of any other party. The price that precludes arbitrage profits is the value of the option. Using that concept, we then proceed to introduce option valuation models using two approaches. The first approach is the binomial model, which is based on discrete time, and the second is the Black–Scholes–Merton (BSM) model, which is based on continuous time.

The reading is organized as follows. Section 2 introduces the principles of the no-arbitrage approach to pricing and valuation of options. In Section 3, the binomial option valuation model is explored, and in Section 4, the BSM model is covered. In Section 5, the Black model, being a variation of the BSM model, is applied to futures options, interest rate options, and swaptions. Finally, in Section 6, the Greeks are reviewed along with implied volatility. Section 7 provides a summary.

2 PRINCIPLES OF A NO-ARBITRAGE APPROACH TO VALUATION

Our approach is based on the concept of arbitrage. Hence, the material will be covered from an arbitrageur's perspective. Key to understanding this material is to think like an arbitrageur. Specifically, like most people, the arbitrageur would rather have more money than less. The arbitrageur, as will be detailed later, follows two fundamental rules:

Rule #1 Do not use your own money.

Rule #2 Do not take any price risk.

Clearly, if we can generate positive cash flows today and abide by both rules, we have a great business—such is the life of an arbitrageur. If traders could create a portfolio with no future liabilities and positive cash flow today, then it would essentially be a money machine that would be attractive to anyone who prefers more cash to less. In the pursuit of these positive cash flows today, the arbitrageur often needs to borrow to satisfy Rule #1. In effect, the arbitrageur borrows the arbitrage profit to capture it today and, if necessary, may borrow to purchase the underlying. Specifically, the arbitrageur will build portfolios using the underlying instrument to synthetically replicate the cash flows of an option. The underlying instrument is the financial instrument whose later value will be referenced to determine the option value. Examples of underlying instruments include shares, indexes, currencies, and interest rates. As we will see, with options we will often rely on a specific trading strategy that changes over time based on the underlying price behavior.

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Based on the concept of comparability, the no-arbitrage valuation approach taken here is built on the concept that if two investments have the same future cash flows regardless of what happens, then these two investments should have the same current price. This principle is known as the **law of one price**. In establishing these foundations of option valuation, the following key assumptions are made: (1) Replicating instruments are identifiable and investable. (2) There are no market frictions, such as transaction costs and taxes. (3) Short selling is allowed with full use of proceeds. (4) The underlying instrument follows a known statistical distribution. (5) Borrowing and lending at a risk-free interest rate is available. When we develop the models in this reading, we will be more specific about what these assumptions mean, in particular what we mean by a known statistical distribution.

In an effort to demonstrate various valuation results based on the absence of arbitrage, we will rely heavily on cash flow tables, which are a representation of the cash flows that occur during the life of an option. For example, if an initial investment requires €100, then from an arbitrageur's perspective, we will present it as a –€100 cash flow. If an option pays off ¥1,000, we will represent it as a +¥1,000 cash flow. That is, cash outflows are treated as negative and inflows as positive.

We first demonstrate how to value options based on a two-period binomial model. The option payoffs can be replicated with a dynamic portfolio of the underlying instrument and financing. A dynamic portfolio is one whose composition changes over time. These changes are important elements of the replicating procedure. Based on the binomial framework, we then turn to exploring interest rate options using a binomial tree. Although more complex, the general approach is shown to be the same.

The multiperiod binomial model is a natural transition to the BSM option valuation model. The BSM model is based on the key assumption that the value of the underlying instrument follows a statistical process called geometric Brownian motion. This characterization is a reasonable way to capture the randomness of financial instrument prices while incorporating a pre-specified expected return and volatility of return. Geometric Brownian motion implies a lognormal distribution of the return, which implies that the continuously compounded return on the underlying is normally distributed.

We also explore the role of carry benefits, meaning the reward or cost of holding the underlying itself instead of holding the derivative on the underlying.

Next we turn to Fischer Black's futures option valuation model (Black model) and note that the model difference, versus the BSM model, is related to the underlying futures contract having no carry costs or benefits. Interest rate options and swaptions are valued based on simple modifications of the Black model.

Finally, we explore the Greeks, otherwise known as delta, gamma, theta, vega, and rho. The Greeks are representations of the sensitivity of the option value to changes in the factors that determine the option value. They provide comparative information essential in managing portfolios containing options. The Greeks are calculated based on an option valuation model, such as the binomial model, BSM model, or the Black model. This information is model dependent, so managers need to carefully select the model best suited for their particular situation. In the last section, we cover implied volatility, which is a measure derived from a market option price and can be interpreted as reflecting what investors believe is the volatility of the underlying.

The models presented here are useful first approximations for explaining observed option prices in many markets. The central theme is that options are generally priced to preclude arbitrage profits, which is not only a reasonable theoretical assumption but is sufficiently accurate in practice.

We turn now to option valuation based on the binomial option valuation model.

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3 BINOMIAL OPTION VALUATION MODEL

The binomial model is a valuable tool for financial analysts. It is particularly useful as a heuristic device to understand the unique valuation approach used with options. This model is extensively used to value path-dependent options, which are options whose values depend not only on the value of the underlying at expiration but also how it got there. The path-dependency feature distinguishes this model from the Black–Scholes–Merton option valuation model (BSM model) presented in the next section. The BSM model values only path-independent options, such as European options, which depend on only the values of their respective underlyings at expiration. One particular type of path-dependent option that we are interested in is American options, which are those that can be exercised prior to expiration. In this section, we introduce the general framework for developing the binomial option valuation models for both European and American options.

The binomial option valuation model is based on the no-arbitrage approach to valuation. Hence, understanding the valuation of options improves if one can understand how an arbitrageur approaches financial markets. An arbitrageur engages in financial transactions in pursuit of an initial positive cash flow with no possibility of a negative cash flow in the future. As it appears, it is a great business if you can find it.[1]

To understand option valuation models, it is helpful to think like an arbitrageur. The arbitrageur seeks to exploit any pricing discrepancy between the option price and the underlying spot price. The arbitrageur is assumed to prefer more money compared with less money, assuming everything else is the same. As mentioned earlier, there are two fundamental rules for the arbitrageur.

Rule #1 Do not use your own money. Specifically, the arbitrageur does not use his or her own money to acquire positions. Also, the arbitrageur does not spend proceeds from short selling transactions on activities unrelated to the transaction at hand.

Rule #2 Do not take any price risk. The focus here is only on market price risk related to the underlying and the derivatives used. We do not consider other risks, such as liquidity risk and counterparty credit risk.

We will rely heavily on these two rules when developing option valuation models. Remember, these rules are general in nature, and as with many things in finance, there are nuances.

In Exhibit 1, the two key dates are the option contract initiation date (identified as Time 0) and the option contract expiration date (identified as Time T). Based on the no-arbitrage approach, the option value from the initiation date onward will be estimated with an option valuation model.

1 There is not a one-to-one correspondence between arbitrage and great investment opportunities. An arbitrage is certainly a great investment opportunity because it produces a risk-free profit with no investment of capital, but all great investment opportunities are not arbitrage. For example, an opportunity to invest €1 today in return for a 99% chance of receiving €1,000,000 tomorrow or a 1% chance of receiving €0 might appear to be a truly great investment opportunity, but it is not arbitrage because it is not risk free and requires the investment of capital.

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Exhibit 1 Illustration of Option Contract Initiation and Expiration

Let S_t denote the underlying instrument price observed at Time t, where t is expressed as a fraction of a year. Similarly, S_T denotes the underlying instrument price observed at the option expiration date, T. For example, suppose a call option had 90 days to expiration when purchased (T = 90/365), but now only has 35 days to expiration (t = 55/365). Further, let c_t denote a European-style call price at Time t and with expiration on Date t = T, where both t and T are expressed in years. Similarly, let C_t denote an American-style call price. At the initiation date, the subscripts are omitted, thus $c = c_0$. We follow similar notation with a put, using the letter p, in place of c. Let X denote the exercise price.[2]

For example, suppose on 15 April a 90-day European-style call option contract with a 14 July expiration is initiated with a call price of c = €2.50 and T = 90/365 = 0.246575.

At expiration, the call and put values will be equal to their intrinsic value or exercise value. These **exercise values** can be expressed as

$$c_T = \text{Max}(0, S_T - X) \text{ and}$$
$$p_T = \text{Max}(0, X - S_T),$$

respectively. If the option values deviate from these expressions, then there will be arbitrage profits available. The option is expiring, there is no uncertainty remaining, and the price must equal the market value obtained from exercising it or letting it expire.

Technically, European options do not have exercise values prior to expiration because they cannot be exercised until expiration. Nonetheless, the notion of the value of the option if it could be exercised, $\text{Max}(0,S_t - X)$ for a call and $\text{Max}(0,X - S_t)$ for a put, forms a basis for understanding the notion that the value of an option declines with the passage of time. Specifically, option values contain an element known as time value, which is just the market valuation of the potential for higher exercise value relative to the potential for lower exercise value. The time value is always non-negative because of the asymmetry of option payoffs at expiration. For example, for a call, the upside is unlimited, whereas the downside is limited to zero. At expiration, time value is zero.

Although option prices are influenced by a variety of factors, the underlying instrument has a particularly significant influence. At this point, the underlying is assumed to be the only uncertain factor affecting the option price. We now look in detail at the one-period binomial option valuation model. The one-period binomial model is foundational for the material that follows.

3.1 One-Period Binomial Model

Exhibit 2 illustrates the one-period binomial process for an asset priced at S. In the figure on the left, each dot represents a particular outcome at a particular point in time in the binomial lattice. The dots are termed nodes. At the Time 0 node, there are only two possible future paths in the binomial process, an up move and a down

2 In financial markets, the exercise price is also commonly called the strike price.

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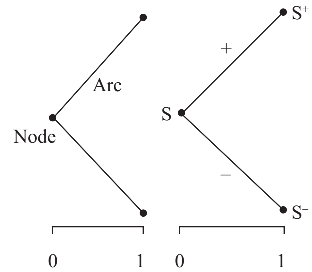
move, termed arcs. The figure on the right illustrates the underlying price at each node. At Time 1, there are only two possible outcomes: S^+ denotes the outcome when the underlying goes up, and S^- denotes the outcome when the underlying goes down.

Exhibit 2 One-Period Binomial Lattice with Underlying Distribution Illustrated



At Time 1, there are only two possible outcomes and two resulting values of the underlying, S^+ (up occurs) and S^- (down occurs). Although the one-period binomial model is clearly unrealistic, it will provide key insights into the more realistic multi-period binomial as well as the BSM model.

We further define the total returns implied by the underlying movements as

$$u = \frac{S^+}{S}\text{(up factor) and}$$

$$d = \frac{S^-}{S}\text{(down factor).}$$

The up factors and down factors are the total returns; that is, one plus the rate of return. The magnitudes of the up and down factors are based on the volatility of the underlying. In general, higher volatility will result in higher up values and lower down values.

We briefly review option valuation within a one-period binomial tree. With this review, we can move quickly to option valuation within a two-period binomial lattice by performing the one-period exercise three times.

We consider the fair value of a two-period call option value measured at Time 1 when an up move occurs, that is c^+. Based on arbitrage forces, we know this option value at expiration is either

$$c^{++} = \text{Max}(0,S^{++} - X) = \text{Max}(0,u^2S - X)\text{, or}$$
$$c^{+-} = \text{Max}(0,S^{+-} - X) = \text{Max}(0,udS - X).$$

At this point, we assume that there are no costs or benefits from owning the underlying instrument. Now consider the transactions illustrated in Exhibit 3. These transactions are presented as cash flows. Thus, if we write a call option, we receive money at Time Step 0 and may have to pay out money at Time Step 1. Suppose the first trade is to write or sell one call option within the single-period binomial model. The value of a call option is positively related to the value of the underlying. That is, they both move up or down together. Hence, by writing a call option, the trader will lose money if the underlying goes up and make money if the underlying falls. Therefore, to execute a hedge, the trader will need a position that will make money

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if the underlying goes up. Thus, the second trade needs to be a long position in the underlying. Specifically, the trader buys a certain number of units, h, of the underlying. The symbol h is used because it represents a hedge ratio.

Note that with these first two trades, neither arbitrage rule is satisfied. The future cash flow could be either $-c^- + hS^-$ or $-c^+ + hS^+$ and can be positive or negative. Thus, the cash flows at the Time Step 1 could result in the arbitrageur having to pay out money if one of these values is less than zero. To resolve both of these issues, we set the Time Step 1 cash flows equal to each other—that is, $-c^+ + hS^+ = -c^- + hS^-$—and solve for the appropriate hedge ratio:

$$h = \frac{c^+ - c^-}{S^+ - S^-} \geq 0 \quad (1)$$

We determine the hedge ratio such that we are indifferent to the underlying going up or down. Thus, we are hedged against moves in the underlying. A simple rule for remembering this formula is that the hedge ratio is the value of the call if the underlying goes up minus the value of the call if the underlying goes down divided by the value of the underlying if it goes up minus the value of the underlying if it goes down. The up and down patterns are the same in the numerator and denominator, but the numerator contains the option and the denominator contains the underlying.

Because call prices are positively related to changes in the underlying price, we know that h is non-negative. As shown in Exhibit 3, we will buy h underlying units as depicted in the second trade, and we will finance the present value of the net cash flows as depicted in the third trade. If we assume r denotes the per period risk-free interest rate, then the present value calculation, denoted as PV, is equal to $1/(1 + r)$. We need to borrow or lend an amount such that the future net cash flows are equal to zero. Therefore, we finance today the present value of $-hS^- + c^-$ which also equals $-hS^+ + c^+$. At this point we do not know if the finance term is positive or negative, thus we may be either borrowing or lending, which will depend on c, h, and S.

Exhibit 3 Writing One Call Hedge with h Units of the Underlying and Finance

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
1) Write one call option	$+c$	$-c^-$	$-c^+$
2) Buy h underlying units	$-hS$	$+hS^-$	$+hS^+$
3) Borrow or lend	$-PV(-hS^- + c^-)$ $= -PV(-hS^+ + c^+)$	$-hS^- + c^-$	$-hS^+ + c^+$
Net Cash Flow	$+c - hS$ $-PV(-hS^- + c^-)$	0	0

The value of the net portfolio at Time Step 0 should be zero or there is an arbitrage opportunity. If the net portfolio has positive value, then arbitrageurs will engage in this strategy, which will push the call price down and the underlying price up until the net is no longer positive. We assume the size of the borrowing will not influence interest rates. If the net portfolio has negative value, then arbitrageurs will engage in the opposite strategy—buy calls, short sell the underlying, and lend—pushing the call price up and the underlying price down until the net cash flow at Time 0 is no longer positive. Therefore, within the single-period binomial model, we have

$$+c - hS - PV(-hS^- + c^-) = 0$$

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or, equivalently,

$$+c - hS - PV(-hS^+ + c^+) = 0.$$

Therefore, the **no-arbitrage approach** leads to the following single-period call option valuation equation:

$$c = hS + PV(-hS^- + c^-) \tag{2}$$

or, equivalently, $c = hS + PV(-hS^+ + c^+)$. In words, long a call option is equal to owning h shares of stock partially financed, where the financed amount is $PV(-hS^- + c^-)$, or using the per period rate, $(-hS^- + c^-)/(1 + r)$.[3]

We will refer to Equation 2 as the no-arbitrage single-period binomial option valuation model. This equation is foundational to understanding the two-period binomial as well as other option valuation models. The option can be replicated with the underlying and financing, a point illustrated in the following example.

EXAMPLE 1

Long Call Option Replicated with Underlying and Financing

Identify the trading strategy that will generate the payoffs of taking a long position in a call option within a single-period binomial framework.

A Buy $h = (c^+ + c^-)/(S^+ + S^-)$ units of the underlying and financing of $-PV(-hS^- + c^-)$

B Buy $h = (c^+ - c^-)/(S^+ - S^-)$ units of the underlying and financing of $-PV(-hS^- + c^-)$

C Short sell $h = (c^+ - c^-)/(S^+ - S^-)$ units of the underlying and financing of $+PV(-hS^- + c^-)$

Solution:

B is correct. The following table shows the terminal payoffs to be identical between a call option and buying the underlying with financing.

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
Buy 1 call option	$-c$	$+c^-$	$+c^+$
OR A REPLICATING PORTFOLIO			
Buy h underlying units	$-hS$	$+hS^-$	$+hS^+$
Borrow or lend	$-PV(-hS^- + c^-)$ $= -PV(-hS^+ + c^+)$	$-hS^- + c^-$	$-hS^+ + c^+$
Net	$-hS - PV(-hS^- + c^-)$	$+c^-$	$+c^+$

Recall that by design, h is selected such that $-hS^- + c^- = -hS^+ + c^+$ or $h = (c^+ - c^-)/(S^+ - S^-)$. Therefore, a call option can be replicated with the underlying and financing. Specifically, the call option is equivalent to a leveraged position in the underlying.

3 Or, by the same logic, $PV(-hS^+ + c^+)$, which is $(-hS^+ + c^+)/(1 + r)$.

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Thus, the no-arbitrage approach is a replicating strategy: A call option is synthetically replicated with the underlying and financing. Following a similar strategy with puts, the no-arbitrage approach leads to the following no-arbitrage single-period put option valuation equation:

$$p = hS + PV(-hS^- + p^-) \tag{3}$$

or, equivalently, $p = hS + PV(-hS^+ + p^+)$ where

$$h = \frac{p^+ - p^-}{S^+ - S^-} \le 0 \tag{4}$$

Because p^+ is less than p^-, the hedge ratio is negative. Hence, to replicate a long put position, the arbitrageur will short sell the underlying and lend a portion of the proceeds. Note that a long put position would be replicated by trading h units of the underlying. With h negative, this trade is a short sale, and because $-h$ is positive, the value $-hS$ results in a positive cash flow at Time Step 0.

EXAMPLE 2

Long Put Option Replicated with Underlying and Financing

Identify the trading strategy that will generate the payoffs of taking a long position in a put option within a single-period binomial framework.

A Short sell $-h = -(p^+ - p^-)/(S^+ - S^-)$ units of the underlying and financing of $-PV(-hS^- + p^-)$

B Buy $-h = (p^+ - p^-)/(S^+ - S^-)$ units of the underlying and financing of $-PV(-hS^- + p^-)$

C Short sell $h = (p^+ - p^-)/(S^+ - S^-)$ units of the underlying and financing of $+PV(-hS^- + p^-)$

Solution:

A is correct. Before illustrating the replicating portfolio, we make a few observations regarding the hedge ratio. Note that by design, h is selected such that $-hS^- + p^- = -hS^+ + p^+$ or $h = (p^+ - p^-)/(S^+ - S^-)$. Unlike calls, the put hedge ratio is not positive (note that $p^+ < p^-$ but $S^+ > S^-$). Remember that taking a position in $-h$ units of the underlying is actually short selling the underlying rather than buying it. The following table shows the terminal payoffs to be identical between a put option and a position in the underlying with financing.

Strategy	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
Buy 1 Put Option	$-p$	$+p^-$	$+p^+$
OR A REPLICATING PORTFOLIO			
Short sell $-h$ Underlying Units	$-hS$	$+hS^-$	$+hS^+$
Borrow or Lend	$-PV(-hS^- + p^-) = -PV(-hS^+ + p^+)$	$-hS^- + p^-$	$-hS^+ + p^+$
Net	$-hS - PV(-hS^- + p^-)$	$+p^-$	$+p^+$

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Therefore, a put option can be replicated with the underlying and financing. Specifically, the put option is simply equivalent to a short position in the underlying with financing in the form of lending.

What we have shown to this point is the no-arbitrage approach. Before turning to the expectations approach, we mention, for the sake of completeness, that the transactions for writing options are the reverse for those of buying them. Thus, for writing a call option, the writer will be selling stock short and investing proceeds, whereas for a put, the writer will be purchasing stock on margin. Once again, we see the powerful result that the same basic conceptual structure is used for puts and calls, whether written or purchased. Only the exercise and expiration conditions vary.

The no-arbitrage results that have been presented can be expressed as the present value of a unique expectation of the option payoffs.[4] Specifically, the **expectations approach** results in an identical value as the no-arbitrage approach, but it is usually easier to compute. The formulas are viewed as follows:

$$c = PV[\pi c^{+} + (1 - \pi)c^{-}] \text{ and} \tag{5}$$

$$p = PV[\pi p^{+} + (1 - \pi)p^{-}] \tag{6}$$

where the probability of an up move is

$$\pi = [FV(1) - d]/(u - d)$$

Recall the future value is simply the reciprocal of the present value or FV(1) = 1/PV(1). Thus, if PV(1) = 1/(1 + r), then FV(1) = (1 + r). Note that the option values are simply the present value of the expected terminal option payoffs. The expected terminal option payoffs can be expressed as

$$E(c_1) = \pi c^{+} + (1 - \pi)c^{-} \text{ and}$$

$$E(p_1) = \pi p^{+} + (1 - \pi)p^{-}$$

where c_1 and p_1 are the values of the options at Time 1. The present value and future value calculations are based on the risk-free rate, denoted r.[5] Thus, the option values based on the expectations approach can be written and remembered concisely as

$$c = PV_r[E(c_1)] \text{ and}$$

$$p = PV_r[E(p_1)]$$

The expectations approach to option valuation differs in two significant ways from the discounted cash flow approach to securities valuation. First, the expectation is not based on the investor's beliefs regarding the future course of the underlying. That is, the probability, π, is objectively determined and not based on the investor's personal view. This probability has taken several different names, including risk-neutral (RN) probability. Importantly, we did not make any assumption regarding the arbitrageur's risk preferences: The expectations approach is a result of this arbitrage process, not an assumption regarding risk preferences. Hence, they are called risk-neutral probabilities. Although we called them probabilities from the very start, they are not the true probabilities of up and down moves.

4 It takes a bit of algebra to move from the no-arbitrage expression to the present value of the expected future payoffs, but the important point is that both expressions yield exactly the same result.

5 We will suppress "r" most of the time and simply denote the calculation as PV. The "r" will be used at times to reinforce that the present value calculation is based on the risk-free interest rate.

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Second, the discount rate is *not* risk adjusted. The discount rate is simply based on the estimated risk-free interest rate. The expectations approach here is often viewed as superior to the discounted cash flow approach because both the subjective future expectation as well as the subjective risk-adjusted discount rate have been replaced with more objective measures.

EXAMPLE 3

Single-Period Binomial Call Value

A non-dividend-paying stock is currently trading at €100. A call option has one year to mature, the periodically compounded risk-free interest rate is 5.15%, and the exercise price is €100. Assume a single-period binomial option valuation model, where u = 1.35 and d = 0.74.

1 The optimal hedge ratio will be *closest* to:
 A 0.57.
 B 0.60.
 C 0.65.

2 The call option value will be *closest* to:
 A €13.
 B €15.
 C €17.

Solution to 1:

A is correct. Given the information provided, we know the following:

$S^+ = uS = 1.35(100) = 135$

$S^- = dS = 0.74(100) = 74$

$c^+ = \text{Max}(0,uS - X) = \text{Max}(0,135 - 100) = 35$

$c^- = \text{Max}(0,dS - X) = \text{Max}(0,74 - 100) = 0$

With this information, we can compute both the hedge ratio as well as the call option value. The hedge ratio is:

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{35 - 0}{135 - 74} = 0.573770$$

Solution to 2:

C is correct. The risk-neutral probability of an up move is

$$\pi = [FV(1) - d]/(u - d) = (1.0515 - 0.74)/(1.35 - 0.74) = 0.510656,$$

where FV(1) = (1 + r) = 1.0515.

Thus the call value by the expectations approach is

$$c = PV[\pi c^+ + (1 - \pi)c^-] = 0.951022[(0.510656)35 + (1 - 0.510656)0] = €16.998,$$

where PV(1) = 1/(1 + r) = 1/(1.0515) = 0.951022.

Note that the call value by the no-arbitrage approach yields the same answer:

$$c = hS + PV(-hS^- + c^-) = 0.573770(100) + 0.951022[-0.573770(74) + 0] = €16.998.$$

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The value of a put option can also be found based on put–call parity. Put–call parity can be remembered as simply two versions of portfolio insurance, long stock and long put or lend and long call, where the exercise prices for the put and call are identical. Put–call parity with symbols is

$$S + p = PV(X) + c \tag{7}$$

Put–call parity holds regardless of the particular valuation model being used. Depending on the context, this equation can be rearranged. For example, a call option can be expressed as a position in a stock, financing, and a put, or

$$c = S - PV(X) + p$$

EXAMPLE 4

Single-Period Binomial Put Value

You again observe a €100 price for a non-dividend-paying stock with the same inputs as the previous box. That is, the call option has one year to mature, the periodically compounded risk-free interest rate is 5.15%, the exercise price is €100, u = 1.35, and d = 0.74. The put option value will be *closest* to:

A €12.00.

B €12.10.

C €12.20.

Solution:

B is correct. For puts, we know the following:

$$p^{+} = \text{Max}(0{,}100 - uS) = \text{Max}(0{,}100 - 135) = 0$$

$$p^{-} = \text{Max}(0{,}100 - dS) = \text{Max}(0{,}100 - 74) = 26$$

With this information, we can compute the put option value based on risk-neutral probability from the previous example or [recall that PV(1) = 0.951022]

$$p = PV[\pi p^{+} + (1 - \pi)p^{-}] = 0.951022[(0.510656)0 + (1 - 0.510656)26] = €12.10$$

Therefore, in summary, option values can be expressed either in terms of replicating portfolios or as the present value of the expected future cash flows. Both expressions yield the same valuations.

3.2 Two-Period Binomial Model

The two-period binomial lattice can be viewed as three one-period binomial lattices, as illustrated in Exhibit 4. Clearly, if we understand the one-period model, then the process can be repeated three times. First, we analyze Box 1 and Box 2. Finally, based on the results of Box 1 and Box 2, we analyze Box 3.

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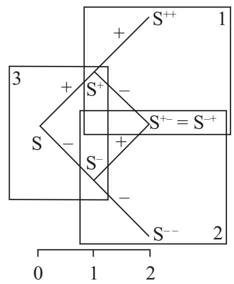
Exhibit 4 Two-Period Binomial Lattice as Three One-Period Binomial Lattices



At Time 2, there are only three values of the underlying, S^{++} (an up move occurs twice), S^{--} (a down move occurs twice), and $S^{+-} = S^{-+}$ (either an up move occurs and then a down move or a down move occurs and then an up move). For computational reasons, it is extremely helpful that the lattice recombines—that is, $S^{+-} = S^{-+}$, meaning that if the underlying goes up and then down, it ends up at the same price as if it goes down and then up. A recombining binomial lattice will always have just one more ending node in the final period than the number of time steps. In contrast, a non-recombining lattice of n time steps will have 2^n ending nodes, which poses a tremendous computational challenge even for powerful computers.

For our purposes here, we assume the up and down factors are constant throughout the lattice, ensuring that the lattice recombines—that is $S^{+-} = S^{-+}$. For example, assume u = 1.25, d = 0.8, and S_0 = 100. Note that S^{+-} = 1.25(0.8)100 = 100 and S^{-+} = 0.8(1.25)100 = 100. So the middle node at Time 2 is 100 and can be reached from either of two paths.

The two-period binomial option valuation model illustrates two important concepts, self-financing and dynamic replication. Self-financing implies that the replicating portfolio will not require any additional funds from the arbitrageur during the life of this dynamically rebalanced portfolio. If additional funds are needed, then they are financed externally. Dynamic replication means that the payoffs from the option can be exactly replicated through a planned trading strategy. Option valuation relies on self-financing, dynamic replication.

Mathematically, the no-arbitrage approach for the two-period binomial model is best understood as working backward through the binomial tree. At Time 2, the payoffs are driven by the option's exercise value.

For calls:

$$c^{++} = \text{Max}(0,S^{++} - X) = \text{Max}(0,u^2S - X),$$
$$c^{+-} = \text{Max}(0,S^{+-} - X) = \text{Max}(0,udS - X), \text{ and}$$
$$c^{--} = \text{Max}(0,S^{--} - X) = \text{Max}(0,d^2S - X)$$

For puts:

$$p^{++} = \text{Max}(0,X - S^{++}) = \text{Max}(0,X - u^2S),$$
$$p^{+-} = \text{Max}(0,X - S^{+-}) = \text{Max}(0,X - udS), \text{ and}$$
$$p^{--} = \text{Max}(0,X - S^{--}) = \text{Max}(0,X - d^2S)$$

At Time 1, the option values are driven by the arbitrage transactions that synthetically replicate the payoffs at Time 2. We can compute the option values at Time 1 based on the option values at Time 2 using the no-arbitrage approach based on

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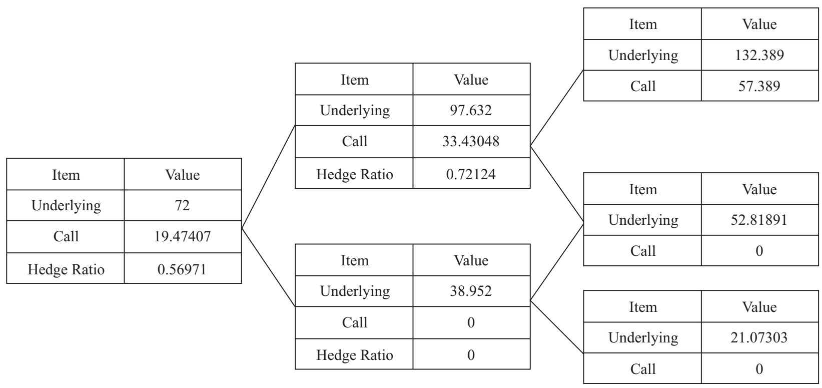
Equations 1 and 2. At Time 0, the option values are driven by the arbitrage transactions that synthetically replicate the value of the options at Time 1 (again based on Equations 1 and 2).

We illustrate the no-arbitrage approach for solving the two-period binomial call value. Suppose the annual interest rate is 3%, the underlying stock is S = 72, u = 1.356, d = 0.541, and the exercise price is X = 75. The stock does not pay dividends. Exhibit 5 illustrates the results.

Exhibit 5 Two-Period Binomial Tree with Call Values and Hedge Ratios



We now verify selected values reported in Exhibit 5. At Time Step 2 and assuming up occurs twice, the underlying stock value is $u^2S = (1.356)^2 72 = 132.389$, and hence, the call value is 57.389 [= Max(0,132.389 – 75)]. The hedge ratio at Time Step 1, assuming up occurs once, is

$$h^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}} = \frac{57.389 - 0}{132.389 - 52.819} = 0.72124$$

The RN probability of an up move throughout this tree is

$$\pi = [FV(1) - d]/(u - d) = (1.03 - 0.541)/(1.356 - 0.541) = 0.6$$

With this information, we can compute the call price at Time 1 when an up move occurs as

$$c = PV[\pi c^{++} + (1 - \pi)c^{+-}] = (1/1.03)[(0.6)57.389 + (1 - 0.6)0] = 33.43048$$

and at Time Step 0,

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{33.43048 - 0}{97.632 - 38.952} = 0.56971$$

Thus, the call price at the start is

$$c = PV[\pi c^+ + (1 - \pi)c^-] = (1/1.03)[(0.6)33.43048 + (1 - 0.6)0] = 19.47$$

From the no-arbitrage approach, the call payoffs can be replicated by purchasing h shares of the underlying and financing $-PV(-hS^- + c^-)$. Therefore, we purchase 0.56971 shares of stock for 41.019 [= 0.56971(72)] and borrow 21.545 {or in cash flow terms, –21.545 = (1/1.03)[–0.56971(38.952) + 0]}, replicating the call values at Time 0. We then illustrate Time 1 assuming that an up move occurs. The stock position will now be worth 55.622 [= 0.56971(97.632)], and the borrowing must be repaid with interest or 22.191 [= 1.03(21.545)]. Note that the portfolio is worth 33.431 (55.622 – 22.191), the same value as the call except for a small rounding error. Therefore, the portfolio of stock and the financing dynamically replicates the value of the call option.

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The final task is to demonstrate that the portfolio is self-financing. Self-financing can be shown by observing that the new portfolio at Time 1, assuming an up move occurs, is equal to the old portfolio that was formed at Time 0 and liquidated at Time 1. Notice that the hedge ratio rose from 0.56971 to 0.72124 as we moved from Time 0 to Time 1, assuming an up move occurs, requiring the purchase of additional shares. These additional shares will be financed with additional borrowing. The total borrowing is 36.98554 {= $-PV(-hS^{+-} + c^{+-}) = -(1/1.03)[-0.72124(52.81891) + 0]$}. The borrowing at Time 0 that is due at Time 1 is 22.191. The funds borrowed at Time 1 grew to 36.98554. Therefore, the strategy is self-financing.

The two-period binomial model can also be represented as the present value of an expectation of future cash flows. Based on the one-period results, it follows by repeated substitutions that

$$c = PV[\pi^2 c^{++} + 2\pi(1-\pi)c^{+-} + (1-\pi)^2 c^{--}] \tag{8}$$

and

$$p = PV[\pi^2 p^{++} + 2\pi(1-\pi)p^{+-} + (1-\pi)^2 p^{--}] \tag{9}$$

Therefore, the two-period binomial model is again simply the present value of the expected future cash flows based on the RN probability. Again, the option values are simply the present value of the expected terminal option payoffs. The expected terminal option payoffs can be expressed as

$$E(c_2) = \pi^2 c^{++} + 2\pi(1-\pi)c^{+-} + (1-\pi)^2 c^{--}$$

and

$$E(p_2) = \pi^2 p^{++} + 2\pi(1-\pi)p^{+-} + (1-\pi)^2 p^{--}$$

Thus, the two-period binomial option values based on the expectations approach can be written and remembered concisely as

$$c = PV_r[E\pi(c_2)] \text{ and}$$

$$p = PV_r[E\pi(p_2)]$$

It is vital to remember that this present value is over two periods, so the discount factor with discrete rates is $PV = [1/(1 + r)^2]$. Recall the subscript "r" just emphasizes the present value calculation and is based on the risk-free interest rate.

EXAMPLE 5

Two-Period Binomial Model Call Valuation

You observe a €50 price for a non-dividend-paying stock. The call option has two years to mature, the periodically compounded risk-free interest rate is 5%, the exercise price is €50, u = 1.356, and d = 0.744. Assume the call option is European-style.

1 The probability of an up move based on the risk-neutral probability is *closest* to:

A 30%.

B 40%.

C 50%.

2 The current call option value is *closest* to:

A €9.53.

B €9.71.

C €9.87.

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3 The current put option value is *closest* to:

A €5.06.

B €5.33.

C €5.94.

Solution to 1:

C is correct. Based on the RN probability equation, we have:

$$\pi = [FV(1) - d]/(u - d) = [(1 + 0.05) - 0.744]/(1.356 - 0.744) = 0.5 \text{ or } 50\%$$

Solution to 2:

B is correct. The current call option value calculations are as follows:

$$c^{++} = \text{Max}(0,u^2S - X) = \text{Max}[0,1.356^2(50) - 50] = 41.9368$$

$$c^{-+} = c^{+-} = \text{Max}(0,udS - X) = \text{Max}[0,1.356(0.744)(50) - 50] = 0.44320$$

$$c^{--} = \text{Max}(0,d^2S - X) = \text{Max}[0,0.744^2(50) - 50] = 0.0$$

With this information, we can compute the call option value:

$$\begin{aligned} c &= PV[E(c_2)] = PV[\pi^2c^{++} + 2\pi(1 - \pi)c^{+-} + (1 - \pi)^2c^{--}] \\ &= [1/(1 + 0.05)]^2[0.5^2 41.9368 + 2(0.5)(1 - 0.5)0.44320 + (1 - 0.5)^2 0.0] \\ &= 9.71 \end{aligned}$$

It is vital to remember that the present value is over two periods, hence the single-period PV is squared. Thus, the current call price is €9.71.

Solution to 3:

A is correct. The put option value can be computed simply by applying put–call parity or $p = c + PV(X) - S = 9.71 + [1/(1 + 0.05)]^2 50 - 50 = 5.06$. Thus, the current put price is €5.06.

We now turn to consider American-style options. It is well-known that non-dividend-paying call options on stock will not be exercised early because the minimum price of the option exceeds its exercise value. To illustrate by example, consider a call on a US$100 stock, with an exercise price of US$10 (that is, very deep in the money). Suppose the call is worth its exercise value of only US$90. To get stock exposure, one could fund and pay US$100 to buy the stock, or fund and pay only US$90 for the call and pay the last US$10 at expiration only if the stock is at or above US$100 at that time. Because the latter choice is preferable, the call must be worth more than the US$90 exercise value. Another way of looking at it is that it would make no sense to exercise this call because you do not believe the stock can go any higher and you would thus simply be obtaining a stock that you believe would go no higher. Moreover, the stock would require that you pay far more money than you have tied up in the call. It is always better to just sell the call in this situation because it will be trading for more than the exercise value.

The same is not true for put options. By early exercise of a put, particularly a deep in-the-money put, the sale proceeds can be invested at the risk-free rate and earn interest worth more than the time value of the put. Thus, we will examine how early exercise influences the value of an American-style put option. As we will see, when early exercise has value, the no-arbitrage approach is the only way to value American-style options.

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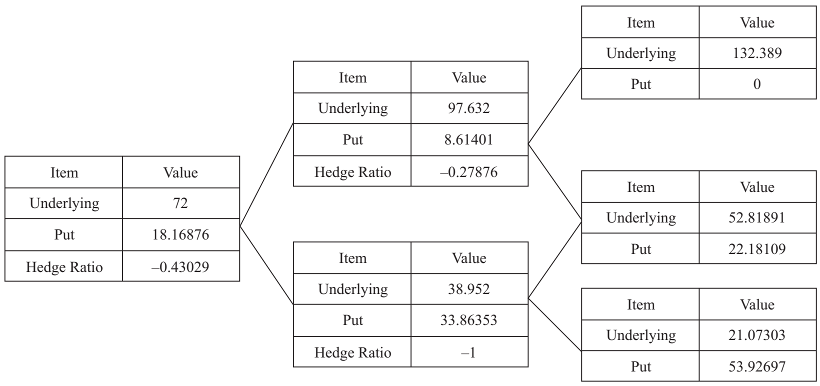
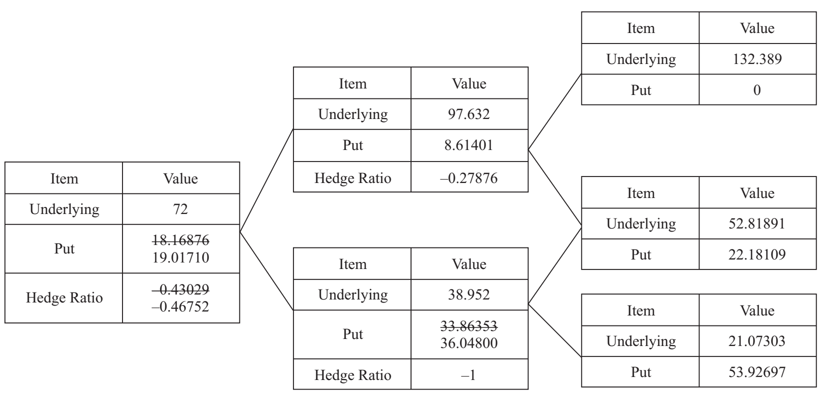
Suppose the periodically compounded interest rate is 3%, the non-dividend-paying underlying stock is currently trading at 72, the exercise price is 75, u = 1.356, d = 0.541, and the put option expires in two years. Exhibit 6 shows the results for a European-style put option.

Exhibit 6 Two-Period Binomial Model for a European-Style Put Option



The Time 1 down move is of particular interest. The exercise value for this put option is 36.048 [= Max(0,75 – 38.952)]. Therefore, the exercise value is higher than the put value. So, if this same option were American-style, then the option would be worth more exercised than not exercised. Thus, the put option should be exercised. Exhibit 7 illustrates how the analysis changes if this put option were American-style. Clearly, the right to exercise early translates into a higher value.

Exhibit 7 Two-Period Binomial Model for an American-Style Put Option



American-style option valuation requires that one work backward through the binomial tree and address whether early exercise is optimal at each step. In Exhibit 7, the early exercise premium at Time 1 when a down move occurs is 2.18447 (36.048 – 33.86353). Also, if we replace 33.86353 with 36.048—in bold below for emphasis—in the Time 0 calculation, we obtain a put value of

$$p = PV[\pi p^{+} + (1 - \pi)p^{-}] = (1/1.03)[(0.6)8.61401 + (1 - 0.6)\mathbf{36.048}] = 19.02$$

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Thus, the early exercise premium at Time 0 is 0.85 (19.02 – 18.17). From this illustration, we see clearly that in a multiperiod setting, American-style put options cannot be valued simply as the present value of the expected future option payouts, as shown in Equation 9. American-style put options can be valued as the present value of the expected future option payout in a single-period setting. Hence, when early exercise is a consideration, we must address the possibility of early exercise as we work backward through the binomial tree.

EXAMPLE 6

Two-Period Binomial American-Style Put Option Valuation

Suppose you are given the following information: S_0 = 26, X = 25, u = 1.466, d = 0.656, n = 2 (time steps), r = 2.05% (per period), and no dividends. The tree is provided in Exhibit 8.

Exhibit 8 Two-Period Binomial American-Style Put Option

Item	Value
Underlying	26
Put	4.01174
Hedge Ratio	–0.35345

Item	Value
Underlying	38.116
Put	0
Hedge Ratio	0

Item	Value
Underlying	17.056
Put	7.44360
Hedge Ratio	–0.99970

Item	Value
Underlying	55.87806
Put	0

Item	Value
Underlying	25.00410
Put	0

Item	Value
Underlying	11.18874
Put	13.81126

The early exercise premium of the above American-style put option is *closest* to:

A 0.27.

B 0.30.

C 0.35.

Solution:

A is correct. The exercise value at Time 1 with a down move is 7.944 [= Max(0,25 – 17.056)]. Thus, we replace this value in the binomial tree and compute the hedge ratio at Time 0. The resulting put option value at Time 0 is thus 4.28143 (see Exhibit 9).

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Exhibit 9 Solution

Item	Value
Underlying	26
Put	~~4.01174~~ 4.28143
Hedge Ratio	~~–0.35345~~ –0.37721

Item	Value
Underlying	38.116
Put	0
Hedge Ratio	0

Item	Value
Underlying	17.056
Put	~~7.44360~~ 7.94400
Hedge Ratio	–0.99970

Item	Value
Underlying	55.87806
Put	0

Item	Value
Underlying	25.00410
Put	0

Item	Value
Underlying	11.18874
Put	13.81126

In Exhibit 9, the early exercise premium at Time 1 when a down move occurs is 0.5004 (7.944 – 7.44360). Thus, if we replace 7.44360 with 7.944—in bold below for emphasis—in the Time 0 calculation, we have the put value of

$$p = PV[\pi p^+ + (1 - \pi)p^-] = (1/1.0205)[(0.45)0 + (1 - 0.45)\mathbf{7.944}] = 4.28$$

Thus, the early exercise premium at Time 0 when a down move occurs 0.27 (= 4.28 – 4.01).

We now briefly introduce the role of dividend payments within the binomial model. Our approach here is known as the escrow method. Because dividends lower the value of the stock, a call option holder is hurt. Although it is possible to adjust the option terms to offset this effect, most option contracts do not provide protection against dividends. Thus, dividends affect the value of an option. We assume dividends are perfectly predictable; hence, we split the underlying instrument into two components: the underlying instrument without the known dividends and the known dividends.[6] For example, the current value of the underlying instrument without dividends can be expressed as

$$\hat{S} = S - \gamma$$

where γ denotes the present value of dividend payments. We use the ^ symbol to denote the underlying instrument without dividends. In this case, we model the uncertainty of the stock based on $\hat{S}$ and not S. At expiration, the underlying instrument value is the same, $\hat{S}_T = S_T$, because we assume any dividends have already been paid. The value of an investment in the stock, however, would be $S_T + \gamma_T$, which assumes the dividend payments are reinvested at the risk-free rate.

To illustrate by example, consider a call on a US$100 stock with exercise price of US$95. The periodically compounded interest rate is 1.0%, the stock will pay a US$3 dividend at Time Step 1, u = 1.224, d = 0.796, and the call option expires in two years. Exhibit 10 shows some results for an American-style call option. The computations in Exhibit 10 involve several technical nuances that are beyond the scope of our objectives. The key objective here is to see how dividend-motivated early exercise influences American options.

6 The reading focuses on regular, “known” dividends. In the case of large, special dividends, option exchanges may adjust the exercise price.

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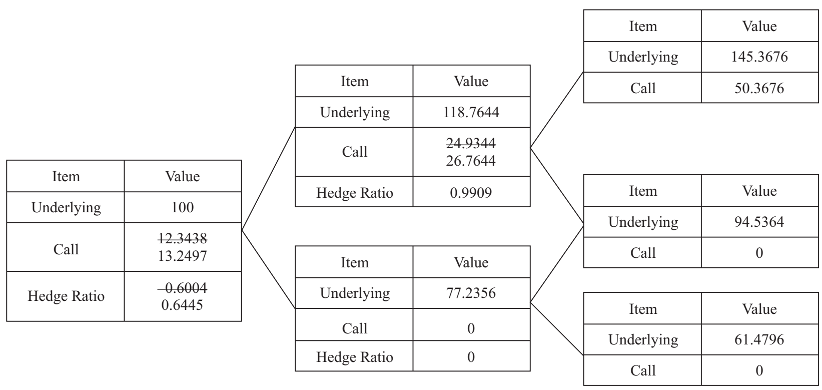
The Time 1 up move is particularly interesting. At Time 0, the present value of the US$3 dividend payment is US$2.970297 (= 3/1.01). Therefore, 118.7644 = (100 – 2.970297)1.224 is the stock value without dividends at Time 1, assuming an up move occurs. The exercise value for this call option, including dividends, is 26.7644 [= Max(0,118.7644 + 3 – 95)], whereas the value of the call option per the binomial model is 24.9344. In other words, the stock price just before it goes ex-dividend is 118.7644 + 3 = 121.7644, so the option can be exercised for 121.7644 – 95 = 26.7644. If not exercised, the stock drops as it goes ex-dividend and the option becomes worth 24.9344 at the ex-dividend price. Thus, by exercising early, the call buyer acquires the stock just before it goes ex-dividend and thus is able to capture the dividend. If the call is not exercised, the call buyer will not receive this dividend. The American-style call option is worth more than the European-style call option because at Time Step 1 when an up move occurs, the call is exercised early, capturing additional value.

Exhibit 10 Two-Period Binomial Model for an American-Style Call Option with Dividends



We now provide a comprehensive binomial option valuation example. In this example, we contrast European-style exercise with American-style exercise.

EXAMPLE 7

Comprehensive Two-Period Binomial Option Valuation Model Exercise

Suppose you observe a non-dividend-paying Australian equity trading for A$7.35. The call and put options have two years to mature, the periodically compounded risk-free interest rate is 4.35%, and the exercise price is A$8.0. Based on an analysis of this equity, the estimates for the up and down moves are u = 1.445 and d = 0.715, respectively.

1 Calculate the European-style call and put option values at Time Step 0 and Time Step 1. Describe and interpret your results.

2 Calculate the European-style call and put option hedge ratios at Time Step 0 and Time Step 1. Based on these hedge ratios, interpret the component terms of the binomial option valuation model.

3 Calculate the American-style call and put option values and hedge ratios at Time Step 0 and Time Step 1. Explain how your results differ from the European-style results.

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Solution to 1:

The expectations approach requires the following preliminary calculations:

$$\begin{aligned}
\text{RN probability: } \pi &= [FV(1) - d]/(u - d) \\
&= [(1 + 0.0435) - 0.715]/(1.445 - 0.715) = 0.45 \\
c^{++} &= \text{Max}(0,u^2S - X) \\
&= \text{Max}[0,1.445^2(7.35) - 8.0] = 7.347 \\
c^{+-} &= \text{Max}(0,udS - X) \\
&= \text{Max}[0,1.445(0.715)7.35 - 8.0] = 0 \\
c^{--} &= \text{Max}(0,d^2S - X) \\
&= \text{Max}[0,0.715^2(7.35) - 8.0] = 0 \\
p^{++} &= \text{Max}(0,X - u^2S) \\
&= \text{Max}[0,8.0 - 1.445^2(7.35)] = 0 \\
p^{+-} &= \text{Max}(0,X - udS) \\
&= \text{Max}[0,8.0 - 1.445(0.715)7.35] = 0.406 \\
p^{--} &= \text{Max}(0,X - d^2S) \\
&= \text{Max}[0,8.0 - 0.715^2(7.35)] = 4.24
\end{aligned}$$

Therefore, at Time Step 1, we have (note that $c_2\big|_1^+$ is read as the call value expiring at Time Step 2 observed at Time Step 1, assuming an up move occurs)

$$E\left(c_2\big|_1^+\right) = \pi c^{++} + (1 - \pi)c^{+-} = 0.45(7.347) + (1 - 0.45)0 = 3.31$$

$$E\left(c_2\big|_1^-\right) = \pi c^{-+} + (1 - \pi)c^{--} = 0.45(0.0) + (1 - 0.45)0.0 = 0.0$$

$$E\left(p_2\big|_1^+\right) = \pi p^{++} + (1 - \pi)p^{+-} = 0.45(0.0) + (1 - 0.45)0.406 = 0.2233$$

$$E\left(p_2\big|_1^-\right) = \pi p^{-+} + (1 - \pi)p^{--} = 0.45(0.406) + (1 - 0.45)4.24 = 2.51$$

Thus, because $PV_{1,2}(1) = 1/(1 + 0.0435) = 0.958313$, we have the Time Step 1 option values of

$$c^+ = PV_{1,2}\left[E\left(c_2\big|_1^+\right)\right] = 0.958313(3.31) = 3.17$$

$$c^- = PV_{1,2}\left[E\left(c_2\big|_1^-\right)\right] = 0.958313(0.0) = 0.0$$

$$p^+ = PV_{1,2}\left[E\left(p_2\big|_1^+\right)\right] = 0.\ 958313(0.2233) = 0.214$$

$$p^- = PV_{1,2}\left[E\left(p_2\big|_1^-\right)\right] = 0.958313(2.51) = 2.41$$

At Time Step 0, we have

$$\begin{aligned}
E\left(c_2\big|_0\right) &= \pi^2c^{++} + 2\pi(1 - \pi)c^{+-} + (1 - \pi)^2c^{--} \\
&= 0.45^2(7.347) + 2(0.45)(1 - 0.45)0 + (1 - 0.45)^2 0 = 1.488
\end{aligned}$$

$$\begin{aligned}
E\left(p_2\big|_0\right) &= \pi^2p^{++} + 2\pi(1 - \pi)p^{+-} + (1 - \pi)^2p^{--} \\
&= 0.45^2(0) + 2(0.45)(1 - 0.45)0.406 + (1 - 0.45)^2 4.24 = 1.484
\end{aligned}$$

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Thus,

$$c = PV_{rf,0,2}\left[E\left(c_2\middle|_0\right)\right] = 0.91836(1.488) = 1.37 \text{ and}$$

$$p = PV_{rf,0,2}\left[E\left(p_2\middle|_0\right)\right] = 0.91836(1.484) = 1.36$$

With the two-period binomial model, the call and put values based on the expectations approach are simply the present values of the expected payoffs. The present value of the expected payoffs is based on the risk-free interest rate and the expectations approach is based on the risk-neutral probability. The parameters in this example were selected so that the European-style put and call would have approximately the same value. Notice that the stock price is less than the exercise price by roughly the present value factor or $7.35 = 8.0/1.0435^2$. One intuitive explanation is put–call parity, which can be expressed as $c - p = S - PV(X)$. Thus, if $S = PV(X)$, then $c = p$.

Solution to 2:

The computation of the hedge ratios at Time Step 1 and Time Step 0 will require the option values at Time Step 1 and Time Step 2. The terminal values of the options are given in Solution 1.

$$S^{++} = u^2S = 1.445^2(7.35) = 15.347$$
$$S^{+-} = udS = 1.445(0.715)7.35 = 7.594$$
$$S^{--} = d^2S = 0.715^2(7.35) = 3.758$$
$$S^{+} = uS = 1.445(7.35) = 10.621$$
$$S^{-} = dS = 0.715(7.35) = 5.255$$

Therefore, the hedge ratios at Time 1 are

$$h_c^+ = \frac{c^{++} - c^{+-}}{S^{++} - S^{+-}} = \frac{7.347 - 0.0}{15.347 - 7.594} = 0.9476$$

$$h_c^- = \frac{c^{-+} - c^{--}}{S^{-+} - S^{--}} = \frac{0.0 - 0.0}{7.594 - 3.758} = 0.0$$

$$h_p^+ = \frac{p^{++} - p^{+-}}{S^{++} - S^{+-}} = \frac{0.0 - 0.406}{15.347 - 7.594} = -0.05237$$

$$h_p^- = \frac{p^{-+} - p^{--}}{S^{-+} - S^{--}} = \frac{0.406 - 4.24}{7.594 - 3.758} = -1.0$$

In the last hedge ratio calculation, both put options are in the money (p^{-+} and p^{--}). In this case, the hedge ratio will be –1, subject to a rounding error. We now turn to interpreting the model's component terms. Based on the no-arbitrage approach, we have for the call price, assuming an up move has occurred, at Time Step 1,

$$c^+ = h_c^+S^+ + PV_{1,2}\left(-h_c^+S^{+-} + c^{+-}\right)$$
$$= 0.9476(10.621) + (1/1.0435)[-0.9476(7.594) + 0.0] = 3.1684$$

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Thus, the call option can be interpreted as a leveraged position in the stock. Specifically, long 0.9476 shares for a cost of 10.0645 [= 0.9476(10.621)] partially financed with a 6.8961 {= (1/1.0435)[–0.9476(7.594) + 0.0]} loan. Note that the loan amount can be found simply as the cost of the position in shares less the option value [6.8961 = 0.9476(10.621) – 3.1684]. Similarly, we have

$$c^- = h_c^- S^- + PV_{1,2}\left(-h_c^- S^{--} + c^{--}\right)$$
$$= 0.0(5.255) + (1/1.0435)[-0.0(3.758) + 0.0] = 0.0$$

Specifically, long 0.0 shares for a cost of 0.0 [= 0.0(5.255)] with no financing. For put options, the interpretation is different. Specifically, we have

$$p^+ = PV_{1,2}\left(-h_p^+ S^{++} + p^{++}\right) + h_p^+ S^+$$
$$= (1/1.0435)[-(-0.05237)15.347 + 0.0] + (-0.05237)10.621 = 0.2140$$

Thus, the put option can be interpreted as lending that is partially financed with a short position in shares. Specifically, short 0.05237 shares for a cost of 0.55622 [= (–0.05237)10.621] with financing of 0.77022 {= (1/1.0435)[–(–0.05237)15.347 + 0.0]}. Note that the lending amount can be found simply as the proceeds from the short sale of shares plus the option value [0.77022 = (0.05237)10.621 + 0.2140]. Again, we have

$$p^- = PV_{1,2}\left(-h_p^- S^{-+} + p^{-+}\right) + h_p^- S^-$$
$$= (1/1.0435)[-(-1.0)7.594 + 0.406] + (-1.0)5.255 = 2.4115$$

Here, we short 1.0 shares for a cost of 5.255 [= (–1.0)5.255] with financing of 7.6665 {= (1/1.0435)[–(–1.0)7.594 + 0.406]}. Again, the lending amount can be found simply as the proceeds from the short sale of shares plus the option value [7.6665 = (1.0)5.255 + 2.4115].

Finally, we have at Time Step 0

$$h_c = \frac{c^+ - c^-}{S^+ - S^-} = \frac{3.1684 - 0}{10.621 - 5.255} = 0.5905$$

$$h_p = \frac{p^+ - p^-}{S^+ - S^-} = \frac{0.2140 - 2.4115}{10.621 - 5.255} = -0.4095$$

The interpretations remain the same at Time Step 0:

$$c = h_c S + PV_{0,1}(-h_c S^- + c^-)$$
$$= 0.5905(7.35) + (1/1.0435)[-0.5905(5.255) + 0.0] = 1.37$$

Here, we are long 0.5905 shares for a cost of 4.3402 [=0.5905(7.35)] partially financed with a 2.97 {= (1/1.0435)[–0.5905(5.255) + 0.0] or = 0.5905(7.35) – 1.37} loan.

$$p = PV_{0,1}(-h_p S^+ + p^+) + h_p S$$
$$= (1/1.0435)\{-[-0.4095(10.621)] + 0.214\} + (-0.4095)7.35 = 1.36$$

Here, we short 0.4095 shares for a cost of 3.01 [= (–0.4095)7.35] with financing of 4.37 (= (1/1.0435){–[–0.4095(10.621)] + 0.214} or = (0.4095)7.35 + 1.36).

Solution to 3:

We know that American-style call options on non-dividend-paying stock are worth the same as European-style call options because early exercise will not occur. Thus, as previously computed, $c^+ = 3.17$, $c^- = 0.0$, and $c = 1.37$. Recall

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that the call exercise value (denoted with EV) is simply the maximum of zero or the stock price minus the exercise price. We note that the EVs are less than or equal to the call model values; that is,

$$c_{EV}^{+} = \text{Max}(0,S^{+} - X) = \text{Max}(0,10.621 - 8.0) = 2.621\ (< 3.1684)$$

$$c_{EV}^{-} = \text{Max}(0,S^{-} - X) = \text{Max}(0,5.255 - 8.0) = 0.0\ (= 0.0)$$

$$c_{EV} = \text{Max}(0,S - X) = \text{Max}(0,7.35 - 8.0) = 0.0\ (< 1.37)$$

Therefore, the American-style feature for non-dividend-paying stocks has no effect on either the hedge ratio or the option value. The binomial model for American-style calls on non-dividend-paying stocks can be described and interpreted the same as a similar European-style call. This point is consistent with what we said earlier. If there are no dividends, an American-style call will not be exercised early.

This result is not true for puts. We know that American-style put options on non-dividend-paying stock may be worth more than the analogous European-style put options. The hedge ratios at Time Step 1 will be the same as European-style puts because there is only one period left. Therefore, as previously shown, $p^{+} = 0.214$ and $p^{-} = 2.41$.

The put exercise values are

$$p_{EV}^{+} = \text{Max}(0,X - S^{+}) = \text{Max}(0,8.0 - 10.621) = 0\ (< 0.214)$$

$$p_{EV}^{-} = \text{Max}(0,X - S^{-}) = \text{Max}(0,8.0 - 5.255) = 2.745\ (> 2.41)$$

Because the exercise value for the put at Time Step 1, assuming a down move occurred, is greater than the model value, we replace the model value with the exercise value. Hence,

$$p^{-} = 2.745$$

and the hedge ratio at Time Step 0 will be affected. Specifically, we now have

$$h_p = \frac{p^{+} - p^{-}}{S^{+} - S^{-}} = \frac{0.2140 - 2.745}{10.621 - 5.255} = -0.4717$$

and thus the put model value is

$$p = (1/1.0435)[0.45(0.214) + 0.55(2.745)] = 1.54$$

Clearly, the early exercise feature has a significant impact on both the hedge ratio and the put option value in this case. The hedge ratio goes from –0.4095 to –0.4717. The put value is raised from 1.36 to 1.54.

We see through the simple two-period binomial model that an option can be viewed as a position in the underlying with financing. Furthermore, this valuation model can be expressed as the present value of the expected future cash flows, where the expectation is taken under the RN probability and the discounting is at the risk-free rate.

Up to this point, we have focused on equity options. The binomial model can be applied to any underlying instrument though often requiring some modifications. For example, currency options would require incorporating the foreign interest rate. Futures options would require a binomial lattice of the futures prices. Interest rate options, however, require somewhat different tools that we now examine.

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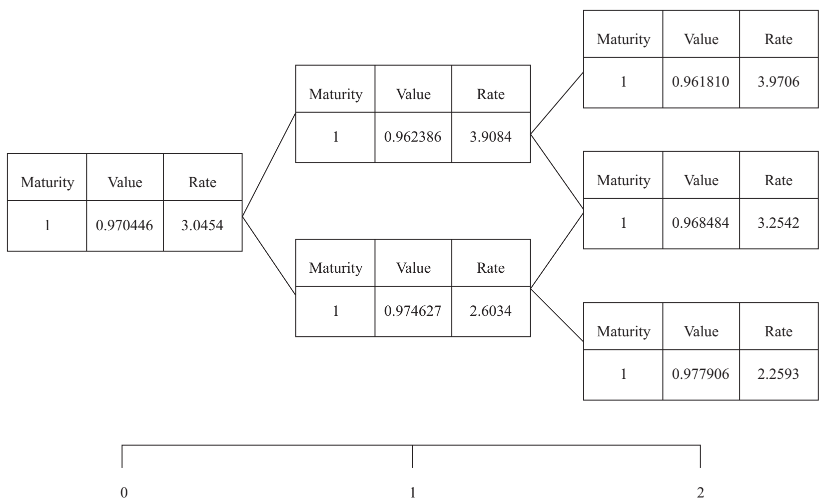
3.3 Interest Rate Options

In this section, we will briefly illustrate how to value interest rate options. There are a wide variety of approaches to valuing interest rate options. We do not delve into how arbitrage-free interest rate trees are generated. The particular approach used here assumes the RN probability of an up move at each node is 50%.

Exhibit 11 presents a binomial lattice of interest rates covering two years along with the corresponding zero-coupon bond values. The rates are expressed in annual compounding. Therefore, at Time 0, the spot rate is (1.0/0.970446) – 1 or 3.04540%.[7] Note that at Time 1, the value in the column labeled "Maturity" reflects time to maturity not calendar time. The lattice shows the rates on one-period bonds, so all bonds have a maturity of 1. The column labeled "Value" is the value of a zero-coupon bond with the stated maturity based on the rates provided.

Exhibit 11 Two-Year Binomial Interest Rate Lattice by Year



The underlying instrument for interest rate options here is the spot rate. A call option on interest rates will be in the money when the current spot rate is above the exercise rate. A put option on interest rates will be in the money when the current spot rate is below the exercise rate. Thus, based on the notation in the previous section, the current spot rate is denoted S. Option valuation follows the expectations approach discussed in the previous section but taken only one period at a time. The procedure is illustrated with an example.

7 The values in the first box from the left are observed at t = 0. The values in the remainder of the lattice are derived by using a technique that is outside the scope of this reading.

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EXAMPLE 8

Option on Interest Rates

This example is based on Exhibit 11. Suppose we seek to value two-year European-style call and put options on the periodically compounded one-year spot interest rate (the underlying). Assume the notional amount of the options is US$1,000,000 and the call and put exercise rate is 3.25% of par. Assume the RN probability is 50% and these option cash settle at Time 2 based on the observed rates.[8]

Solution:

Using the expectations approach introduced in the last section, we have (per US$1) at Time Step 2

$$
\begin{aligned}
c^{++} &= \text{Max}(0,S^{++} - X) = \text{Max}[0,0.039706 - 0.0325] = 0.007206\\
c^{+-} &= \text{Max}(0,S^{+-} - X) = \text{Max}[0,0.032542 - 0.0325] = 0.000042\\
c^{--} &= \text{Max}(0,S^{--} - X) = \text{Max}[0,0.022593 - 0.0325] = 0.0\\
p^{++} &= \text{Max}(0,X - S^{++}) = \text{Max}[0,0.0325 - 0.039706] = 0.0\\
p^{+-} &= \text{Max}(0,X - S^{+-}) = \text{Max}[0,0.0325 - 0.032542] = 0.0\\
p^{--} &= \text{Max}(0,X - S^{--}) = \text{Max}[0,0.0325 - 0.022593] = 0.009907
\end{aligned}
$$

At Time Step 1, we have

$$
\begin{aligned}
c^{+} &= PV_{1,2}[\pi c^{++} + (1 - \pi)c^{+-}]\\
&= 0.962386[0.5(0.007206) + (1 - 0.5)0.000042]\\
&= 0.003488\\
c^{-} &= PV_{1,2}[\pi c^{+-} + (1 - \pi)c^{--}]\\
&= 0.974627[0.5(0.000042) + (1 - 0.5)0.0]\\
&= 0.00002\\
p^{+} &= PV_{1,2}[\pi p^{++} + (1 - \pi)p^{+-}]\\
&= 0.962386[0.5(0.0) + (1 - 0.5)0.0]\\
&= 0.0\\
p^{-} &= PV_{1,2}[\pi p^{+-} + (1 - \pi)p^{--}]\\
&= 0.974627[0.5(0.0) + (1 - 0.5)0.009907]\\
&= 0.004828
\end{aligned}
$$

Notice how the present value factors are different for the up and down moves. At Time Step 1 in the + outcome, we discount by a factor of 0.962386, and in the – outcome, we discount by the factor 0.974627. Because this is an option on interest rates, it should not be surprising that we have to allow the interest rate to vary.

8 In practice, interest rate options usually have a settlement procedure that results in a deferred payoff. The deferred payoff arises from the fact that the underlying interest rate is based on an instrument that pays interest at the end of its life. For the instrument underlying the interest rate, the interest payment occurs after the interest has accrued. To accommodate this reality in this problem, we would have to introduce an instrument that matures at time three. The purpose of this example is merely to illustrate the procedure for rolling backward through an interest rate tree when the underlying is the interest rate. We simplify this example by omitting this deferred settlement. In Section 5.2, we discuss in detail the deferred settlement procedure and incorporate it into the pricing model.

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Therefore, at Time Step 0, we have

$$\begin{aligned} c &= PV_{rf,0,1}[\pi c^{+} + (1-\pi)c^{-}] \\ &= 0.970446[0.5(0.003488) + (1-0.5)0.00002] \\ &= 0.00170216 \\ p &= PV_{rf,0,1}[\pi p^{+} + (1-\pi)p^{-}] \\ &= 0.970446[0.5(0.0) + (1-0.5)0.004828] \\ &= 0.00234266 \end{aligned}$$

Because the notional amount is US$1,000,000, the call value is US$1,702.16 [= US$1,000,000(0.00170216)] and the put value is US$2,342.66 [= US$1,000,000(0.00234266)]. The key insight is to just work a two-period binomial model as three one-period binomial models.

We turn now to briefly generalize the binomial model as it leads naturally to the Black–Scholes–Merton option valuation model.

3.4 Multiperiod Model

The multiperiod binomial model provides a natural bridge to the Black–Scholes–Merton option valuation model presented in the next section. The idea is to take the option's expiration and slice it up into smaller and smaller periods. The two-period model divides the expiration into two periods. The three-period model divides expiration into three periods and so forth. The process continues until you have a large number of time steps. The key feature is that each time step is of equal length. Thus, with a maturity of T, if there are n time steps, then each time step is T/n in length.

For American-style options, we must also test at each node whether the option is worth more exercised or not exercised. As in the two-period case, we work backward through the binomial tree testing the model value against the exercise value and always choosing the higher one.

The binomial model is an important and useful methodology for valuing options. The expectations approach can be applied to European-style options and will lead naturally to the BSM model in the next section. This approach simply values the option as the present value of the expected future payoffs, where the expectation is taken under the risk-neutral probability and the discounting is based on the risk-free rate. The no-arbitrage approach can be applied to either European-style or American-style options because it provides the intuition for the fair value of options.

4 BLACK–SCHOLES–MERTON OPTION VALUATION MODEL

The BSM model, although very complex in its derivation, is rather simple to use and interpret. The objective here is to illustrate several facets of the BSM model with the objective of highlighting its practical usefulness. After a brief introduction, we examine the assumptions of the BSM model and then delve into the model itself.

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4.1 Introductory Material

Louis Bachelier published the first known mathematically rigorous option valuation model in 1900. By the late 1960s, there were several published quantitative option models. Fischer Black, Myron Scholes, and Robert Merton introduced the BSM model in 1973 in two published papers, one by Black and Scholes and the other by Merton. The innovation of the BSM model is essentially the no-arbitrage approach introduced in the previous section but applied with a continuous time process, which is equivalent to a binomial model in which the length of the time step essentially approaches zero. It is also consistent with the basic statistical fact that the binomial process with a "large" number of steps converges to the standard normal distribution. Myron Scholes and Robert Merton won the 1997 Nobel Prize in Economics based, in part, on their work related to the BSM model.[9] Let us now examine the BSM model assumptions.

4.2 Assumptions of the BSM Model

The key assumption for option valuation models is how to model the random nature of the underlying instrument. This characteristic of how an asset evolves randomly is called a stochastic process. Many financial instruments enjoy limited liability; hence, the values of instruments cannot be negative, but they certainly can be zero. In 1900, Bachelier proposed the normal distribution. The key advantages of the normal distribution are that zero is possible, meaning that bankruptcy is allowable, it is symmetric, it is relatively easy to manipulate, and it is additive (which means that sums of normal distributions are normally distributed). The key disadvantage is that negative stock values are theoretically possible, which violates the limited liability principal of stock ownership. Based on research on stock prices in the 1950s and 1960s, a preference emerged for the lognormal distribution, which means that log returns are distributed normally. Black, Scholes, and Merton chose to use the lognormal distribution.

Recall that the no-arbitrage approach requires self-financing and dynamic replication; we need more than just an assumption regarding the terminal distribution of the underlying instrument. We need to model the value of the instrument as it evolves over time, which is what we mean by a stochastic process. The stochastic process chosen by Black, Scholes, and Merton is called geometric Brownian motion (GBM).

Exhibit 12 illustrates GBM, assuming the initial stock price is S = 50. We assume the stock will grow at 3% (μ = 3% annually, geometrically compounded rate). This GBM process also reflects a random component that is determined by a volatility (σ) of 45%. This volatility is the annualized standard deviation of continuously compounded percentage change in the underlying, or in other words, the log return. Note that as a particular sample path drifts upward, we observe more variability on an absolute basis, whereas when the particular sample path drifts downward, we observe less variability on an absolute basis. For example, examine the highest and lowest lines shown in Exhibit 12. The highest line is much more erratic than the lowest line. Recall that a 10% move in a stock with a price of 100 is 10 whereas a 10% move in a stock with a price of 10 is only 1. Thus, GBM can never hit zero nor go below it. This property is appealing because many financial instruments enjoy limited liability and cannot be negative. Finally, note that although the stock movements are rather erratic, there are no large jumps—a common feature with marketable financial instruments.

9 Fischer Black passed away in 1995 and the Nobel Prize is not awarded posthumously.

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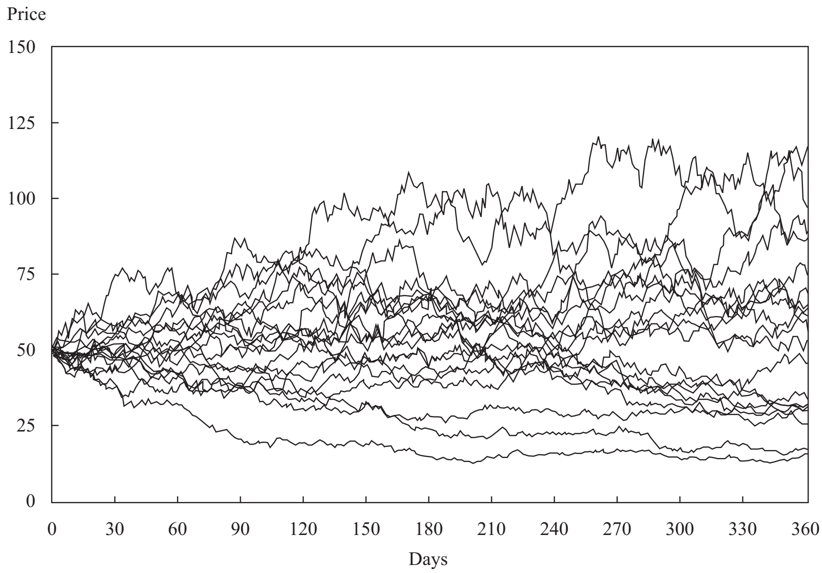
Exhibit 12 Geometric Brownian Motion Simulation (S = 50, μ = 3%, σ = 45%)



Within the BSM model framework, it is assumed that all investors agree on the distributional characteristics of GBM except the assumed growth rate of the underlying. This growth rate depends on a number of factors, including other instruments and time. The standard BSM model assumes a constant growth rate and constant volatility.

The specific assumptions of the BSM model are as follows:

- The underlying follows a statistical process called geometric Brownian motion, which implies that the continuously compounded return is normally distributed.
- Geometric Brownian motion implies continuous prices, meaning that the price of underlying instrument does not jump from one value to another; rather, it moves smoothly from value to value.
- The underlying instrument is liquid, meaning that it can be easily bought and sold.
- Continuous trading is available, meaning that in the strictest sense one must be able to trade at every instant.
- Short selling of the underlying instrument with full use of the proceeds is permitted.
- There are no market frictions, such as transaction costs, regulatory constraints, or taxes.
- No arbitrage opportunities are available in the marketplace.
- The options are European-style, meaning that early exercise is not allowed.
- The continuously compounded risk-free interest rate is known and constant; borrowing and lending is allowed at the risk-free rate.
- The volatility of the return on the underlying is known and constant.
- If the underlying instrument pays a yield, it is expressed as a continuous known and constant yield at an annualized rate.

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Naturally, the foregoing assumptions are not absolutely consistent with real financial markets, but, as in all financial models, the question is whether they produce models that are tractable and useful in practice, which they do.

EXAMPLE 9

BSM Model Assumptions

Which is the *correct* pair of statements? The BSM model assumes:

A the return on the underlying has a normal distribution. The price of the underlying can jump abruptly to another price.

B brokerage costs are factored into the BSM model. It is impossible to trade continuously.

C volatility can be predicted with certainty. Arbitrage is non-existent in the marketplace.

Solution:

C is correct. All four of the statements in A and B are incorrect within the BSM model paradigm.

We turn now to a careful examination of the BSM model.

4.3 BSM Model

The BSM model is a continuous time version of the discrete time binomial model. Given that the BSM model is based on continuous time, it is customary to use a continuously compounded interest rate rather than some discretely compounded alternative. Thus, when an interest rate is used here, denoted simply as r, we mean solely the annualized continuously compounded rate.[10] The volatility, denoted as σ, is also expressed in annualized percentage terms. Initially, we focus on a non-dividend-paying stock. The BSM model, with some adjustments, applies to other underlying instruments, which will be examined later.

The BSM model for stocks can be expressed as

$$c = SN(d_1) - e^{-rT}XN(d_2) \tag{10}$$

and

$$p = e^{-rT}XN(-d_2) - SN(-d_1) \tag{11}$$

where

$$d_1 = \frac{\ln(S/X) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

N(x) denotes the standard normal cumulative distribution function, which is the probability of obtaining a value of less than x based on a standard normal distribution. In our context, x will have the value of d_1 or d_2. N(x) reflects the likelihood of observing values less than x from a random sample of observations taken from the standard normal distribution.

10 Note $e^r = 1 + r_d$, where r_d is the annually compounded rate.

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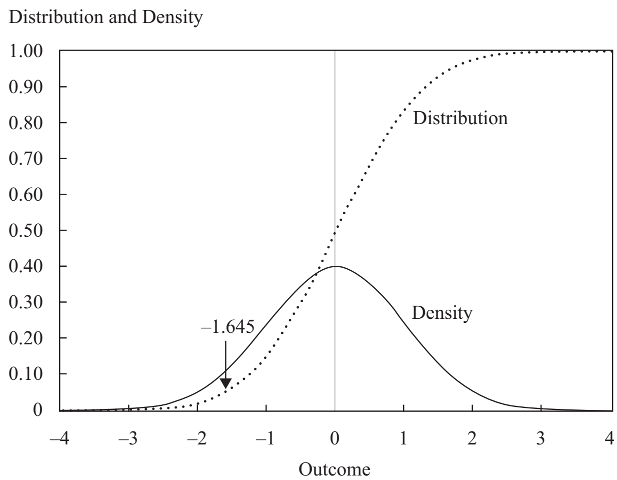
Although the BSM model appears very complicated, it has straightforward interpretations that will be explained. N(x) can be estimated by a computer program or a spreadsheet or approximated from a lookup table. The normal distribution is a symmetric distribution with two parameters, the mean and standard deviation. The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1.

Exhibit 13 illustrates the standard normal probability density function (the standard bell curve) and the cumulative distribution function (the accumulated probability and range of 0 to 1). Note that even though GBM is lognormally distributed, the N(x) functions in the BSM model are based on the standard normal distribution. In Exhibit 13, we see that if $x = -1.645$, then $N(x) = N(-1.645) = 0.05$. Thus, if the model value of d is -1.645, the corresponding probability is 5%. Clearly, values of d that are less than 0 imply values of N(x) that are less than 0.5. As a result of the symmetry of the normal distribution, we note that $N(-x) = 1 - N(x)$.

Exhibit 13 Standard Normal Distribution



The BSM model can be described as the present value of the expected option payoff at expiration. Specifically, we can express the BSM model for calls as $c = PV_r[E(c_T)]$ and for puts as $p = PV_r[E(p_T)]$, where $E(c_T) = Se^{rT}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{rT}N(-d_1)$. The present value term in this context is simply e^{-rT}. As with most valuation tasks in finance, the value today is simply the present value of the expected future cash flows. It is important to note that the expectation is based on the risk-neutral probability measure defined in Section 3.1. The expectation is not based on the investor's subjective beliefs, which reflect an aversion to risk. Also, the present value function is based on the risk-free interest rate not on the investor's required return on invested capital, which of course is a function of risk.

Alternatively, the BSM model can be described as having two components: a stock component and a bond component. For call options, the stock component is $SN(d_1)$ and the bond component is $e^{-rT}XN(d_2)$. The BSM model call value is the stock component minus the bond component. For put options, the stock component is $SN(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$. The BSM model put value is the bond component minus the stock component.

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The BSM model can be interpreted as a dynamically managed portfolio of the stock and zero-coupon bonds.[11] The goal is to replicate the option payoffs with stocks and bonds. For both call and put options, we can represent the initial cost of this replicating strategy as

Replicating strategy cost = $n_S S + n_B B$

where the equivalent number of underlying shares is $n_S = N(d_1) > 0$ for calls and $n_S = -N(-d_1) < 0$ for puts. The equivalent number of bonds is $n_B = -N(d_2) < 0$ for calls and $n_B = N(-d_2) > 0$ for puts. The price of the zero-coupon bond is $B = e^{-rT}X$. Note, if n is positive, we are buying the underlying and if n is negative we are selling (short selling) the underlying. The cost of the portfolio will exactly equal either the BSM model call value or the BSM model put value.

For calls, we are simply buying stock with borrowed money because $n_S > 0$ and $n_B < 0$. Again the cost of this portfolio will equal the BSM model call value, and if appropriately rebalanced, then this portfolio will replicate the payoff of the call option. Therefore, a call option can be viewed as a leveraged position in the stock.

Similarly, for put options, we are simply buying bonds with the proceeds from short selling the underlying because $n_S < 0$ and $n_B > 0$. The cost of this portfolio will equal the BSM model put value, and if appropriately rebalanced, then this portfolio will replicate the payoff of the put option. Note that a short position in a put will result in receiving money today and $n_S > 0$ and $n_B < 0$. Therefore, a short put can be viewed as an over-leveraged or over-geared position in the stock because the borrowing exceeds 100% of the cost of the underlying.

Exhibit 14 illustrates the direct comparison between the no-arbitrage approach to the single-period binomial option valuation model and the BSM option valuation model. The parallel between the h term in the binomial model and $N(d_1)$ is easy to see. Recall that the term hedge ratio was used with the binomial model because we were creating a no-arbitrage portfolio. Note for call options, $-N(d_2)$ implies borrowing money or short selling $N(d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$. For put options, $N(-d_2)$ implies lending money or buying $N(-d_2)$ shares of a zero-coupon bond trading at $e^{-rT}X$.

Exhibit 14 BSM and Binomial Option Valuation Model Comparison

	Call Option		Put Option	
Option Valuation Model Terms	**Underlying**	**Financing**	**Underlying**	**Financing**
Binomial Model	hS	$PV(-hS^- + c^-)$	hS	$PV(-hS^- + p^-)$
BSM Model	$N(d_1)S$	$-N(d_2)e^{-rT}X$	$-N(-d_1)S$	$N(-d_2)e^{-rT}X$

If the value of the underlying, S, increases, then the value of $N(d_1)$ also increases because S has a positive effect on d_1. Thus, the replicating strategy for calls requires continually buying shares in a rising market and selling shares in a falling market.

Within the BSM model theory, the aggregate losses from this "buy high/sell low" strategy, over the life of the option, adds up exactly to the BSM model option premium received for the option at inception.[12] This result must be the case; otherwise there would be arbitrage profits available. Because transaction costs are not, in fact, zero,

11 When covering the binomial model, the bond component was generically termed financing. This component is typically handled with bank borrowing or lending. With the BSM model, it is easier to understand as either buying or short selling a risk-free zero-coupon bond.

12 The validity of this claim does not rest on the validity of the BSM model assumptions; rather the validity depends only on whether the BSM model accurately predicts the replication cost.

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the frequent rebalancing by buying and selling the underlying adds significant costs for the hedger. Also, markets can often move discontinuously, contrary to the BSM model's assumption that prices move continuously, thus allowing for continuous hedging adjustments. Hence, in reality, hedges are imperfect. For example, if a company announces a merger, then the company's stock price may jump substantially higher, contrary to the BSM model's assumption.

In addition, volatility cannot be known in advance. For these reasons, options are typically more expensive than they would be as predicted by the BSM model theory. In order to continue using the BSM model, the volatility parameter used in the formula is usually higher (by, say, 1% or 2%, but this can vary a lot) than the volatility of the stock actually expected by market participants. We will ignore this point for now, however, as we focus on the mechanics of the model.

EXAMPLE 10

Illustration of BSM Model Component Interpretation

Suppose we are given the following information on call and put options on a stock: S = 100, X = 100, r = 5%, T = 1.0, and σ = 30%. Thus, based on the BSM model, it can be demonstrated that PV(X) = 95.123, d_1 = 0.317, d_2 = 0.017, $N(d_1)$ = 0.624, $N(d_2)$ = 0.507, $N(-d_1)$ = 0.376, $N(-d_2)$ = 0.493, c = 14.23, and p = 9.35.

1 The initial trading strategy required by the no-arbitrage approach to replicate the call option payoffs for a buyer of the option is:

   A buy 0.317 shares of stock and short sell –0.017 shares of zero-coupon bonds.

   B buy 0.624 shares of stock and short sell 0.507 shares of zero-coupon bonds.

   C short sell 0.317 shares of stock and buy 0.017 shares of zero-coupon bonds.

2 Identify the initial trading strategy required by the no-arbitrage approach to replicate the put option payoffs for a buyer of the put.

   A Buy 0.317 shares of stock and short sell –0.017 shares of zero-coupon bonds.

   B Buy 0.624 shares of stock and short sell 0.507 shares of zero-coupon bonds.

   C Short sell 0.376 shares of stock and buy 0.493 shares of zero-coupon bonds.

Solution to 1:

B is correct. The no-arbitrage approach to replicating the call option involves purchasing $n_S = N(d_1) = 0.624$ shares of stock partially financed with $n_B = -N(d_2) = -0.507$ shares of zero-coupon bonds priced at $B = Xe^{-rT} = 95.123$ per bond. Note that by definition the cost of this replicating strategy is the BSM call model value or $n_S S + n_B B = 0.624(100) + (-0.507)95.123 = 14.17$. Without rounding errors, the option value is 14.23.

Solution to 2:

C is correct. The no-arbitrage approach to replicating the put option is similar. In this case, we trade $n_S = -N(-d_1) = -0.376$ shares of stock—specifically, short sell 0.376 shares—and buy $n_B = N(-d_2) = 0.493$ shares of zero-coupon bonds. Again, the cost of the replicating strategy is $n_S S + n_B B = -0.376(100) + (0.493)95.123 =$

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9.30. Without rounding errors, the option value is 9.35. Thus, to replicate a call option based on the BSM model, we buy stock on margin. To replicate a put option, we short the stock and buy zero-coupon bonds.

Note that the $N(d_2)$ term has an additional important interpretation. It is a unique measure of the probability that the call option expires in the money, and correspondingly, $1 - N(d_2) = N(-d_2)$ is the probability that the put option expires in the money. Specifically, the probability based on the RN probability of being in the money, not one's own estimate of the probability of being in the money nor the market's estimate. That is, $N(d_2) = \text{Prob}(S_T > X)$ based on the unique RN probability.

We now turn to incorporating various carry benefits into the BSM model. Carry benefits include dividends for stock options, foreign interest rates for currency options, and coupon payments for bond options. For other underlying instruments, there are carry costs that can easily be treated as negative carry benefits, such as storage and insurance costs for agricultural products. Because the BSM model is established in continuous time, it is common to model these carry benefits as a continuous yield, denoted generically here as γ^c or simply γ.

The BSM model requires a few adjustments to accommodate carry benefits. The carry benefit-adjusted BSM model is

$$c = Se^{-\gamma T}N(d_1) - e^{-rT}XN(d_2) \tag{12}$$

and

$$p = e^{-rT}XN(-d_2) - Se^{-\gamma T}N(-d_1) \tag{13}$$

where

$$d_1 = \frac{\ln(S/X) + \left(r - \gamma + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

Note that d_2 can be expressed again simply as $d_2 = d_1 - \sigma\sqrt{T}$. The value of a put option can also be found based on the carry benefit-adjusted put–call parity:

$$p + Se^{-\gamma T} = c + e^{-rT}X \tag{14}$$

The carry benefit-adjusted BSM model can again be described as the present value of the expected option payoff at expiration. Now, however, $E(c_T) = Se^{(r-\gamma)T}N(d_1) - XN(d_2)$ and $E(p_T) = XN(-d_2) - Se^{(r-\gamma)T}N(-d_1)$. The present value term remains simply e^{-rT}. Carry benefits will have the effect of lowering the expected future value of the underlying

Again, the carry benefit adjusted BSM model can be described as having two components, a stock component and a bond component. For call options, the stock component is $Se^{-\gamma T}N(d_1)$ and the bond component is again $e^{-rT}XN(d_2)$. For put options, the stock component is $Se^{-\gamma T}N(-d_1)$ and the bond component is again $e^{-rT}XN(-d_2)$. Although both d_1 and d_2 are reduced by carry benefits, the general approach to valuation remains the same. An increase in carry benefits will lower the value of the call option and raise the value of the put option.

Note that $N(d_2)$ term continues to be interpreted as the RN probability of a call option being in the money. The existence of carry benefits has the effect of lowering d_1 and d_2, hence the probability of being in the money with call options declines as the carry benefit rises. This RN probability is an important element to describing how the BSM model is used in various valuation tasks.

For stock options, $\gamma = \delta$, which is the continuously compounded dividend yield. The dividend-yield BSM model can again be interpreted as a dynamically managed portfolio of the stock and zero coupon bonds. Based on the call model above applied to a dividend yielding stock, the equivalent number of units of stock is now $n_S = e^{-\delta T}N(d_1)$

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> 0 and the equivalent number of units of bonds remains $n_B = -N(d_2) < 0$. Similarly with puts, the equivalent number of units of stock is now $n_S = -e^{-\delta T}N(-d_1) < 0$ and the equivalent number of units of bonds again remains $n_B = N(-d_2) > 0$.

With dividend paying stocks, the arbitrageur is able to receive the benefits of dividend payments when long the stock and has to pay dividends when short the stock. Thus, the burden of carrying the stock is diminished for a long position. The key insight is that dividends influence the dynamically managed portfolio by lowering the number of shares to buy for calls and lowering the number of shares to short sell for puts. Higher dividends will lower the value of d_1, thus lowering $N(d_1)$. Also, higher dividends will lower the number of bonds to short sell for calls and lower the number of bonds to buy for puts.

EXAMPLE 11

BSM Model Applied to Equities

Suppose we are given the following information on an underlying stock and options: S = 60, X = 60, r = 2%, T = 0.5, δ = 2%, and σ = 45%. Assume we are examining European-style options.

1 Which answer *best* describes how the BSM model is used to value a call option with the parameters given?

A The BSM model call value is the exercise price times $N(d_1)$ less the present value of the stock price times $N(d_2)$.

B The BSM model call value is the stock price times $e^{-\delta T}N(d_1)$ less the exercise price times $e^{-rT}N(d_2)$.

C The BSM model call value is the stock price times $e^{-\delta T}N(-d_1)$ less the present value of the exercise price times $e^{-rT}N(-d_2)$.

2 Which answer *best* describes how the BSM model is used to value a put option with the parameters given?

A The BSM model put value is the exercise price times $N(d_1)$ less the present value of the stock price times $N(d_2)$.

B The BSM model put value is the exercise price times $e^{-\delta T}N(-d_2)$ less the stock price times $e^{-rT}N(-d_2)$.

C The BSM model put value is the exercise price times $e^{-rT}N(-d_2)$ less the stock price times $e^{-\delta T}N(-d_1)$.

3 Suppose now that the stock does not pay a dividend—that is, δ = 0%. Identify the correct statement.

A The BSM model option value is the same as the previous problems because options are not dividend adjusted.

B The BSM model option values will be different because there is an adjustment term applied to the exercise price, that is $e^{-\delta T}$, which will influence the option values.

C The BSM model option value will be different because d_1, d_2, and the stock component are all adjusted for dividends.

Solution to 1:

B is correct. The BSM call model for a dividend-paying stock can be expressed as $Se^{-\delta T}N(d_1) - Xe^{-rT}N(d_2)$.

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Solution to 2:

C is correct. The BSM put model for a dividend-paying stock can be expressed as $Xe^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1)$.

Solution to 3:

C is correct. The BSM model option value will be different because d_1, d_2, and the stock component are all adjusted for dividends.

EXAMPLE 12

How the BSM Model Is Used to Value Stock Options

Suppose that we have some Bank of China shares that are currently trading on the Hong Kong Stock Exchange at HKD4.41. Our view is that the Bank of China's stock price will be steady for the next three months, so we decide to sell some three-month out-of-the-money calls with exercise price at 4.60 in order to enhance our returns by receiving the option premium. Risk-free government securities are paying 1.60% and the stock is yielding HKD 0.24%. The stock volatility is 28%. We use the BSM model to value the calls.

Which statement is correct? The BSM model inputs (underlying, exercise, expiration, risk-free rate, dividend yield, and volatility) are:

A 4.60, 4.41, 3, 0.0160, 0.0024, and 0.28.

B 4.41, 4.60, 0.25, 0.0160, 0.0024, and 0.28.

C 4.41, 4.41, 0.3, 0.0160, 0.0024, and 0.28.

Solution:

B is correct. The spot price of the underlying is HKD4.41. The exercise price is HKD4.60. The expiration is 0.25 years (three months). The risk-free rate is 0.016. The dividend yield is 0.0024. The volatility is 0.28.

For foreign exchange options, $\gamma = r^f$, which is the continuously compounded foreign risk-free interest rate. When quoting an exchange rate, we will give the value of the domestic currency per unit of the foreign currency. For example, Japanese yen (¥) per unit of the euro (€) will be expressed as the euro trading for ¥135 or succinctly 135¥/€. This is called the foreign exchange spot rate. Thus, the foreign currency, the euro, is expressed in terms of the Japanese yen, which is in this case the domestic currency. This is logical, for example, when a Japanese firm would want to express its foreign euro holdings in terms of its domestic currency, Japanese yen.

With currency options, the underlying instrument is the foreign exchange spot rate. Again, the carry benefit is the interest rate in the foreign country because the foreign currency could be invested in the foreign country's risk-free instrument. Also, with currency options, the underlying and the exercise price must be quoted in the same currency unit. Lastly, the volatility in the model is the volatility of the log return of the spot exchange rate. Each currency option is for a certain quantity of foreign currency, termed the notional amount, a concept analogous to the number of shares of stock covered in an option contract. The total cost of the option would be obtained by multiplying the formula value by the notional amount in the same way that one would multiply the formula value of an option on a stock by the number of shares the option contract covers.

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The BSM model applied to currencies can be described as having two components, a foreign exchange component and a bond component. For call options, the foreign exchange component is $Se^{-r^f T}N(d_1)$ and the bond component is $e^{-rT}XN(d_2)$, where r is the domestic risk-free rate. The BSM call model applied to currencies is simply the foreign exchange component minus the bond component. For put options, the foreign exchange component is $Se^{-r^f T}N(-d_1)$ and the bond component is $e^{-rT}XN(-d_2)$. The BSM put model applied to currencies is simply the bond component minus the foreign exchange component. Remember that the underlying is expressed in terms of the domestic currency.

EXAMPLE 13

BSM Model Applied to Value Options on Currency

A Japanese camera exporter to Europe has contracted to receive fixed euro (€) amounts each quarter for his goods. The spot price of the currency pair is 135¥/€. If the exchange rate falls to, say, 130¥/€, then the yen will have strengthened because it will take fewer yen to buy one euro. The exporter is concerned that the yen will strengthen because in this case, his forthcoming fixed euro will buy fewer yen. Hence, the exporter is considering buying an at-the-money spot euro put option to protect against this fall; this in essence is a call on yen. The Japanese risk-free rate is 0.25% and the European risk-free rate is 1.00%.

1 What are the underlying and exercise prices to use in the BSM model to get the euro put option value?

 A 1/135; 1/135

 B 135; 135

 C 135; 130

2 What are the risk-free rate and the carry rate to use in the BSM model to get the euro put option value?

 A 0.25%; 1.00%

 B 0.25%; 0.00%

 C 1.00%; 0.25%

Solution to 1:

B is correct. The underlying is the spot FX price of 135 ¥/€. Because the put is at-the-money spot, the exercise price equals the spot price.

Solution to 2:

A is correct. The risk-free rate to use is the Japanese rate because the Japanese yen is the domestic currency unit per the exchange rate quoting convention. The carry rate is the foreign currency's risk-free rate, which is the European rate.

We turn now to examine a modification of the BSM model when the underlying is a forward or futures contract.

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5 BLACK OPTION VALUATION MODEL

In 1976, Fischer Black introduced a modified version of the BSM model approach that is applicable to options on underlying instruments that are costless to carry, such as options on futures contracts—for example, equity index futures—and options on forward contracts. The latter include interest rate-based options, such as caps, floors, and swaptions.

5.1 European Options on Futures

We assume that the futures price also follows geometric Brownian motion. We ignore issues like margin requirements and marking to market. Black proposed the following model for European-style futures options:

$$c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)] \tag{15}$$

and

$$p = e^{-rT}[XN(-d_2) - F_0(T)N(-d_1)] \tag{16}$$

where

$$d_1 = \frac{\ln\left[F_0(T)/X\right] + \left(\sigma^2/2\right)T}{\sigma\sqrt{T}} \text{ and}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Note that $F_0(T)$ denotes the futures price at Time 0 that expires at Time T, and σ denotes the volatility related to the futures price. The other terms are as previously defined. Black's model is simply the BSM model in which the futures contract is assumed to reflect the carry arbitrage model. Futures option put–call parity can be expressed as

$$c = e^{-rT}[F_0(T) - X] + p \tag{17}$$

As we have seen before, put–call parity is a useful tool for describing the valuation relationship between call and put values within various option valuation models.

The Black model can be described in a similar way to the BSM model. The Black model has two components, a futures component and a bond component. For call options, the futures component is $F_0(T)e^{-rT}N(d_1)$ and the bond component is again $e^{-rT}XN(d_2)$. The Black call model is simply the futures component minus the bond component. For put options, the futures component is $F_0(T)e^{-rT}N(-d_1)$ and the bond component is again $e^{-rT}XN(-d_2)$. The Black put model is simply the bond component minus the futures component.

Alternatively, futures option valuation, based on the Black model, is simply computing the present value of the difference between the futures price and the exercise price. The futures price and exercise price are appropriately adjusted by the N(d) functions. For call options, the futures price is adjusted by $N(d_1)$ and the exercise price is adjusted by $-N(d_2)$ to arrive at difference. For put options, the futures price is adjusted by $-N(-d_1)$ and the exercise price is adjusted by $+N(-d_2)$.

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EXAMPLE 14

European Options on Futures Index

The S&P 500 Index (a spot index) is presently at 1,860 and the 0.25 expiration futures contract is trading at 1,851.65. Suppose further that the exercise price is 1,860, the continuously compounded risk-free rate is 0.2%, time to expiration is 0.25, volatility is 15%, and the dividend yield is 2.0%. Based on this information, the following results are obtained for options on the futures contract.[13]

Options on Futures	
Calls	**Puts**
$N(d_1)$ =0.491	$N(-d_1)$ = 0.509
$N(d_2)$ = 0.461	$N(-d_2)$ = 0.539
c = US$51.41	p = US$59.76

1 Identify the statement that *best* describes how the Black model is used to value a European call option on the futures contract just described.

- **A** The call value is the present value of the difference between the exercise price times 0.461 and the current futures price times 0.539.
- **B** The call value is the present value of the difference between the current futures price times 0.491 and the exercise price times 0.461.
- **C** The call value is the present value of the difference between the current spot price times 0.491 and the exercise price times 0.461.

2 Which statement *best* describes how the Black model is used to value a European put options on the futures contract just described?

- **A** The put value is the present value of the difference between the exercise price times 0.539 and the current futures price times 0.509.
- **B** The put value is the present value of the difference between the current futures price times 0.491 and the exercise price times 0.461.
- **C** The put value is the present value of the difference between the current spot price times 0.491 and the exercise price times 0.461.

3 What are the underlying and exercise prices to use in the Black futures option model?

- **A** 1,851.65; 1,860
- **B** 1,860; 1,860
- **C** 1,860; 1,851.65

Solution to 1:

B is correct. Recall Black's model for call options can be expressed as $c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)]$.

Solution to 2:

A is correct. Recall Black's model for put options can be expressed as $p = e^{-rT}[XN(-d_2) - F_0(T)N(-d_1)]$.

13 We ignore the effect of the multiplier. As of this writing, the S&P 500 futures option contract has a multiplier of 250. The prices reported here have not been scaled up by this amount. In practice, the option cost would by 250 times the option value.

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Solution to 3:

A is correct. The underlying is the futures price of 1,851.65 and the exercise price was given as 1,860.

5.2 Interest Rate Options

With interest rate options, the underlying instrument is a reference interest rate, such as three-month Libor. An interest rate call option gains when the reference interest rate rises and an interest rate put option gains when the reference interest rate falls. Interest rate options are the building blocks of many other instruments.

For an interest rate call option on three-month Libor with one year to expiration, the underlying interest rate is a forward rate agreement (FRA) rate that expires in one year. This FRA is observed today and is the underlying rate used in the Black model. The underlying rate of the FRA is a 3-month Libor deposit that is investable in 12 months and matures in 15 months. Thus, in one year, the FRA rate typically converges to the three-month spot Libor.

Interest rates are typically set in advance, but interest payments are made in arrears, which is referred to as advanced set, settled in arrears. For example, with a bank deposit, the interest rate is usually set when the deposit is made, say t_{j-1}, but the interest payment is made when the deposit is withdrawn, say t_j. The deposit, therefore, has $t_m = t_j - t_{j-1}$ time until maturity. Thus, the rate is advanced set, but the payment is settled in arrears. Likewise with a floating rate loan, the rate is usually set and the interest accrues at this known rate, but the payment is made later. Similarly, with some interest rate options, the time to option expiration (t_{j-1}) when the interest rate is set does not correspond to the option settlement (t_j) when the cash payment is made, if any. For example, if an interest rate option payment based on three-month Libor is US$5,000 determined on January 15th, the actual payment of the US$5,000 would occur on April 15.

Interest rates are quoted on an annual basis, but the underlying implied deposit is often less than a year. Thus, the annual rates must be adjusted for the accrual period. Recall that the accrual period for a quarterly reset 30/360 day count FRA is 0.25 (= 90/360). If the day count is on an actual (ACT) number of days divided by 360 (ACT/360), then the accrual period may be something like 0.252778 (= 91/360), assuming 91 days in the period. Typically, the accrual period in FRAs is based on 30/360 whereas the accrual period based on the option is actual number of days in the contract divided by the actual number of days in the year (identified as ACT/ACT or ACT/365).

The model presented here is known as the standard market model and is a variation of Black's futures option valuation model. Again, let t_{j-1} denote the time to option expiration (ACT/365), whereas let t_j denote the time to the maturity date of the underlying FRA. Note that the interest accrual on the underlying begins at the option expiration (Time t_{j-1}). Let $FRA(0,t_{j-1},t_m)$ denote the fixed rate on a FRA at Time 0 that expires at Time t_{j-1}, where the underlying matures at Time t_j (= $t_{j-1} + t_m$), with all times expressed on an annual basis. We assume the FRA is 30/360 day count. For example, FRA(0,0.25,0.5) = 2% denotes the 2% fixed rate on a forward rate agreement that expires in 0.25 years with settlement amount being paid in 0.75 (= 0.25 + 0.5) years.[14] Let R_X denote the exercise rate expressed on an annual basis. Finally, let σ denote the interest rate volatility. Specifically, σ is the annualized standard deviation of the continuously compounded percentage change in the underlying FRA rate.

14 Note that in other contexts the time periods are expressed in months. For example with months, this FRA would be expressed as FRA(0,3,6). Note that the third term in parentheses denotes the maturity of the underlying deposit from the expiration of the FRA.

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Interest rate options give option buyers the right to certain cash payments based on observed interest rates. For example, an interest rate call option gives the call buyer the right to a certain cash payment when the underlying interest rate exceeds the exercise rate. An interest rate put option gives the put buyer the right to a certain cash payment when the underlying interest rate is below the exercise rate.

With the standard market model, the prices of interest rate call and put options can be expressed as

$$c = (AP)e^{-r(t_{j-1}+t_m)}\left[FRA(0,t_{j-1},t_m)N(d_1) - R_X N(d_2)\right] \tag{18}$$

and

$$p = (AP)e^{-r(t_{j-1}+t_m)}\left[R_X N(-d_2) - FRA(0,t_{j-1},t_m)N(-d_1)\right] \tag{19}$$

where

AP denotes the accrual period in years

$$d_1 = \frac{\ln\left[FRA(0,t_{j-1},t_m)/R_X\right] + (\sigma^2/2)t_{j-1}}{\sigma\sqrt{t_{j-1}}}$$

$$d_2 = d_1 - \sigma\sqrt{t_{j-1}}$$

The formulas here give the value of the option for a notional amount of 1. In practice, the notional would be more than one, so the full cost of the option is obtained by multiplying these formula amounts by the notional amount. Of course, this point is just the same as finding the value of an option on a single share of stock and then multiplying that value by the number of shares covered by the option contract.

Immediately, we note that the standard market model requires an adjustment when compared with the Black model for the accrual period. In other words, a value such as $FRA(0,t_{j-1},t_m)$ or the strike rate, R_X, as appearing in the formula given earlier, is stated on an annual basis, as are interest rates in general. The actual option premium would have to be adjusted for the accrual period. After accounting for this adjustment, this model looks very similar to the Black model, but there are important but subtle differences. First, the discount factor, $e^{-r(t_{j-1}+t_m)}$, does not apply to the option expiration, t_{j-1}. Rather, the discount factor is applied to the maturity date of the FRA or t_j (= $t_{j-1} + t_m$). We express this maturity as ($t_{j-1} + t_m$) rather than t_j to emphasize the settlement in arrears nature of this option. Second, rather than the underlying being a futures price, the underlying is an interest rate, specifically a forward rate based on a forward rate agreement or $FRA(0,t_{j-1},t_m)$. Third, the exercise price is really a rate and reflects an interest rate, not a price. Fourth, the time to the option expiration, t_{j-1}, is used in the calculation of d_1 and d_2. Finally, both the forward rate and the exercise rate should be expressed in decimal form and not as percent (for example, 0.02 and not 2.0). Alternatively, if expressed as a percent, then the notional amount adjustment could be divided by 100.

As with other option models, the standard market model can be described as simply the present value of the expected option payoff at expiration. Specifically, we can express the standard market model for calls as $c = PV[E(c_{tj})]$ and for puts as $p = PV[E(p_{tj})]$, where $E(c_{tj}) = (AP)[FRA(0,t_{j-1},t_m)N(d_1) - R_X N(d_2)]$ and $E(p_{tj}) = (AP)[R_X N(-d_2) - FRA(0,t_{j-1},t_m)N(-d_1)]$. The present value term in this context is simply $e^{-rt_j} = e^{-r(t_{j-1}+t_m)}$. Again, note we discount from Time t_j, the time when the cash flows are settled on the FRA.

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There are several interesting and useful combinations that can be created with interest rate options. We focus on a few that will prove useful for understanding swaptions in the next section. First, if the exercise rate is selected so as to equal the current FRA rate, then long an interest rate call option and short an interest rate put option is equivalent to a receive-floating, pay-fixed FRA.

Second, if the exercise rate is again selected so it is equal to the current FRA rate, then long an interest rate put option and short an interest rate call option is equivalent to a receive-fixed, pay-floating FRA. Note that FRAs are the building blocks of interest rate swaps.

Third, an interest rate cap is a portfolio or strip of interest rate call options in which the expiration of the first underlying corresponds to the expiration of the second option and so forth. The underlying interest rate call options are termed caplets. Thus, a set of floating-rate loan payments can be hedged with a long position in an interest rate cap encompassing a series of interest rate call options.

Fourth, an interest rate floor is a portfolio or strip of interest rate put options in which the expiration of the first underlying corresponds with the expiration of the second option and so forth. The underlying interest rate put options are termed floorlets. Thus, a floating-rate bond investment or any other floating-rate lending situation can be hedged with an interest rate floor encompassing a series of interest rate put options.

Fifth, applying put–call parity as discussed earlier, long an interest rate cap and short an interest rate floor with the exercise prices set at the swap rate is equivalent to a receive-floating, pay-fixed swap. On a settlement date, when the underlying rate is above the strike, both the cap and the swap pay off to the party. When the underlying rate is below the strike on a settlement date, the party must make a payment on the short floor, just as the case with a swap. For the opposite position, long an interest rate floor and short an interest rate cap result in the party making a payment when the underlying rate is above the strike and receiving one when the underlying rate is below the strike, just as is the case for a pay-floating, receive-fixed swap.

Finally, if the exercise rate is set equal to the swap rate, then the value of the cap must be equal to the value of the floor at the start. When an interest rate swap is initiated, its current value is zero and is known as an at-market swap. When an exercise rate is selected such that the cap value equals the floor value, then the initial cost of being long a cap and short the floor is also zero. This occurs when the cap and floor strike are equal to the swap rate.

EXAMPLE 15

European Interest Rate Options

Suppose you are a speculative investor in Singapore. On 15 May, you anticipate that some regulatory changes will be enacted, and you want to profit from this forecast. On 15 June, you intend to borrow 10,000,000 Singapore dollars to fund the purchase of an asset, which you expect to resell at a profit three months after purchase, say on 15 September. The current three-month Sibor (that is, Singapore Libor) is 0.55%. The appropriate FRA rate over the period of 15 June to 15 September is currently 0.68%. You are concerned that rates will rise, so you want to hedge your borrowing risk by purchasing an interest rate call option with an exercise rate of 0.60%.

1 In using the Black model to value this interest rate call option, what would the underlying rate be?

   A 0.55%

   B 0.68%

   C 0.60%

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2 The discount factor used in pricing this option would be over what period of time?

A 15 May–15 June

B 15 June–15 September

C 15 May–15 September

Solution to 1:

B is correct. In using the Black model, a forward or futures price is used as the underlying. This approach is unlike the BSM model in which a spot price is used as the underlying.

Solution to 2:

C is correct. You are pricing the option on 15 May. An option expiring 15 June when the underlying is three-month Sibor will have its payoff determined on 15 June, but the payment will be made on 15 September. Thus, the expected payment must be discounted back from 15 September to 15 May.

Interest rate option values are linked in an important way with interest rate swap values through caps and floors. As we will see in the next section, an interest rate swap serves as the underlying for swaptions. Thus, once again, we see that important links exist between interest rate options, swaps, and swaptions.

5.3 Swaptions

A swap option or swaption is simply an option on a swap. It gives the holder the right, but not the obligation, to enter a swap at the pre-agreed swap rate—the exercise rate. Interest rate swaps can be either receive fixed, pay floating or receive floating, pay fixed. A payer swaption is an option on a swap to pay fixed, receive floating. A receiver swaption is an option on a swap to receive fixed, pay floating. Note that the terms "call" and "put" are often avoided because of potential confusion over the nature of the underlying. Notice also that the terminology focuses on the fixed swap rate.

A payer swaption buyer hopes the fixed rate goes up before the swaption expires. When exercised, the payer swaption buyer is able to enter into a pay-fixed, receive-floating swap at the predetermined exercise rate, R_X. The buyer can then immediately enter an offsetting at-market receive-fixed, pay-floating swap at the current fixed swap rate. The floating legs of both swaps will offset, leaving the payer swaption buyer with an annuity of the difference between the current fixed swap rate and the swaption exercise rate. Thus, swaption valuation will reflect an annuity.

Swap payments are advanced set, settled in arrears. Let the swap reset dates be expressed as t_0, t_1, t_2, ..., t_n. Let R_{FIX} denote the fixed swap rate starting when the swaption expires, denoted as before with T, quoted on an annual basis, and R_X denote the exercise rate starting at Time T, again quoted on an annual basis. As before, we will assume a notional amount of 1.

Because swap rates are quoted on an annual basis, let AP denote the accrual period. Finally, we need some measure of uncertainty. Let σ denote the volatility of the forward swap rate. More precisely, σ denotes annualized, standard deviation of the continuously compounded percentage changes in the forward swap rate.

The swaption model presented here is a modification of the Black model. Let the present value of an annuity matching the forward swap payment be expressed as

$$\text{PVA} = \sum_{j=1}^{n} \text{PV}_{0,t_j}(1)$$

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This term is equivalent to what is sometimes referred to as an annuity discount factor. It applies here because a swaption creates a series of equal payments of the difference in the market swap rate at expiration and the chosen exercise rate. Therefore, the payer swaption valuation model is

$$PAY_{SWN} = (AP)PVA[R_{FIX}N(d_1) - R_XN(d_2)] \tag{20}$$

and the receiver swaption valuation model

$$REC_{SWN} = (AP)PVA[R_XN(-d_2) - R_{FIX}N(-d_1)] \tag{21}$$

where

$$d_1 = \frac{\ln(R_{FIX}/R_X) + (\sigma^2/2)T}{\sigma\sqrt{T}}, \text{ and as always,}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

As noted with interest rate options, the actual premium would need to be scaled by the notional amount. Once again, we can see the similarities to the Black model. We note that the swaption model requires two adjustments, one for the accrual period and one for the present value of an annuity. After accounting for these adjustments, this model looks very similar to the Black model but there are important subtle differences. First, the discount factor is absent. The payoff is not a single payment but a series of payments. Thus, the present value of an annuity used here embeds the option-related discount factor. Second, rather than the underlying being a futures price, the underlying is the fixed rate on a forward interest rate swap. Third, the exercise price is really expressed as an interest rate. Finally, both the forward swap rate and the exercise rate should be expressed in decimal form and not as percent (for example, 0.02 and not 2.0).

As with other option models, the swaption model can be described as simply the present value of the expected option payoff at expiration. Specifically, we can express the payer swaption model value as

$$PAY_{SWN} = PV[E(PAY_{SWN,T})]$$

and the receiver swaption model value as

$$REC_{SWN} = PV[E(REC_{SWN,T})],$$

where

$$E(PAY_{SWN,T}) = e^{rT}PAY_{SWN} \text{ and}$$
$$E(REC_{SWN,T}) = e^{rT}REC_{SWN}.$$

The present value term in this context is simply e^{-rT}. Because the annuity term embedded the discounting over the swaption life, the expected swaption values are the current swaption values grossed up by the current risk-free interest rate.

Alternatively, the swaption model can be described as having two components, a swap component and a bond component. For payer swaptions, the swap component is $(AP)PVA(R_{FIX})N(d_1)$ and the bond component is $(AP)PVA(R_X)N(d_2)$. The payer swaption model value is simply the swap component minus the bond component. For receiver swaptions, the swap component is $(AP)PVA(R_{FIX})N(-d_1)$ and the bond component is $(AP)PVA(R_X)N(-d_2)$. The receiver swaption model value is simply the bond component minus the swap component.

As with nearly all derivative instruments, there are many useful equivalence relationships. Recall that long an interest rate cap and short an interest rate floor with the same exercise rate is equal to a receive-floating, pay-fixed interest rate swap. Also, short an interest rate cap and long an interest rate floor with the same exercise rate is equal to a pay-floating, receive-fixed interest rate swap. There are also equivalence

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relationships with swaptions. In a similar way, long a receiver swaption and short a payer swaption with the same exercise rate is equivalent to entering a receive-fixed, pay-floating forward swap. Long a payer swaption and short a receiver swaption with the same exercise rate is equivalent to entering a receive-floating, pay-fixed forward swap. Note that if the exercise rate is selected such that the receiver and payer swaptions have the same value, then the exercise rate is equal to the at-market forward swap rate. Thus, there is again a put–call parity relationship important for valuation.

In addition, being long a callable fixed-rate bond can be viewed as being long a straight fixed-rate bond and short a receiver swaption. A receiver swaption gives the buyer the right to receive a fixed rate. Hence, the seller will have to pay the fixed rate when this right is exercised in a lower rate environment. Recall that the bond issuer has the right to call the bonds. If the bond issuer sells a receiver swaption with similar terms, then the bond issuer has essentially converted the callable bond into a straight bond. The bond issuer will now pay the fixed rate on the underlying swap and the floating rate received will be offset by the floating-rate loan created when the bond was refinanced. Specifically, the receiver swaption buyer will benefit when rates fall and the swaption is exercised. Thus, the embedded call feature is similar to a receiver swaption.

EXAMPLE 16

European Swaptions

Suppose you are an Australian company and have ongoing floating-rate debt. You have profited for some time by paying at a floating rate because rates have been falling steadily for the last few years. Now, however, you are concerned that within three months the Australian central bank may tighten its monetary policy and your debt costs will thus increase. Rather than lock in your borrowing via a swap, you prefer to hedge by buying a swaption expiring in three months, whereby you will have the choice, but not the obligation, to enter a five-year swap locking in your borrowing costs. The current three-month forward, five-year swap rate is 2.65%. The current five-year swap rate is 2.55%. The current three-month risk-free rate is 2.25%.

With reference to the Black model to value the swaption, which statement is correct?

A The underlying is the three-month forward, five-year swap rate.

B The discount rate to use is 2.55%.

C The swaption time to expiration, T, is five years.

Solution:

A is correct. The current five-year swap rate is not used as a discount rate with swaptions. The swaption time to expiration is 0.25, not the life of the swap.

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6 OPTION GREEKS AND IMPLIED VOLATILITY

With option valuation models, such as the binomial model, BSM model, and Black's model, we are able to estimate a wide array of comparative information, such as how much the option value will change for a small change in a particular parameter.[15] We will explore this derived information as well as implied volatility in this section. These topics are essential for those managing option positions and in general in obtaining a solid understanding of how option prices change. Our discussion will be based on stock options, though the material covered in this section applies to all types of options.

The measures examined here are known as the Greeks and include, delta, gamma, theta, vega, and rho. With these calculations, we seek to address how much a particular portfolio will change for a given small change in the appropriate parameter. These measures are sometimes referred to as static risk measures in that they capture movements in the option value for a movement in one of the factors that affect the option value, while holding all other factors constant.

Our focus here is on European stock options in which the underlying stock is assumed to pay a dividend yield (denoted δ). Note that for non-dividend-paying stocks, δ = 0.

6.1 Delta

Delta is defined as the change in a given instrument for a given small change in the value of the stock, holding everything else constant. Thus, the delta of long one share of stock is by definition +1.0, and the delta of short one share of stock is by definition −1.0. The concept of the option delta is similarly the change in an option value for a given small change in the value of the underlying stock, holding everything else constant. The option deltas for calls and puts are, respectively,

$$\text{Delta}_c = e^{-\delta T}N(d_1) \quad \textbf{(22)}$$

and

$$\text{Delta}_p = -e^{-\delta T}N(-d_1) \quad \textbf{(23)}$$

Note that the deltas are a simple function of $N(d_1)$. The delta of an option answers the question of how much the option will change for a given change in the stock, holding everything else constant. Therefore, delta is a static risk measure. It does not address how likely this particular change would be. Recall that $N(d_1)$ is a value taken from the cumulative distribution function of a standard normal distribution. As such, the range of values is between 0 and 1. Thus, the range of call delta is 0 and $e^{-\delta T}$ and the range of put delta is $-e^{-\delta T}$ and 0. As the stock price increases, the call option goes deeper in the money and the value of $N(d_1)$ is moving toward 1. As the stock price decreases, the call option goes deeper out of the money and the value of $N(d_1)$ is moving toward zero. When the option gets closer to maturity, the delta will drift either toward 0 if it is out of the money or drift toward 1 if it is in the money. Clearly, as the stock price changes and as time to maturity changes, the deltas are also changing.

Delta hedging an option is the process of establishing a position in the underlying stock of a quantity that is prescribed by the option delta so as to have no exposure to very small moves up or down in the stock price. Hence, to execute a single option delta hedge, we first calculate the option delta and then buy or sell delta units of stock. In practice, rarely does one have only one option position to manage. Thus, in general, delta hedging refers to manipulating the underlying portfolio delta by

15 Parameters in the BSM model, for example, include the stock price, exercise price, volatility, time to expiration, and the risk-free interest rate.

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appropriately changing the positions in the portfolio. A delta neutral portfolio refers to setting the portfolio delta all the way to zero. In theory, the delta neutral portfolio will not change in value for small changes in the stock instrument. Let N_H denote the number of units of the hedging instrument and Delta_H denote the delta of the hedging instrument, which could be the underlying stock, call options, or put options. Delta neutral implies the portfolio delta plus $N_H\text{Delta}_H$ is equal to zero. The optimal number of hedging units, N_H, is

$$N_H = -\frac{\text{Portfolio delta}}{\text{Delta}_H}$$

Note that if N_H is negative, then one must short the hedging instrument, and if N_H is positive, then one must go long the hedging instrument. Clearly, if the portfolio is options and the hedging instrument is stock, then we will buy or sell shares to offset the portfolio position. For example, if the portfolio consists of 100,000 shares of stock at US$10 per share, then the portfolio delta is 100,000. The delta of the hedging instrument, stock, is +1. Thus, the optimal number of hedging units, N_H, is −100,000 (= −100,000/1) or short 100,000 shares. Alternatively, if the portfolio delta is 5,000 and a particular call option with delta of 0.5 is used as the hedging instrument, then to arrive at a delta neutral portfolio, one must sell 10,000 call options (= −5,000/0.5). Alternatively, if a portfolio of options has a delta of −1,500, then one must buy 1,500 shares of stock to be delta neutral [= −(−1,500)/1]. If the hedging instrument is stock, then the delta is +1 per share.

EXAMPLE 17

Delta Hedging

Apple stock is trading at US$125. We write calls (that is, we sell calls) on 1,000 Apple shares and now are exposed to an increase in the price of the Apple stock. That is, if Apple rises, we will lose money because the calls we sold will go up in value, so our liability will increase. Correspondingly, if Apple falls, we will make money. We want to neutralize our exposure to Apple. Say the call delta is 0.50, which means that if Apple goes up by US$0.10, a call on one Apple share will go up US$0.05. We need to trade in such a way as to make money if Apple goes up, to offset our exposure. Hence, we buy 500 Apple shares to hedge. Now, if Apple goes up US$0.10, the sold calls will go up US$50 (our liability goes up), but our long 500 Apple hedge will profit by US$50. Hence, we are delta hedged.

Identify the *incorrect* statement:

A If we sell Apple puts, we need to buy Apple stock to delta hedge.

B Call delta is non-negative (≥ 0); put delta is non-positive (≤ 0).

C Delta hedging is the process of neutralizing exposure to the underlying.

Solution:

A is the correct answer because statement A is incorrect. If we sell puts, we need to short sell stock to delta hedge.

One final interpretation of option delta is related to forecasting changes in option prices. Let $\hat{c}$, $\hat{p}$, and $\hat{S}$ denote some new value for the call, put, and stock. Based on an approximation method, the change in the option price can be estimated with a concept known as a delta approximation or

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$\hat{c} - c \cong \text{Delta}_c\left(\hat{S} - S\right)$ for calls and

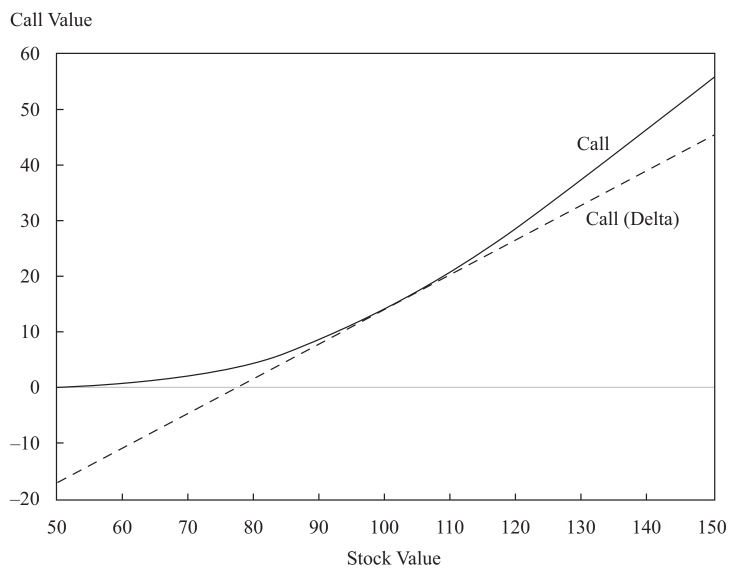
$\hat{p} - p \cong \text{Delta}_p\left(\hat{S} - S\right)$ for puts.[16]

We can now illustrate the actual call values as well as the estimated call values based on delta. Exhibit 15 illustrates the call value based on the BSM model and the call value based on the delta approximation,

$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right)$$

Notice for very small changes in the stock, the delta approximation is fairly accurate. For example, if the stock value rises from 100 to 101, notice that both the call line and the call (delta) estimated line are almost the same value. If, however, the stock value rises from 100 to 150, the call line is now significantly above the call (delta) estimated line. Thus, we see that as the change in the stock increases, the estimation error also increases. The delta approximation is biased low for both a down move and an up move.

Exhibit 15 Call Values and Delta Estimated Call Values (S = 100 = X, r = 5%, σ = 30%, δ = 0)



We see that delta hedging is imperfect and gets worse as the underlying moves further away from its original value of 100. Based on the graph, the BSM model assumption of continuous trading is essential to avoid hedging risk. This hedging risk is related to the difference between these two lines and the degree to which the underlying price experiences large changes.

16 The symbol ≅ denotes approximately. The approximation method is known as a Taylor series. Also note that the put delta is non-positive (≤ 0).

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EXAMPLE 18

Delta Hedging

Suppose we know S = 100, X = 100, r = 5%, T = 1.0, σ = 30%, and δ = 5%. We have a short position in put options on 10,000 shares of stock. Based on this information, we note $\text{Delta}_c = 0.532$, and $\text{Delta}_p = -0.419$. Assume each stock option contract is for one share of stock.

1 The appropriate delta hedge, assuming the hedging instrument is stock, is executed by which of the following transactions? Select the *closest* answer.
   - A Buy 5,320 shares of stock.
   - B Short sell 4,190 shares of stock.
   - C Buy 4,190 shares of stock.

2 The appropriate delta hedge, assuming the hedging instrument is calls, is executed by which of the following transactions? Select the *closest* answer.
   - A Sell 7,876 call options.
   - B Sell 4,190 call options.
   - C Buy 4,190 call options.

3 Identify the correct interpretation of an option delta.
   - A Option delta measures the curvature in the option price with respect to the stock price.
   - B Option delta is the change in an option value for a given small change in the stock's value, holding everything else constant.
   - C Option delta is the probability of the option expiring in the money.

Solution to 1:

B is correct. Recall that $N_H = -\dfrac{\text{Portfolio delta}}{\text{Delta}_H}$. The put delta is given as –0.419, thus the short put delta is 0.419. In this case, Portfolio delta = 10,000(0.419) = 4,190 and $\text{Delta}_H = 1.0$. Thus, the number of number of hedging units is –4,190 [= –(4,190/1)] or short sell 4,190 shares of stock.

Solution to 2:

A is correct. Again the Portfolio delta = 4,190 but now $\text{Delta}_H = 0.532$. Thus, the number of hedging units is –7,875.9 [= –(4,190/0.532)] or sell 7,876 call options.

Solution to 3:

B is correct. Delta is defined as the change in a given portfolio for a given small change in the stock's value, holding everything else constant. Option delta is defined as the change in an option value for a given small change in the stock's value, holding everything else constant.

6.2 Gamma

Recall that delta is a good approximation of how an option price will change for a small change in the stock. For larger changes in the stock, we need better accuracy. **Gamma** is defined as the change in a given instrument's delta for a given small change in the stock's value, holding everything else constant. Option gamma is similarly defined as the change in a given option delta for a given small change in the stock's value, holding everything else constant. Option gamma is a measure of the curvature in the option price in relationship to the stock price. Thus, the gamma of a long or short position

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in one share of stock is zero because the delta of a share of stock never changes. A stock always moves one-for-one with itself. Thus, its delta is always +1 and, of course, −1 for a short position in the stock. The gamma for a call and put option are the same and can be expressed as

$$\text{Gamma}_c = \text{Gamma}_p = \frac{e^{-´T}}{S\sigma\sqrt{T}}n(d_1) \quad (24)$$

where $n(d_1)$ is the standard normal probability density function. The lowercase "n" is distinguished from the cumulative normal distribution—which the density function generates—and that we have used elsewhere in this reading denoted by uppercase "N". The gamma of a call equals the gamma of a similar put based on put–call parity or $c - p = S_0 - e^{-rT}X$. Note that neither S_0 nor $e^{-rT}X$ is a direct function of delta. Hence, the right-hand side of put–call parity has a delta of 1. Thus, the right-hand side delta is not sensitive to changes in the underlying. Therefore, the gamma of a call must equal the gamma of a put.

Gamma is always non-negative. Gamma takes on its largest value near at the money. Options deltas do not change much for small changes in the stock price if the option is either deep in or deep out of the money. Also, as the stock price changes and as time to expiration changes, the gamma is also changing.

Gamma measures the rate of change of delta as the stock changes. Gamma approximates the estimation error in delta for options because the option price with respect to the stock is non-linear and delta is a linear approximation. Thus, gamma is a risk measure; specifically, gamma measures the non-linearity risk or the risk that remains once the portfolio is delta neutral. A gamma neutral portfolio implies the gamma is zero. For example, gamma can be managed to an acceptable level first and then delta is neutralized as a second step. This hedging approach is feasible because options have gamma but a stock does not. Thus, in order to modify gamma, one has to include additional option trades in the portfolio. Once the revised portfolio, including any new option trades, has the desired level of gamma, then the trader can get the portfolio delta to its desired level as step two. To alter the portfolio delta, the trader simply buys or sells stock. Because stock has a positive delta, but zero gamma, the portfolio delta can be brought to its desired level with no impact on the portfolio gamma.

One final interpretation of gamma is related to improving the forecasted changes in option prices. Again, let $\hat{c}$, $\hat{p}$, and $\hat{S}$ denote new values for the call, put, and stock. Again based on an approximation method, the change in the option price can be estimated by a delta-plus-gamma approximation or

$$\hat{c} - c \approx \text{Delta}_c\left(\hat{S} - S\right) + \frac{\text{Gamma}_c}{2}\left(\hat{S} - S\right)^2 \text{ for calls and}$$

$$\hat{p} - p \approx \text{Delta}_p\left(\hat{S} - S\right) + \frac{\text{Gamma}_p}{2}\left(\hat{S} - S\right)^2 \text{ for puts.}$$

Exhibit 16 illustrates the call value based on the BSM model; the call value based on the delta approximation,

$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right)$$

and the call value based on the delta-plus-gamma approximation,

$$\hat{c} = c + \text{Delta}_c\left(\hat{S} - S\right) + \frac{\text{Gamma}_c}{2}\left(\hat{S} - S\right)^2$$

Notice again that for very small changes in the stock, the delta approximation and the delta-plus-gamma approximations are fairly accurate. If the stock value rises from 100 to 150, the call line is again significantly above the delta estimated line but is below the delta-plus-gamma estimated line. Importantly, the call delta-plus-gamma

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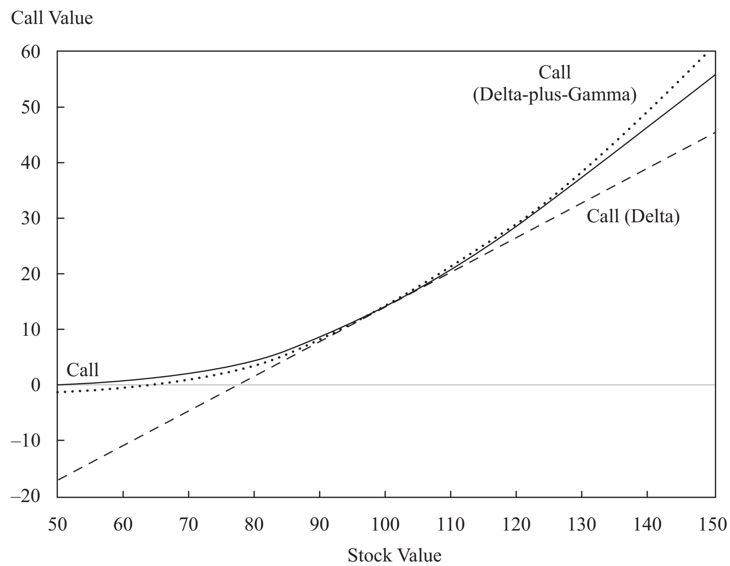
estimated line is significantly closer to the BSM model call values. Thus, we see that even for fairly large changes in the stock, the delta-plus-gamma approximation is accurate. As the change in the stock increases, the estimation error also increases. From Exhibit 16, we see the delta-plus-gamma approximation is biased low for a down move but biased high for an up move. Thus, when estimating how the call price changes when the underlying changes, we see how the delta-plus-gamma approximation is an improvement when compared with using the delta approximation on its own.

Exhibit 16 Call Values, Delta Estimated Call Values, and Delta-Plus-Gamma Estimated Call Values (S = 100 = X, r = 5%, σ = 30%, δ = 0)



If the BSM model assumptions hold, then we would have no risk in managing option positions. In reality, however, stock prices often jump rather than move continuously and smoothly, which creates "gamma risk." Gamma risk is so-called because gamma measures the risk of stock prices jumping when hedging an option position, and thus leaving us suddenly unhedged.

EXAMPLE 19

Gamma Risk in Option Trading

Suppose we are options traders and have only one option position—a short call option. We also hold some stock such that we are delta hedged. Which one of the following statements is true?

A We are gamma neutral.

B Buying a call will increase our overall gamma.

C Our overall position is a positive gamma, which will make large moves profitable for us, whether up or down.

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Solution:

B is correct. Buying options (calls or puts) will always increase net gamma. A is incorrect because we are short gamma, not gamma neutral. C is also incorrect because we are short gamma. We can only become gamma neutral from a short gamma position by purchasing options.

6.3 Theta

Theta is defined as the change in a portfolio for a given small change in calendar time, holding everything else constant. Option theta is similarly defined as the change in an option value for a given small change in calendar time, holding everything else constant. Option theta is the rate at which the option time value declines as the option approaches expiration. To understand theta, it is important to remember the "holding everything else constant" assumption. Specifically, the theta calculation assumes nothing changes except calendar time. Clearly, if calendar time passes, then time to expiration declines. Because stocks do not have an expiration date, the stock theta is zero. Like gamma, theta cannot be adjusted with stock trades.

The gain or loss of an option portfolio in response to the mere passage of calendar time is known as time decay. Particularly with long options positions, often the mere passage of time without any change in other variables, such as the stock, will result is significant losses in value. Therefore, investment managers with significant option positions carefully monitor theta and their exposure to time decay. Time decay is essentially the measure of profit and loss of an option position as time passes, holding everything else constant.

Note that theta is fundamentally different from delta and gamma in the sense that the passage of time does not involve any uncertainty. There is no chance that time will go backward. Time marches on, but it is important to understand how your investment position will change with the mere passage of time.

Typically, theta is negative for options. That is, as calendar time passes, expiration time declines and the option value also declines. Exhibit 17 illustrates the option value with respect to time to expiration. Remember, as calendar time passes, the time to expiration declines. Both the call and the put option are at the money and eventually are worthless if the stock does not change. Notice, however, how the speed of the option value decline increases as time to expiration decreases.

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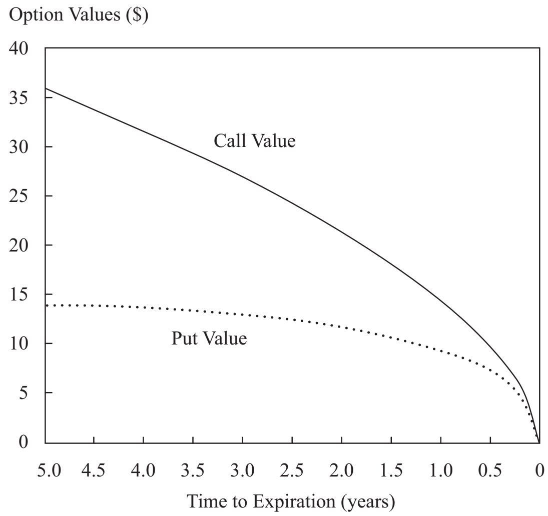
Exhibit 17 Option Values and Time to Expiration (S = 100 = X, r = 5%, σ = 30%, δ = 0)



6.4 Vega

Vega is defined as the change in a given portfolio for a given small change in volatility, holding everything else constant. Vega measures the sensitivity of a given portfolio to volatility. The vega of an option is positive. An increase in volatility results in an increase in the option value for both calls and puts.

The vega of a call equals the vega of a similar put based on put–call parity or $c - p = S_0 - e^{-rT}X$. Note that neither S_0 nor $e^{-rT}X$ is a direct function of volatility. Therefore, the vega of a call must offset the vega of a put so that the vega of the right-hand side is zero.

Unlike the Greeks we have already discussed, vega is based on an unobservable parameter, future volatility. Although historical volatility can be calculated, there is no objective measure of future volatility. Similar to the concept of expected value, future volatility is subjective. Thus, vega measures the sensitivity of a portfolio to changes in the volatility used in the option valuation model. Option values are generally quite sensitive to volatility. In fact, of the five variables in the BSM, an option's value is most sensitive to volatility changes.

At extremely low volatility, the option values tend toward their lower bounds. The lower bound of a European-style call option is zero or the stock less the present value of the exercise price, whichever is greater. The lower bound of a European-style put option is zero or the present value of the exercise price less the stock, whichever is greater. Exhibit 18 illustrates the option values with respect to volatility. In this case, the call lower bound is 4.88 and the put lower bound is 0. The difference between the call and put can be explained by put–call parity.

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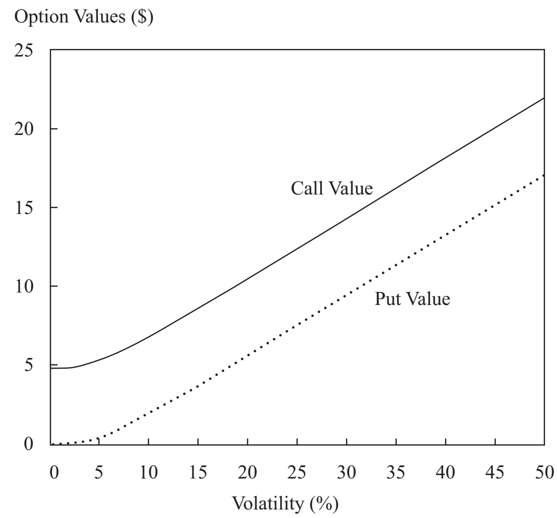
Exhibit 18 Option Values and Volatility (S = 100 = X, r = 5%, T = 1, δ = 0)



Vega is very important in managing an options portfolio because option values can be very sensitive to volatility changes. Vega is high when options are at or near the money. Volatility is usually only hedged with other options and volatility itself can be quite volatile. Volatility is sometimes considered a separate asset class or a separate risk factor. Because it is rather exotic and potentially dangerous, exposure to volatility needs to be managed, bearing in mind that risk managers, board members, and clients may not understand or appreciate losses if volatility is the source.

6.5 Rho

Rho is defined as the change in a given portfolio for a given small change in the risk-free interest rate, holding everything else constant. Thus, rho measures the sensitivity of the portfolio to the risk-free interest rate.

The rho of a call is positive. Intuitively, buying an option avoids the financing costs involved with purchasing the stock. In other words, purchasing a call option allows an investor to earn interest on the money that otherwise would have gone to purchasing the stock. The higher the interest rate, the higher the call value.

The rho of a put is negative. Intuitively, the option to sell the stock delays the opportunity to earn interest on the proceeds from the sale. For example, purchasing a put option rather than selling the stock deprives an investor of the potential interest that would have been earned from the proceeds of selling the stock. The higher the interest rate, the lower the put value.

When interest rates are zero, the call and put option values are the same for at-the-money options. Recall that with put–call parity, we have $c - p = S_0 - e^{-rT}X$, and when interest rates are zero, then the present value function has no effect. As interest rates rise, the difference between call and put options increases as illustrated in

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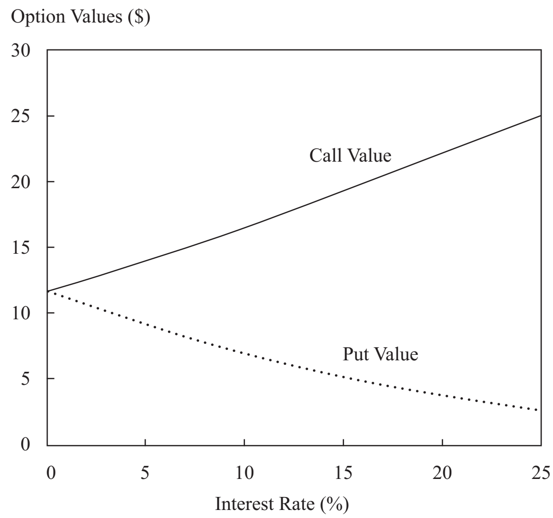
Exhibit 19. The impact on option prices when interest rates change is relatively small when compared with that for volatility changes and that for changes in the stock. Hence, the influence of interest rates is generally not a major concern.[17]

Exhibit 19 Option Values and Interest Rates (S = 100 = X, r = 5%, T = 1, δ = 0)



6.6 Implied Volatility

As we have already touched on in Section 6.4, for most options, the value is particularly sensitive to volatility. Unlike the price of the underlying, however, volatility, is not an observable value in the marketplace. Volatility can be, and often is estimated, based on a sample of historical data. For example, for a three-month option, we might look back over the last three months and calculate the actual historical stock volatility. We can then use this figure as an estimate of volatility over the next three months. The volatility parameter in the BSM model, however, is the *future* volatility. As we know, history is a very frail guide of the future, so the option may appear to be "mispriced" with respect to the actual future volatility experienced. Different investors will have different views of the future volatility. The one with the most accurate forecast will have the most accurate assessment of the option value.

Much like yield to maturity with bonds, volatility can be inferred from option prices. This inferred volatility is called the **implied volatility**. Thus, one important use of the BSM model is to invert the model and estimate implied volatility. The key advantage is that implied volatility provides information regarding the perceived uncertainty going forward and thereby allows us to gain an understanding of the collective opinions of investors on the volatility of the underlying and the demand for options. If the demand for options increases and the no-arbitrage approach is not perfectly reflected

17 An exception to this rule is that with interest rate options, the interest rate is not constant and serves as the underlying. The relationship between the option value and the underlying interest rate is, therefore, captured by the delta, not the rho. Rho is really more generally the relationship between the option value and the rate used to discount cash flows.

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in market prices—for example, because of transaction costs—then the preference for buying options will drive option prices up, and hence, the observed implied volatility. This kind of information is of great value to traders in options.

Recall that one assumption of the BSM model is that all investors agree on the value of volatility and that this volatility is non-stochastic. Note that the original BSM model assumes the underlying instrument volatility is constant in our context. That is, when we calculate option values, we have assumed a single volatility number, like 30%. In practice, it is very common to observe different implied volatilities for different exercise prices and observe different implied volatilities for calls and puts with the same terms. Implied volatility also varies across time to expiration as well as across exercise prices. The implied volatility with respect to time to expiration is known as the term structure of volatility, whereas the implied volatility with respect to the exercise price is known as the volatility smile or sometimes skew depending on the particular shape. It is common to construct a three dimensional plot of the implied volatility with respect to both expiration time and exercise prices, a visualization known as the volatility surface. If the BSM model assumptions were true, then one would expect to find the volatility surface flat.

Implied volatility is also not constant through calendar time. As implied volatility increases, market participants are communicating an increased market price of risk. For example, if the implied volatility of a put increases, it is more expensive to buy downside protection with a put. Hence, the market price of hedging is rising. With index options, various volatility indexes have been created, and these indexes measure the collective opinions of investors on the volatility in the market. Investors can now trade futures and options on various volatility indexes in an effort to manage their vega exposure in other options.

Exhibit 20 provides a look at a couple of decades of one such volatility index, the Chicago Board Options Exchange S&P 500 Volatility Index, known as the VIX. The VIX is quoted as a percent and is intended to approximate the implied volatility of the S&P 500 over the next 30 days. VIX is often termed the fear index because it is viewed as a measure of market uncertainty. Thus, an increase in the VIX index is regarded as greater investor uncertainty. From this figure, we see that the implied volatility of the S&P 500 is not constant and goes through periods when the VIX is low and periods when the VIX is high. In the 2008 global financial crisis, the VIX was extremely high, indicating great fear and uncertainty in the equity market. Remember that implied volatility reflects both beliefs regarding future volatility as well as a preference for risk mitigating products like options. Thus, during the crisis, the higher implied volatility reflected both higher expected future volatility as well as increased preference for buying rather than selling options.

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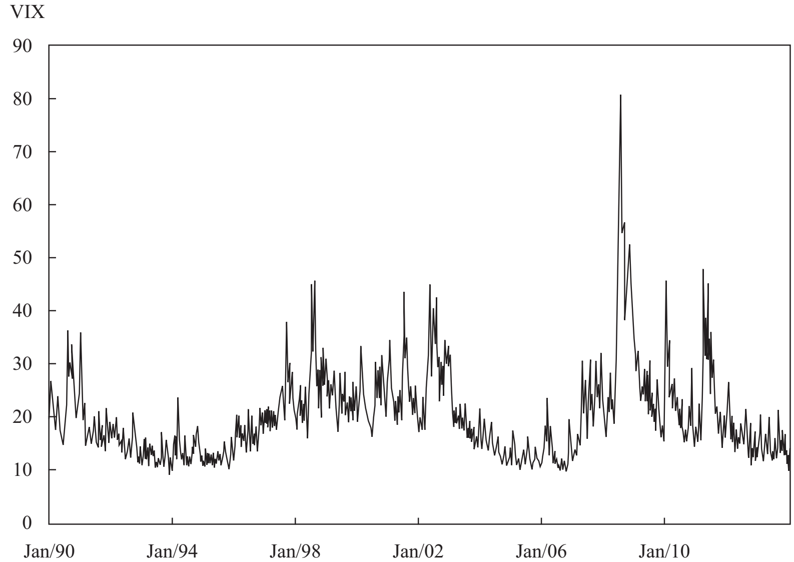
Exhibit 20 VIX Daily Values, 2 January 1990–18 July 2014



Implied volatility has several uses in option trading. An understanding of implied volatility is essential in managing an options portfolio. This reading explains the valuation of options as a function of the value of the underlying, the exercise price, the expiration date, the risk-free rate, dividends or other benefits paid by the underlying, and the volatility of the underlying. Note that each of these parameters is observable except the volatility of the underlying over the option term looking ahead. This volatility has to be estimated in some manner, such as by calculating historical volatility. But as noted, historical volatility involves looking back in time. There are, however, a vast number of liquid options traded on exchanges around the world so that a wide variety of option prices are observable. Because we know the price and all the parameters except the volatility, we can back out the volatility needed by the option valuation model to get the known price. This volatility is the implied volatility.

Hence, implied volatility can be interpreted as the market's view of how to value options. In the option markets, participants use volatility as the medium in which to quote options. The price is simply calculated by the use of an agreed model with the quoted volatility. For example, rather than quote a particular call option as trading for €14.23, it may be quoted as 30.00, where 30.00 denotes in percentage points the implied volatility based on a €14.23 option price. Note that there is a one-to-one relationship between the implied volatility and the option price, ignoring rounding errors.

The benefit of quoting via implied volatility (or simply volatility), rather than price, is that it allows volatility to be traded in its own right. Volatility is the "guess factor" in option pricing. All other inputs—value of the underlying, exercise price, expiration, risk-free rate, and dividend yield—are agreed.[18] Volatility is often the same order of magnitude across exercise prices and expiration dates. This means that traders can compare the values of two options, which may have markedly different exercise prices and expiration dates, and therefore, markedly different prices in a common unit of measure, specifically implied volatility.

18 The risk-free rate and dividend yield may not be entirely agreed, but the impact of variations to these parameters is generally very small compared with the other inputs.

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EXAMPLE 20

Implied Volatility in Option Trading within One Market

Suppose we hold portfolio of options all tied to FTSE 100 futures contracts. Let the current futures price be 6,850. A client calls to request our offer prices on out-of-the-money puts and at-the-money puts, both with the same agreed expiration date. We calculate the prices to be respectively, 190 and 280 futures points. The client wants these prices quoted in implied volatility as well as in futures points because she wants to compare prices by comparing the quoted implied volatilities. The implied volatilities are 16% for the out-of-the-money puts and 15.2% for the at-the-money puts. Why does the client want the quotes in implied volatility?

A Because she can better compare the two options for value—that is, she can better decide which is cheap and which is expensive.

B Because she can assess where implied volatility is trading at that time, and thus consider revaluing her options portfolio at the current market implied volatilities for the FTSE 100.

C Both A and B are valid reasons for quoting options in volatility units.

Solution:

C is correct. Implied volatility can be used to assess the relative value of different options, neutralizing the moneyness and time to expiration effects. Also, implied volatility is useful for revaluing existing positions over time.

EXAMPLE 21

Implied Volatility in Option Trading Across Markets

Suppose an options dealer offers to sell a three-month at-the-money call on the FTSE index option at 19% implied volatility and a one-month in-the-money put on Vodaphone (VOD) at 24%. An option trader believes that based on the current outlook, FTSE volatility should be closer to 25% and VOD volatility should be closer to 20%. What actions might the trader take to benefit from her views?

A Buy the FTSE call and the VOD put.

B Buy the FTSE call and sell the VOD put.

C Sell the FTSE call and sell the VOD puts.

Solution:

B is correct. The trader believes that the FTSE call volatility is understated by the dealer and that the VOD put volatility is overstated. Thus, the trader would expect FTSE volatility to rise and VOD volatility to fall. As a result, the FTSE call would be expected to increase in value and the VOD put would be expected to decrease in value. The trader would take the positions as indicated in B.

Regulators, banks, compliance officers, and most option traders use implied volatilities to communicate information related to options portfolios. This is because implied volatilities, together with standard pricing models, give the "market consensus" valuation, in the same way that other assets are valued using market prices.

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In summary, as long as all market participants agree on the underlying option model and how other parameters are calculated, then implied volatility can be used as a quoting mechanism. Recall that there are calls and puts, various exercise prices, various maturities, American and European, and exchange-traded and OTC options. Thus, it is difficult to conceptualize all these different prices. For example, if two call options on the same stock had different prices, but one had a longer expiration and lower exercise price and the other had a shorter expiration and higher exercise, which should be the higher priced option? It is impossible to tell on the surface. But if one option implied a higher volatility than the other, we know that after taking into account the effects of time and exercise, one option is more expensive than the other. Thus, by converting the quoted price to implied volatility, it is easier to understand the current market price of various risk exposures.

SUMMARY

This reading on the valuation of contingent claims provides a foundation for understanding how a variety of different options are valued. Key points include the following:

- The arbitrageur would rather have more money than less and abides by two fundamental rules: Do not use your own money and do not take any price risk.
- The no-arbitrage approach is used for option valuation and is built on the key concept of the law of one price, which says that if two investments have the same future cash flows regardless of what happens in the future, then these two investments should have the same current price.
- Throughout this reading, the following key assumptions are made:
  - Replicating instruments are identifiable and investable.
  - Market frictions are nil.
  - Short selling is allowed with full use of proceeds.
  - The underlying instrument price follows a known distribution.
  - Borrowing and lending is available at a known risk-free rate.
- The two-period binomial model can be viewed as three one-period binomial models, one positioned at Time 0 and two positioned at Time 1.
- In general, European-style options can be valued based on the expectations approach in which the option value is determined as the present value of the expected future option payouts, where the discount rate is the risk-free rate and the expectation is taken based on the risk-neutral probability measure.
- Both American-style options and European-style options can be valued based on the no-arbitrage approach, which provides clear interpretations of the component terms; the option value is determined by working backward through the binomial tree to arrive at the correct current value.
- For American-style options, early exercise influences the option values and hedge ratios as one works backward through the binomial tree.
- Interest rate option valuation requires the specification of an entire term structure of interest rates, so valuation is often estimated via a binomial tree.
- A key assumption of the Black–Scholes–Merton option valuation model is that the return of the underlying instrument follows geometric Brownian motion, implying a lognormal distribution of the return.
- The BSM model can be interpreted as a dynamically managed portfolio of the underlying instrument and zero-coupon bonds.

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- BSM model interpretations related to $N(d_1)$ are that it is the basis for the number of units of underlying instrument to replicate an option, that it is the primary determinant of delta, and that it answers the question of how much the option value will change for a small change in the underlying.
- BSM model interpretations related to $N(d_2)$ are that it is the basis for the number of zero-coupon bonds to acquire to replicate an option and that it is the basis for estimating the risk-neutral probability of an option expiring in the money.
- The Black futures option model assumes the underlying is a futures or a forward contract.
- Interest rate options can be valued based on a modified Black futures option model in which the underlying is a forward rate agreement (FRA), there is an accrual period adjustment as well as an underlying notional amount, and that care must be given to day-count conventions.
- An interest rate cap is a portfolio of interest rate call options termed caplets, each with the same exercise rate and with sequential maturities.
- An interest rate floor is a portfolio of interest rate put options termed floorlets, each with the same exercise rate and with sequential maturities.
- A swaption is an option on a swap.
- A payer swaption is an option on a swap to pay fixed and receive floating.
- A receiver swaption is an option on a swap to receive fixed and pay floating.
- Long a callable fixed-rate bond can be viewed as long a straight fixed-rate bond and short a receiver swaption.
- Delta is a static risk measure defined as the change in a given portfolio for a given small change in the value of the underlying instrument, holding everything else constant.
- Delta hedging refers to managing the portfolio delta by entering additional positions into the portfolio.
- A delta neutral portfolio is one in which the portfolio delta is set and maintained at zero.
- A change in the option price can be estimated with a delta approximation.
- Because delta is used to make a linear approximation of the non-linear relationship that exists between the option price and the underlying price, there is an error that can be estimated by gamma.
- Gamma is a static risk measure defined as the change in a given portfolio delta for a given small change in the value of the underlying instrument, holding everything else constant.
- Gamma captures the non-linearity risk or the risk—via exposure to the underlying—that remains once the portfolio is delta neutral.
- A gamma neutral portfolio is one in which the portfolio gamma is maintained at zero.
- The change in the option price can be better estimated by a delta-plus-gamma approximation compared with just a delta approximation.
- Theta is a static risk measure defined as the change in the value of an option given a small change in calendar time, holding everything else constant.
- Vega is a static risk measure defined as the change in a given portfolio for a given small change in volatility, holding everything else constant.

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- Rho is a static risk measure defined as the change in a given portfolio for a given small change in the risk-free interest rate, holding everything else constant.
- Although historical volatility can be estimated, there is no objective measure of future volatility.
- Implied volatility is the BSM model volatility that yields the market option price.
- Implied volatility is a measure of future volatility, whereas historical volatility is a measure of past volatility.
- Option prices reflect the beliefs of option market participant about the future volatility of the underlying.
- The volatility smile is a two dimensional plot of the implied volatility with respect to the exercise price.
- The volatility surface is a three dimensional plot of the implied volatility with respect to both expiration time and exercise prices.
- If the BSM model assumptions were true, then one would expect to find the volatility surface flat, but in practice, the volatility surface is not flat.

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PRACTICE PROBLEMS

The following information relates to Questions 1–9

Bruno Sousa has been hired recently to work with senior analyst Camila Rocha. Rocha gives him three option valuation tasks.

Alpha Company

Sousa's first task is to illustrate how to value a call option on Alpha Company with a one-period binomial option pricing model. It is a non-dividend-paying stock, and the inputs are as follows.

- The current stock price is 50, and the call option exercise price is 50.
- In one period, the stock price will either rise to 56 or decline to 46.
- The risk-free rate of return is 5% per period.

Based on the model, Rocha asks Sousa to estimate the hedge ratio, the risk-neutral probability of an up move, and the price of the call option. In the illustration, Sousa is also asked to describe related arbitrage positions to use if the call option is overpriced relative to the model.

Beta Company

Next, Sousa uses the two-period binomial model to estimate the value of a European-style call option on Beta Company's common shares. The inputs are as follows.

- The current stock price is 38, and the call option exercise price is 40.
- The up factor (u) is 1.300, and the down factor (d) is 0.800.
- The risk-free rate of return is 3% per period.

Sousa then analyzes a put option on the same stock. All of the inputs, including the exercise price, are the same as for the call option. He estimates that the value of a European-style put option is 4.53. Exhibit 1 summarizes his analysis. Sousa next must determine whether an American-style put option would have the same value.

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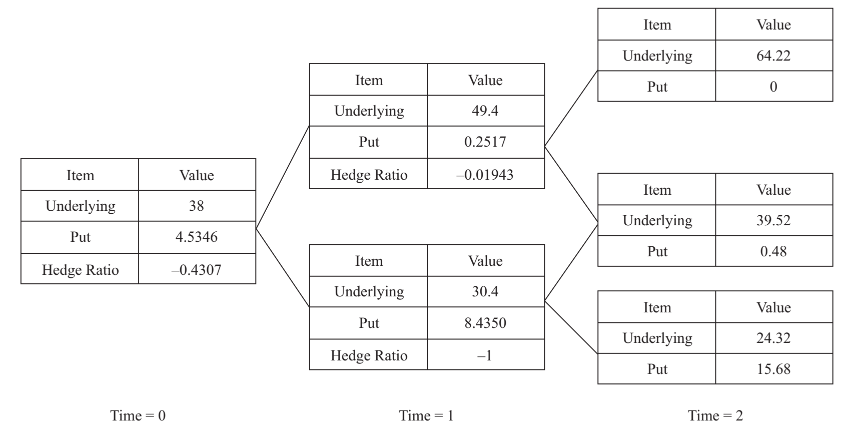
Exhibit 1 Two-Period Binomial European-Style Put Option on Beta Company



Sousa makes two statements with regard to the valuation of a European-style option under the expectations approach.

Statement 1 The calculation involves discounting at the risk-free rate.

Statement 2 The calculation uses risk-neutral probabilities instead of true probabilities.

Rocha asks Sousa whether it is ever profitable to exercise American options prior to maturity. Sousa answers, "I can think of two possible cases. The first case is the early exercise of an American call option on a dividend-paying stock. The second case is the early exercise of an American put option."

Interest Rate Option

The final option valuation task involves an interest rate option. Sousa must value a two-year, European-style call option on a one-year spot rate. The notional value of the option is 1 million, and the exercise rate is 2.75%. The risk-neutral probability of an up move is 0.50. The current and expected one-year interest rates are shown in Exhibit 2, along with the values of a one-year zero-coupon bond of 1 notional value for each interest rate.

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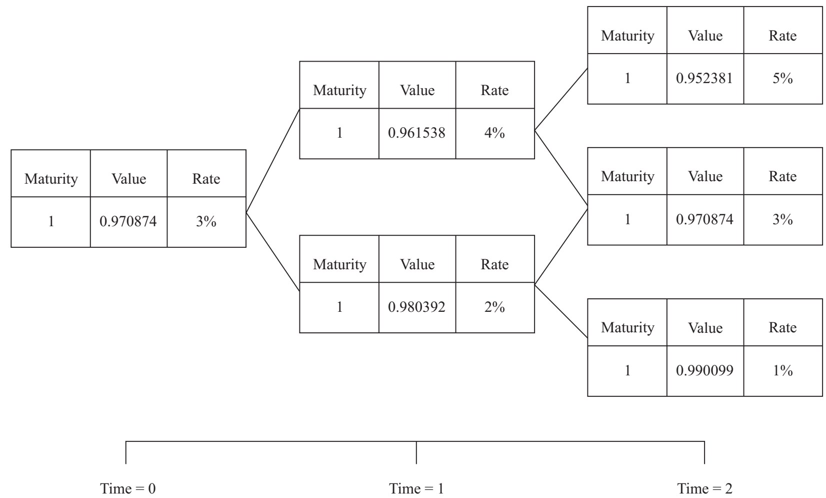
Exhibit 2 Two-Year Interest Rate Lattice for an Interest Rate Option



Rocha asks Sousa why the value of a similar in-the-money interest rate call option decreases if the exercise price is higher. Sousa provides two reasons.

Reason 1 The exercise value of the call option is lower.

Reason 2 The risk-neutral probabilities are changed.

1 The optimal hedge ratio for the Alpha Company call option using the one-period binomial model is *closest* to:

A 0.60.

B 0.67.

C 1.67.

2 The risk-neutral probability of the up move for the Alpha Company stock is *closest* to:

A 0.06.

B 0.40.

C 0.65.

3 The value of the Alpha Company call option is *closest* to:

A 3.71.

B 5.71.

C 6.19.

4 For the Alpha Company option, the positions to take advantage of the arbitrage opportunity are to write the call and:

A short shares of Alpha stock and lend.

B buy shares of Alpha stock and borrow.

C short shares of Alpha stock and borrow.

5 The value of the European-style call option on Beta Company shares is *closest* to:

A 4.83.

B 5.12.

C 7.61.

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6 The value of the American-style put option on Beta Company shares is *closest* to:

A 4.53.

B 5.15.

C 9.32.

7 Which of Sousa's statements about binomial models is correct?

A Statement 1 only

B Statement 2 only

C Both Statement 1 and Statement 2

8 Based on Exhibit 2 and the parameters used by Sousa, the value of the interest rate option is *closest* to:

A 5,251.

B 6,236.

C 6,429.

9 Which of Sousa's reasons for the decrease in the value of the interest rate option is correct?

A Reason 1 only

B Reason 2 only

C Both Reason 1 and Reason 2

The following information relates to Questions 10–17

Trident Advisory Group manages assets for high-net-worth individuals and family trusts.

Alice Lee, chief investment officer, is meeting with a client, Noah Solomon, to discuss risk management strategies for his portfolio. Solomon is concerned about recent volatility and has asked Lee to explain options valuation and the use of options in risk management.

Options on Stock

Lee uses the BSM model to price TCB, which is one of Solomon's holdings. Exhibit 1 provides the current stock price (S), exercise price (X), risk-free interest rate (r), volatility (σ), and time to expiration (T) in years as well as selected outputs from the BSM model. TCB does not pay a dividend.

Exhibit 1 BSM Model for European Options on TCB

BSM Inputs

S	*X*	*r*	Σ	*T*
$57.03	55	0.22%	32%	0.25

(continued)

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Exhibit 1 (Continued)

BSM Outputs

d_1	$N(d_1)$	d_2	$N(d_2)$	BSM Call Price	BSM Put Price
0.3100	0.6217	0.1500	0.5596	\$4.695	\$2.634

Options on Futures

The Black model valuation and selected outputs for options on another of Solomon's holdings, the GPX 500 Index (GPX), are shown in Exhibit 2. The spot index level for the GPX is 187.95, and the index is assumed to pay a continuous dividend at a rate of 2.2% (δ) over the life of the options being valued, which expire in 0.36 years. A futures contract on the GPX also expiring in 0.36 years is currently priced at 186.73.

Exhibit 2 Black Model for European Options on the GPX Index

Black Model Inputs

GPX Index	X	r	σ	T	δ Yield
187.95	180	0.39%	24%	0.36	2.2%

Black Model Call Value	Black Model Put Value	Market Call Price	Market Put Price
\$14.2089	\$7.4890	\$14.26	\$7.20

Option Greeks

Delta (call)	Delta (put)	Gamma (call or put)	Theta (call) daily	Rho (call) per %	Vega per % (call or put)
0.6232	–0.3689	0.0139	–0.0327	0.3705	0.4231

After reviewing Exhibit 2, Solomon asks Lee which option Greek letter best describes the changes in an option's value as time to expiration declines.

Solomon observes that the market price of the put option in Exhibit 2 is \$7.20. Lee responds that she used the historical volatility of the GPX of 24% as an input to the BSM model, and she explains the implications for the implied volatility for the GPX.

Options on Interest Rates

Solomon forecasts the three-month Libor will exceed 0.85% in six months and is considering using options to reduce the risk of rising rates. He asks Lee to value an interest rate call with a strike price of 0.85%. The current three-month Libor is 0.60%, and an FRA for a three-month Libor loan beginning in six months is currently 0.75%.

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Hedging Strategy for the Equity Index

Solomon's portfolio currently holds 10,000 shares of an exchange-traded fund (ETF) that tracks the GPX. He is worried the index will decline. He remarks to Lee, "You have told me how the BSM model can provide useful information for reducing the risk of my GPX position." Lee suggests a delta hedge as a strategy to protect against small moves in the GPX Index.

Lee also indicates that a long position in puts could be used to hedge larger moves in the GPX. She notes that although hedging with either puts or calls can result in a delta-neutral position, they would need to consider the resulting gamma.

10 Based on Exhibit 1 and the BSM valuation approach, the initial portfolio required to replicate the long call option payoff is:

- **A** long 0.3100 shares of TCB stock and short 0.5596 shares of a zero-coupon bond.
- **B** long 0.6217 shares of TCB stock and short 0.1500 shares of a zero-coupon bond.
- **C** long 0.6217 shares of TCB stock and short 0.5596 shares of a zero-coupon bond.

11 To determine the long put option value on TCB stock in Exhibit 1, the correct BSM valuation approach is to compute:

- **A** 0.4404 times the present value of the exercise price minus 0.6217 times the price of TCB stock.
- **B** 0.4404 times the present value of the exercise price minus 0.3783 times the price of TCB stock.
- **C** 0.5596 times the present value of the exercise price minus 0.6217 times the price of TCB stock.

12 What are the correct spot value (S) and the risk-free rate (r) that Lee should use as inputs for the Black model?

- **A** 186.73 and 0.39%, respectively
- **B** 186.73 and 2.20%, respectively
- **C** 187.95 and 2.20%, respectively

13 Which of the following is the correct answer to Solomon's question regarding the option Greek letter?

- **A** Vega
- **B** Theta
- **C** Gamma

14 Based on Solomon's observation about the model price and market price for the put option in Exhibit 2, the implied volatility for the GPX is *most likely*:

- **A** less than the historical volatility.
- **B** equal to the historical volatility.
- **C** greater than the historical volatility.

15 The valuation inputs used by Lee to price a call reflecting Solomon's interest rate views should include an underlying FRA rate of:

- **A** 0.60% with six months to expiration.
- **B** 0.75% with nine months to expiration.
- **C** 0.75% with six months to expiration.

16 The strategy suggested by Lee for hedging small moves in Solomon's ETF position would *most likely* involve:

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A selling put options.
B selling call options.
C buying call options.

17 Lee's put-based hedge strategy for Solomon's ETF position would *most likely* result in a portfolio gamma that is:

A negative.
B neutral.
C positive.

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SOLUTIONS

1 A is correct. The hedge ratio requires the underlying stock and call option values for the up move and down move. $S^+ = 56$, and $S^- = 46$. $c^+ = \text{Max}(0,S^+ - X) = \text{Max}(0,56 - 50) = 6$, and $c^- = \text{Max}(0,S^- - X) = \text{Max}(0,46 - 50) = 0$. The hedge ratio is

$$h = \frac{c^+ - c^-}{S^+ - S^-} = \frac{6 - 0}{56 - 46} = \frac{6}{10} = 0.60$$

2 C is correct. For this approach, the risk-free rate is $r = 0.05$, the up factor is $u = S^+/S = 56/50 = 1.12$, and the down factor is $d = S^-/S = 46/50 = 0.92$. The risk-neutral probability of an up move is

$$\pi = [\text{FV}(1) - d]/(u - d) = (1 + r - d]/(u - d)$$
$$\pi = (1 + 0.05 - 0.92)/(1.12 - 0.92) = 0.13/0.20 = 0.65$$

3 A is correct. The call option can be estimated using the no-arbitrage approach or the expectations approach. With the no-arbitrage approach, the value of the call option is

$$c = hS + \text{PV}(-hS^- + c^-).$$
$$h = (c^+ - c^-)/(S^+ - S^-) = (6 - 0)/(56 - 46) = 0.60.$$
$$c = (0.60 \times 50) + (1/1.05) \times [(-0.60 \times 46) + 0].$$
$$c = 30 - [(1/1.05) \times 27.6] = 30 - 26.286 = 3.714.$$

Using the expectations approach, the risk-free rate is $r = 0.05$, the up factor is $u = S^+/S = 56/50 = 1.12$, and the down factor is $d = S^-/S = 46/50 = 0.92$. The value of the call option is

$$c = \text{PV} \times [\pi c^+ + (1 - \pi)c^-].$$
$$\pi = [\text{FV}(1) - d]/(u - d) = (1.05 - 0.92)/(1.12 - 0.92) = 0.65.$$
$$c = (1/1.05) \times [0.65(6) + (1 - 0.65)(0)] = (1/1.05)(3.9) = 3.714.$$

Both approaches are logically consistent and yield identical values.

4 B is correct. You should sell (write) the overpriced call option and then go long (buy) the replicating portfolio for a call option. The replicating portfolio for a call option is to buy h shares of the stock and borrow the present value of $(hS^- - c^-)$.

$$c = hS + \text{PV}(-hS^- + c^-).$$
$$h = (c^+ - c^-)/(S^+ - S^-) = (6 - 0)/(56 - 46) = 0.60.$$

For the example in this case, the value of the call option is 3.714. If the option is overpriced at, say, 4.50, you short the option and have a cash flow at Time 0 of +4.50. You buy the replicating portfolio of 0.60 shares at 50 per share (giving you a cash flow of –30) and borrow (1/1.05) × [(0.60 × 46) – 0] = (1/1.05) × 27.6 = 26.287. Your cash flow for buying the replicating portfolio is –30 + 26.287 = –3.713. Your net cash flow at Time 0 is + 4.50 – 3.713 = 0.787. Your net cash flow at Time 1 for either the up move or down move is zero. You have made an arbitrage profit of 0.787.

In tabular form, the cash flows are as follows:

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Transaction	Time Step 0	Time Step 1 Down Occurs	Time Step 1 Up Occurs
Sell the call option	4.50	0	−6.00
Buy h shares	−0.6 × 50 = −30	0.6 × 46 = 27.6	0.6 × 56 = 33.6
Borrow $-\text{PV}(-hS^- + c^-)$	−(1/1.05) × [(−0.6 × 46) + 0] = 26.287	−0.6 × 46 = −27.6	−0.6 × 46 = −27.6
Net cash flow	0.787	0	0

5 A is correct. Using the expectations approach, the risk-neutral probability of an up move is

$$\pi = [\text{FV}(1) - d]/(u - d) = (1.03 - 0.800)/(1.300 - 0.800) = 0.46.$$

The terminal value calculations for the exercise values at Time Step 2 are

$$c^{++} = \text{Max}(0,u^2S - X) = \text{Max}[0,1.30^2(38) - 40] = \text{Max}(0,24.22) = 24.22.$$
$$c^{-+} = \text{Max}(0,udS - X) = \text{Max}[0,1.30(0.80)(38) - 40] = \text{Max}(0,-0.48) = 0.$$
$$c^{--} = \text{Max}(0,d^2S - X) = \text{Max}[0,0.80^2(38) - 40] = \text{Max}(0,-15.68) = 0.$$

Discounting back for two years, the value of the call option at Time Step 0 is

$$c = \text{PV}[\pi^2c^{++} + 2\pi(1 - \pi)c^{-+} + (1 - \pi)^2c^{--}].$$
$$c = [1/(1.03)]^2[0.46^2(24.22) + 2(0.46)(0.54)(0) + 0.54^2(0)].$$
$$c = [1/(1.03)]^2[5.1250] = 4.8308.$$

6 B is correct. Using the expectations approach, the risk-neutral probability of an up move is

$$\pi = [\text{FV}(1) - d]/(u - d) = (1.03 - 0.800)/(1.300 - 0.800) = 0.46.$$

An American-style put can be exercised early. At Time Step 1, for the up move, p^+ is 0.2517 and the put is out of the money and should not be exercised early ($X < S$, 40 < 49.4). However, at Time Step 1, p^- is 8.4350 and the put is in the money by 9.60 ($X - S$ = 40 − 30.40). So, the put is exercised early, and the value of early exercise (9.60) replaces the value of not exercising early (8.4350) in the binomial tree. The value of the put at Time Step 0 is now

$$p = \text{PV}[\pi p^+ + (1 - \pi)p^-] = [1/(1.03)][0.46(0.2517) + 0.54(9.60)] = 5.1454.$$

Following is a supplementary note regarding Exhibit 1.

The values in Exhibit 1 are calculated as follows.

At Time Step 2:

$$p^{++} = \text{Max}(0,X - u^2S) = \text{Max}[0,40 - 1.300^2(38)] = \text{Max}(0,40 - 64.22) = 0.$$
$$p^{-+} = \text{Max}(0,X - udS) = \text{Max}[0,40 - 1.300(0.800)(38)] = \text{Max}(0,40 - 39.52) = 0.48.$$
$$p^{--} = \text{Max}(0,X - d^2S) = \text{Max}[0,40 - 0.800^2(38)] = \text{Max}(0,40 - 24.32) = 15.68.$$

At Time Step 1:

$$p^+ = \text{PV}[\pi p^{++} + (1 - \pi)p^{-+}] = [1/(1.03)][0.46(0) + 0.54(0.48)] = 0.2517.$$
$$p^- = \text{PV}[\pi p^{-+} + (1 - \pi)p^{--}] = [1/(1.03)][0.46(0.48) + 0.54(15.68)] = 8.4350.$$

At Time Step 0:

$$p = \text{PV}[\pi p^+ + (1 - \pi)p^-] = [1/(1.03)][0.46(0.2517) + 0.54(8.4350)] = 4.5346.$$

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7 C is correct. Both statements are correct. The expected future payoff is calculated using risk-neutral probabilities, and the expected payoff is discounted at the risk-free rate.

8 C is correct. Using the expectations approach, per 1 of notional value, the values of the call option at Time Step 2 are

c^{++} = Max(0,S^{++} – X) = Max(0,0.050 – 0.0275) = 0.0225.
c^{+-} = Max(0,S^{+-} – X) = Max(0,0.030 – 0.0275) = 0.0025.
c^{--} = Max(0,S^{--} – X) = Max(0,0.010 – 0.0275) = 0.

At Time Step 1, the call values are

c^{+} = PV[πc^{++} + $(1 - \pi)c^{+-}$] .
c^{+} = 0.961538[0.50(0.0225) + (1 – 0.50)(0.0025)] = 0.012019.
c^{-} = PV[πc^{+-} + $(1 - \pi)c^{--}$].
c^{-} = 0.980392[0.50(0.0025) + (1 – 0.50)(0)] = 0.001225.

At Time Step 0, the call option value is

c = PV[πc^{+} + $(1 - \pi)c^{-}$].
c = 0.970874[0.50(0.012019) + (1 – 0.50)(0.001225)] = 0.006429.

The value of the call option is this amount multiplied by the notional value, or 0.006429 × 1,000,000 = 6,429.

9 A is correct. Reason 1 is correct: A higher exercise price does lower the exercise value (payoff) at Time 2. Reason 2 is not correct because the risk-neutral probabilities are based on the paths that interest rates take, which are determined by the market and not the details of a particular option contract.

10 C is correct. The no-arbitrage approach to creating a call option involves buying Delta = $N(d_1)$ = 0.6217 shares of the underlying stock and financing with $-N(d_2)$ = –0.5596 shares of a risk-free bond priced at exp($-rt$)(X) = exp(–0.0022 × 0.25)(55) = $54.97 per bond. Note that the value of this replicating portfolio is $n_S S + n_B B$ = 0.6217(57.03) – 0.5596(54.97) = $4.6943 (the value of the call option with slight rounding error).

11 B is correct. The formula for the BSM price of a put option is $p = e^{-rt}XN(-d_2) - SN(-d_1)$. $N(-d_1) = 1 - N(d_1)$ = 1 – 0.6217 = 0.3783, and $N(-d_2) = 1 - N(d_2)$ = 1 – 0.5596 = 0.4404.

Note that the BSM model can be represented as a portfolio of the stock ($n_S S$) and zero-coupon bonds ($n_B B$). For a put, the number of shares is $n_S = -N(-d_1)$ < 0 and the number of bonds is $n_B = -N(d_2) > 0$. The value of the replicating portfolio is $n_S S + n_B B$ = –0.3783(57.03) + 0.4404(54.97) = $2.6343 (the value of the put option with slight rounding error). B is a risk-free bond priced at exp(–rt)(X) = exp(–0.0022 × 0.25)(55) = $54.97.

12 A is correct. Black's model to value a call option on a futures contract is $c = e^{-rT}[F_0(T)N(d_1) - XN(d_2)]$. The underlying F_0 is the futures price (186.73). The correct discount rate is the risk-free rate, r = 0.39%.

13 B is correct. Lee is pointing out the option price's sensitivity to small changes in time. In the BSM approach, option price sensitivity to changes in time is given by the option Greek theta.

14 A is correct. The put is priced at $7.4890 by the BSM model when using the historical volatility input of 24%. The market price is $7.20. The BSM model overpricing suggests the implied volatility of the put must be lower than 24%.

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15 C is correct. Solomon's forecast is for the three-month Libor to exceed 0.85% in six months. The correct option valuation inputs use the six-month FRA rate as the underlying, which currently has a rate of 0.75%.

16 B is correct because selling call options creates a short position in the ETF that would hedge his current long position in the ETF.

Exhibit 2 could also be used to answer the question. Solomon owns 10,000 shares of the GPX, each with a delta of +1; by definition, his portfolio delta is +10,000. A delta hedge could be implemented by selling enough calls to make the portfolio delta neutral:

$$N_H = -\frac{\text{Portfolio delta}}{\text{Delta}_H} = -\frac{+10{,}000}{+0.6232} = -16{,}046 \text{ calls.}$$

17 C is correct. Because the gamma of the stock position is 0 and the put gamma is always non-negative, adding a long position in put options would most likely result in a positive portfolio gamma.

Gamma is the change in delta from a small change in the stock's value. A stock position always has a delta of +1. Because the delta does not change, gamma equals 0.

The gamma of a call equals the gamma of a similar put, which can be proven using put–call parity.