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Hyponormal measurable operators, affiliated to a semifinite von Neumann algebra

Airat Bikchentaev[1]

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Abstract

Let $\mathcal{M}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and τ be a faithful normal semifinite trace on $\mathcal{M}$, $S(\mathcal{M}, \tau)$ be the *-algebra of all τ-measurable operators. Assume that an operator $T \in S(\mathcal{M}, \tau)$ is paranormal or *-paranormal. If T^n is τ-compact for some $n \in \mathbb{N}$ then T is τ-compact; if $T^n = 0$ for some $n \in \mathbb{N}$ then $T = 0$; if $T^3 = T$ then $T = T^*$; if $T^2 \in L_1(\mathcal{M}, \tau)$ then $T \in L_2(\mathcal{M}, \tau)$ and $\|T\|_2^2 = \|T^2\|_1$. If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and $T^{*p}T^q$ is τ-compact for some $p, q \in \mathbb{N} \cup \{0\}$, $p+q \geq 1$ then T is normal. If $T \in S(\mathcal{M}, \tau)$ is p-hyponormal for some $0 < p \leq 1$ then the operator $(T^*T)^p - (TT^*)^p$ cannot have the inverse in $\mathcal{M}$. If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal (or cohyponormal) and the operator T^2 is Hermitian then T is normal.

Keywords Hilbert space · Von Neumann algebra · Normal trace · Measurable operator · Hyponormal operator · Paranormal operator

Mathematics Subject Classification 46L51 · 47B20

1 Introduction

Let a von Neumann algebra $\mathcal{M}$ of operators act on a Hilbert space $\mathcal{H}$, let I be the unit of $\mathcal{M}$, let τ be a faithful normal semifinite trace on $\mathcal{M}$. Let $S(\mathcal{M}, \tau)$ be the *-algebra of all τ-measurable operators, $|A| = \sqrt{A^*A}$ for $A \in S(\mathcal{M}, \tau)$. Hyponormal (i.e. $A^*A \geq AA^*$) bounded operators have many good properties and have long attracted attention of large group of investigators, see, for example, [1, 19, 21–23, 25, 27, 31, 33, 36, 37, 44, 46]. In 2014 the author started the research of unbounded hyponormal operators from $S(\mathcal{M}, \tau)$, see [6]. An operator $A \in S(\mathcal{M}, \tau)$ is called *p-hyponormal* for some number $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$; *quasinormal*, if it commutes with

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✉ Airat Bikchentaev
Airat.Bikchentaev@kpfu.ru

1 Kazan Federal University, 18 Kremlyovskaya str, P.O. Box 420008, Kazan, Russia

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A^*A. Every quasinormal operator $A \in S(\mathcal{M}, \tau)$ is hyponormal [7, Theorem 2.9]. Every p-hyponormal operator $A \in S(\mathcal{M}, \tau)$ is paranormal [11, Theorem 4.4], i.e.,

$$2|A|^2 \le \lambda^{-1}|A^2|^2 + \lambda I \quad \text{for all} \quad \lambda > 0. \tag{1.1}$$

If a paranormal operator $A \in S(\mathcal{M}, \tau)$ has the inverse $A^{-1} \in \mathcal{M}$ then A^{-1} is also paranormal [8, item (iii) of Theorem 2], see also [9]. If an operator $A \in S(\mathcal{M}, \tau)$ is hyponormal and $(\lambda I + A)^{-1} \in \mathcal{M}$ for some $\lambda \in \mathbb{C}$ then $(\lambda I + A)^{-1}$ is hyponormal [8, Proposition 2]. If a hyponormal operator $A \in S(\mathcal{M}, \tau)$ has the inverse $A^{-1} \in S(\mathcal{M}, \tau)$ then A^{-1} is also hyponormal, see Solution 9.16 in [35]. Every hyponormal operator $A \in S(\mathcal{M}, \tau)$ is *-paranormal [11, item (i) of Theorem 3.6], i.e.

$$2|A^*|^2 \le \lambda^{-1}|A^2|^2 + \lambda I \quad \text{for all} \quad \lambda > 0. \tag{1.2}$$

If a hyponormal operator $A \in S(\mathcal{M}, \tau)$ has a right inverse in $S(\mathcal{M}, \tau)$ then A is invertible in $S(\mathcal{M}, \tau)$ [13, Theorem 3]. Every τ-compact p-hyponormal operator is normal [6, Theorem 2.2]. If an operator $A \in S(\mathcal{M}, \tau)$ is p-hyponormal and $|A^*| \ge \mu(\infty; A)I$ then A is normal [9, Theorem 4.1]. Let $\mathcal{M}$ be a *factor*, i.e., $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. If an operator $A \in \mathcal{M}$ is hyponormal and compact relative $\mathcal{M}$ then A is normal [2, Theorem]; see also [10, Section 4].

We obtain the following results. Let an operator $T \in S(\mathcal{M}, \tau)$ be paranormal or *-paranormal. If T^n is τ-compact for some $n \in \mathbb{N}$ then T is τ-compact; if $T^n = 0$ for some $n \in \mathbb{N}$ then $T = 0$; if $T^3 = T$ then $T = T^*$; if $T^2 \in L_1(\mathcal{M}, \tau)$ then $T \in L_2(\mathcal{M}, \tau)$ and $\|T\|_2^2 = \|T^2\|_1$ (Theorem 3.3). If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and $T^{*p}T^q$ is τ-compact for some $p, q \in \mathbb{N} \cup \{0\}$, $p + q \ge 1$ then T is normal (Theorem 3.7). For $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau = \mathrm{tr}$ and $p, q \in \mathbb{N}$ this assertion was proved by Istrăţescu [32, Theorem 1.2] in different way. If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal for some $0 < p \le 1$ then the operator $(T^*T)^p - (TT^*)^p$ cannot have the inverse in $\mathcal{M}$ (Theorem 3.9). In particular, a positive self-commutator cannot have the inverse in $\mathcal{M}$. For $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau = \mathrm{tr}$ and $p = 1$ Theorem 3.9 was proved by Putnam [38] in different way. For Hermitian operators $A, B \in S(\mathcal{M}, \tau)$ the following conditions are equivalent: (i) an operator $A + iB$ is hyponormal; (ii) an operator $aA + ibB$ is hyponormal for some numbers $a, b > 0$. If an operator A is invertible in $S(\mathcal{M}, \tau)$ then (i) and (ii) are equivalent to the following condition: (iii) an operator $A^{-1} - iB$ is hyponormal (Theorem 3.19). If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal (or cohyponormal) and the operator T^2 is Hermitian then T is normal (Theorem 3.21, Corollary 3.22). Let Hermitian operators $A, B \in S(\mathcal{M}, \tau)$ and $A^2 = I$. If an operator $A + iB$ is hyponormal (or cohyponormal) then it is normal (Theorem 3.23). The results are mostly new even for the pair $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau = \mathrm{tr}$.

2 Notation, definitions and preliminaries

Let $\mathcal{M}$ be a von Neumann algebra of operators on the Hilbert space $\mathcal{H}$, $\mathcal{M}^{\mathrm{pr}}$ be the lattice of projections ($P = P^2 = P^*$) in $\mathcal{M}$, I be the unit of $\mathcal{M}$, $P^\perp = I - P$ for

$P \in \mathcal{M}^{\text{pr}}$. An operator $U \in \mathcal{M}$ is called an *isometry*, if $U^*U = I$. Let $\mathcal{M}^{\text{sym}} = \{S \in \mathcal{M} : S^2 = I\}$.

An operator on $\mathcal{H}$ (not necessarily bounded or densely defined) is said to be *affiliated to the von Neumann algebra* $\mathcal{M}$ if it commutes with any unitary operator from the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. Let τ be a faithful normal semifinite trace on $\mathcal{M}$ [24, Chap. 1, §1.15]. A closed operator X, affiliated to $\mathcal{M}$ and possessing a domain $\mathfrak{D}(X)$ everywhere dense in $\mathcal{H}$ is said to be τ-*measurable* if, for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{M}^{\text{pr}}$ such that $P\mathcal{H} \subset \mathfrak{D}(X)$ and $\tau(P^{\perp}) < \varepsilon$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a *-algebra under passage to the adjoint operator, multiplication by a scalar, and operations of strong addition and multiplication resulting from the closure of the ordinary operations [24, Chap. 2, §2.3].

Let $\mathcal{L}^+$, $\mathcal{L}^{\text{h}}$ and $\mathcal{L}^{\text{id}}$ denote the positive part, the Hermitian part and the idempotent part ($A^2 = A$) of a family $\mathcal{L} \subset S(\mathcal{M}, \tau)$, respectively. We denote by $\leq$ the partial order in $S(\mathcal{M}, \tau)^{\text{h}}$ generated by its proper cone $S(\mathcal{M}, \tau)^+$. If $X \in S(\mathcal{M}, \tau)$ and $X = U|X|$ is the polar decomposition of X, then $U \in \mathcal{M}$ and $|X| = \sqrt{X^*X} \in S(\mathcal{M}, \tau)^+$.

Let $[A, B] = AB - BA$ be the commutator of operators $A, B \in S(\mathcal{M}, \tau)$, $\sigma(A)$ and $[A^*, A]$ be the spectrum and the self-commutator of an operator A, respectively. An operator $A \in S(\mathcal{M}, \tau)$ is called p-*hyponormal* for some number $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$; p-*cohyponormal*, if A^* is p-hyponormal. The sets

$$U(\varepsilon, \delta) = \{X \in S(\mathcal{M}, \tau) : \ \|XP\| \leq \varepsilon \ \text{ and } \ \tau(P^{\perp}) \leq \delta \ \text{ for some } \ P \in \mathcal{M}^{\text{pr}}\},$$

where $\varepsilon > 0, \ \delta > 0$, form a base at 0 for a metrizable vector topology t_τ on $S(\mathcal{M}, \tau)$, called *the measure topology* [24, Chap. 2, §2.5]. Equipped with this topology, $S(\mathcal{M}, \tau)$ is a complete topological *-algebra in which $\mathcal{M}$ is dense. We will write $X_n \xrightarrow{\tau} X$ if a sequence of τ-measurable operators $\{X_n\}_{n=1}^{\infty}$ converges to $X \in S(\mathcal{M}, \tau)$ in the measure topology on $S(\mathcal{M}, \tau)$. The generalized singular value function $\mu(\cdot; X) : t \to \mu(t; X)$ of the τ-measurable operator X is defined by setting

$$\mu(t; X) = \inf\{\|XP\| : \ \ P \in \mathcal{M}^{\text{pr}} \ \text{ and } \ \tau(P^{\perp}) \leq t\}, \quad t > 0.$$

It is a non-increasing right-continuous function.

Lemma 2.1 [26] *Let* $X, Y \in S(\mathcal{M}, \tau)$. *Then,*

(i) $\mu(t; X) = \mu(t; |X|) = \mu(t; X^*)$ *for all* $t > 0$*;*
(ii) $\mu(t; \lambda X) = |\lambda|\mu(t; X)$ *for all* $\lambda \in \mathbb{C}$ *and* $t > 0$*;*
(iii) *if* $|X| \leq |Y|$, *then* $\mu(t; X) \leq \mu(t; Y)$ *for all* $t > 0$*;*
(iv) $\mu(s + t; X + Y) \leq \mu(s; X) + \mu(t; Y)$ *for all* $s, t > 0$*;*
(v) $\mu(t; |X|^p) = \mu(t; X)^p$ *for all* $0 < p < +\infty$ *and* $t > 0$.

Let m be the linear Lebesgue measure on $\mathbb{R}$. Noncommutative Lebesgue L_p-space ($0 < p < \infty$), associated with $(\mathcal{M}, \tau)$, may be defined as

$$L_p(\mathcal{M}, \tau) = \{X \in S(\mathcal{M}, \tau) : \ \mu(\cdot; X) \in L_p(\mathbb{R}^+, m)\}$$

with the F-norm (norm for $1 \leq p < \infty$) $\|X\|_p = \|\mu(\cdot; X)||_p$, $X \in L_p(\mathcal{M}, \tau)$. The extension of τ to the unique linear functional on the whole space $L_1(\mathcal{M}, \tau)$ we denote by the same letter τ. The set

$$S_0(\mathcal{M},\tau)=\{X\in S(\mathcal{M},\tau):\mu(\infty;X):=\lim_{t\to+\infty}\mu(t;X)=0\}$$

of τ-compact operators is a t_τ-closed ideal in $S(\mathcal{M},\tau)$.

If $\mathcal{M}=\mathcal{B}(\mathcal{H})$, the *-algebra of all bounded linear operators on $\mathcal{H}$, and $\tau=\mathrm{tr}$, then $S(\mathcal{M},\tau)$ coincides with $\mathcal{B}(\mathcal{H})$, $S_0(\mathcal{M},\tau)$ coincides with the ideal $S_\infty(\mathcal{H})$ of compact (i.e., completely continuous) operators on $\mathcal{H}$, the topology t_τ coincides with the $\|\cdot\|$-topology, the space $L_p(\mathcal{M},\tau)$ coincides with the Shatten–von Neumann *-ideal $S_p(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ and

$$\mu(t;X)=\sum_{n=1}^{\infty}s_n(X)\chi_{[n-1,n)}(t),\quad t>0,$$

where $\{s_n(X)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator X; χ_A is the indicator function of the set $A\subset\mathbb{R}$ [28, Chap. II].

3 When a hyponormal τ-measurable operator is normal?

Let τ be a faithful normal semifinite trace on a von Neumann algebra $\mathcal{M}$.

Lemma 3.1 (cf. [34, Lemma]) *If numbers $p,q,r>0$ with $1/p+1/q=1/r$ and $A\in L_p(\mathcal{M},\tau)$, $B\in L_q(\mathcal{M},\tau)$, $X\in\mathcal{M}$ then $AXB\in L_r(\mathcal{M},\tau)$ and $\|AXB\|_r\le\|X\|\|A\|_p\|B\|_q$.*

Corollary 3.2 *If $p>0$ and $A\in L_p(\mathcal{M},\tau)$ then $A^n\in L_{p/n}(\mathcal{M},\tau)$ and $\|A^n\|_{p/n}\le\|A\|_p^n$ for all $n\in\mathbb{N}$.*

In particular, if $A=A^n\in L_p(\mathcal{M},\tau)$ then $A\in L_{p/n}(\mathcal{M},\tau)$; for $B=A^{n-1}$ we have $B^2=A^nA^{n-2}=AA^{n-2}=B$ for $n\ge3$. If $A=A^2\in S(\mathcal{M},\tau)$ then $\mu(t;A)\in\{0\}\cup[1,+\infty)$ for all $t>0$ [16, Theorem 3]. Therefore, if $A=A^2\in L_p(\mathcal{M},\tau)$ and $0<q<p$ then $A\in L_q(\mathcal{M},\tau)$ and $\|A\|_q^q\le\|A\|_p^p$.

Theorem 3.3 *Let an operator $T\in S(\mathcal{M},\tau)$ be paranormal or *-paranormal.*

(i) *If $T^n\in S_0(\mathcal{M},\tau)$ for some $n\in\mathbb{N}$ then $T\in S_0(\mathcal{M},\tau)$;*
(ii) *if $T^n=0$ for some $n\in\mathbb{N}$ then $T=0$;*
(iii) *if $T^3=T$ then $T=T^*$;*
(iv) *if $T^2\in L_1(\mathcal{M},\tau)$ then $T\in L_2(\mathcal{M},\tau)$ and $\|T\|_2^2=\|T^2\|_1$.*

Proof (i). Let $T\in S(\mathcal{M},\tau)$ be paranormal and $T^n\in S_0(\mathcal{M},\tau)$, a number $\varepsilon>0$ be arbitrary. *Step 1*. If $n=2$ then from (1.1) by items (i), (ii), (iv) and (v) of Lemma 2.1 we have

$$\begin{aligned}2\mu(2t;T)^2&=2\mu(2t;T^*T)\le\lambda^{-1}\mu(t;T^{2*}T^2)+\lambda\mu(t;I)=\lambda^{-1}\mu(2t;T^2)^2+\lambda\\&<\varepsilon^{-1}\varepsilon^2+\varepsilon=2\varepsilon\end{aligned}$$

for all $t > t_0$ and numbers $\lambda = \varepsilon$ and $t_0 > 0$ such that $\mu(t; T^2) < \varepsilon$ for $t > t_0$. Therefore, $T \in S_0(\mathcal{M}, \tau)$.

Step 2. For $n \geq 3$ we show that $T^{n-1} \in S_0(\mathcal{M}, \tau)$. If $T^{n-2} \in S_0(\mathcal{M}, \tau)$ then $T^{n-1} = T \cdot T^{n-2} \in S_0(\mathcal{M}, \tau)$. Assume that $T^{n-2} \notin S_0(\mathcal{M}, \tau)$. Then $a := \mu(\infty; T^{n-2}) > 0$. Multiply both sides of inequality (1.1) from the left by the operator $(T^*)^{n-2}$ and from the right by the operator T^{n-2} and obtain

$$2|T^{n-1}|^2 \leq \lambda^{-1} T^{*n} T^n + \lambda\, (T^*)^{n-2} T^{n-2} \quad \text{for all} \quad \lambda > 0. \tag{3.1}$$

Let a number $t_1 > 0$ with $\mu(t; T^n)^2 < \frac{\varepsilon^2}{8a^2}$ for $t > t_1$. Put $\lambda := \frac{\varepsilon}{4a^2}$ and choose a number $t_2 > 0$ such that $\mu(t; T^{n-2}) < 2a$ for $t > t_2$. Then from (3.1) and items (i), (ii), (iv) and (v) of Lemma 2.1 we have for all $t > \max\{t_1, t_2\}$ the estimate

$$\begin{aligned} 2\mu(2t; T^{n-1})^2 &= 2\mu(2t; (T^*)^{n-1} T^{n-1}) \leq \lambda^{-1} \mu(t; T^n)^2 + \lambda \mu(t; T^{n-2})^2 \\ &< \frac{4a^2}{\varepsilon} \cdot \frac{\varepsilon^2}{8a^2} + \frac{\varepsilon}{4a^2} \cdot 4a^2 = \frac{3}{2}\varepsilon. \end{aligned}$$

Thus, $T \in S_0(\mathcal{M}, \tau)$. Repeating Step 2 $n-3$ times, we obtain $T^2 \in S_0(\mathcal{M}, \tau)$ and apply Step 1.

Let now an operator $T \in S(\mathcal{M}, \tau)$ be *-paranormal and $T^n \in S_0(\mathcal{M}, \tau)$.

Step 1a. If $n = 2$ then from (1.2) by items (i), (ii), (iv) and (v) of Lemma 2.1 we have

$$\begin{aligned} 2\mu(2t; T)^2 &= 2\mu(2t; TT^*) \leq \lambda^{-1} \mu(t; T^{2*} T^2) + \lambda \mu(t; I) = \lambda^{-1} \mu(2t; T^2)^2 + \lambda \\ &< \varepsilon^{-1} \varepsilon^2 + \varepsilon = 2\varepsilon \end{aligned}$$

for all $t > t_0$ and numbers $\lambda = \varepsilon$ and $t_0 > 0$ such that $\mu(t; T^2) < \varepsilon$ for $t > t_0$. Therefore, $T \in S_0(\mathcal{M}, \tau)$.

Step 2a. For $n \geq 3$ we show that $T^{n-2} \in S_0(\mathcal{M}, \tau)$. Multiply both sides of inequality (1.2) from the left by the operator $(T^*)^{n-2}$ and from the right by the operator T^{n-2}, and achieve

$$2(T^*)^{n-2} T \cdot T^* T^{n-2} \leq \lambda^{-1} T^{n*} T^n + \lambda\, (T^*)^{n-2} T^{n-2} \quad \text{for all} \quad \lambda > 0.$$

Assume that $T^{n-2} \notin S_0(\mathcal{M}, \tau)$. Then $a := \mu(\infty; T^{n-2}) > 0$. Almost verbatim repetition of reasoning of Step 2 yields that

$$2\mu(2t; T^* T^{n-2})^2 \leq \frac{3}{2}\varepsilon \quad \text{for all} \quad t > \max\{t_1, t_2\},$$

the numbers t_1, t_2 were defined in Step 2. Therefore, $T^* T^{n-2} \in S_0(\mathcal{M}, \tau)$. If $n = 3$ then $T^* T^{n-2} = T^* T \in S_0(\mathcal{M}, \tau)$ and $T \in S_0(\mathcal{M}, \tau)$ by definition of the ideal $S_0(\mathcal{M}, \tau)$ and items (i) and (v) of Lemma 2.1. If $n > 3$ then

$$|T^{n-2}|^2 = (T^*)^{n-3} \cdot T^* T^{n-2} \in S_0(\mathcal{M}, \tau)$$

and again $T^{n-2} \in S_0(\mathcal{M}, \tau)$. By repeating above mentioned reasoning for τ-compact operator T^{n-2}, for even number $n = 2k$ through $k-1$ steps we obtain $T^2 \in S_0(\mathcal{M}, \tau)$ and apply Step 1a. If $n = 2k+1$ is odd then through k steps we obtain $T \in S_0(\mathcal{M}, \tau)$.
(ii). Let an operator $T \in S(\mathcal{M}, \tau)$ be paranormal and $T^n = 0$. If $n = 2$ then from (1.1) we obtain

$$0 \le 2T^*T \le \lambda I \quad \text{for all} \quad \lambda > 0. \tag{3.2}$$

Let $\lambda \to 0+$ and pass to limits in the topology t_τ in inequalities (3.2), we have $T^*T = 0$ and $T = 0$. If $n \ge 3$ then multiply both sides of inequality (1.1) from the left by the operator $(T^*)^{n-2}$ and from the right by the operator T^{n-2}, and achieve

$$0 \le 2(T^*)^{n-1}T^{n-1} \le \lambda\,(T^*)^{n-2}T^{n-2} \quad \text{for all} \quad \lambda > 0.$$

Again let $\lambda \to 0+$ and pass to limits in the topology t_τ in these inequalities, we have $T^{n-1} = 0$. By repeating above mentioned procedure several times, we obtain $T^2 = 0$.

The case of a *-paranormal operator $T \in S(\mathcal{M}, \tau)$ with $T^n = 0$ is dealt with in a similar way.

(iii). Let $T \in S(\mathcal{M}, \tau)$ and $T^3 = T$. Then $T = P - Q$ for some $P, Q \in S(\mathcal{M}, \tau)^{\text{id}}$ with $PQ = QP = 0$ [18, Proposition 1]. Note that $T^2 = P + Q \in S(\mathcal{M}, \tau)^{\text{id}}$.

For a paranormal operator T from (1.1) with $\lambda = 1$ we obtain

$$2(P-Q)^*(P-Q) \le (P+Q)^*(P+Q) + I. \tag{3.3}$$

Multiply both sides of inequality (3.3) from the left by the operator $(P-Q)^*$ and from the right by the operator $P - Q$, and achieve

$$(P+Q)^*(P+Q) \le (P-Q)^*(P-Q).$$

Hence $P^*Q + Q^*P \le 0$. Now from (3.3) follows the inequality

$$0 \le P^*P + Q^*Q \le 3(P^*Q + Q^*P) + I \le I,$$

in particular, we have $P^*P \le I$ and $Q^*Q \le I$. Therefore, $\|P^*P\| = \|P\|^2 \le 1$ and $P \in \mathcal{M}^{\text{pr}}$; analogously we have $Q \in \mathcal{M}^{\text{pr}}$. Thus, $T = P - Q \in S(\mathcal{M}, \tau)^{\text{h}}$.

For a *-paranormal operator T from (1.2) for $\lambda = 1$ we obtain

$$2TT^* \le T^{*2}T^2 + I. \tag{3.4}$$

Note that T^* is also a tripotent, i. e., $T^{*3} = T^*$. Multiply both sides of inequality (3.4) from the left by the operator T^* and from the right by the operator T and obtain

$$(T^*T)^2 \le T^*T.$$

Hence by functional calculus of self-adjoint operators we have $T^*T \le I$ and $\|T^*T\| = \|T\|^2 \le 1$. Thus,

$$\|P - Q\| = \|T\| \le 1, \quad \|P + Q\| = \|T^2\| \le \|T\|\,\|T\| \le 1$$

by submultiplicativity of the C^*-norm $\|\cdot\|$ on $\mathcal{M}$. Now by the triangle inequality for the norm $\|\cdot\|$ we obtain

$$2\|P\| = \|2P\| = \|(P - Q) + (P + Q)\| \le \|P - Q\| + \|P + Q\| \le 2$$

and $P \in \mathcal{M}^{\mathrm{pr}}$; analogously we have $Q \in \mathcal{M}^{\mathrm{pr}}$. Thus, $T = P - Q \in S(\mathcal{M}, \tau)^{\mathrm{h}}$.

(iv). If $A \in L_2(\mathcal{M}, \tau)$ then $A^2 \in L_1(\mathcal{M}, \tau)$ and $\|A^2\|_1 \le \|A\|_2^2$ by Corollary 3.2. We have

$$\mu(t; T)^2 \le \mu(t; T^2) \quad \text{for all} \quad t > 0$$

by [9, Proposition 3.5] and [11, Proposition 3.9]. Therefore,

$$\|T\|_2^2 = \int_0^{+\infty} \mu(t; T)^2\, dt \le \int_0^{+\infty} \mu(t; T^2)\, dt = \|T^2\|_1 \le \|T\|_2^2 < +\infty$$

and $\|T\|_2^2 = \|T^2\|_1$. Theorem is proved. □

Corollary 3.4 (cf. [11, Corollary 3.10(iii)]) *Let an operator $T \in S(\mathcal{M}, \tau)$ be paranormal or *-paranormal. Then we have the equivalence $T \in S_0(\mathcal{M}, \tau) \Leftrightarrow T^n \in S_0(\mathcal{M}, \tau)$ for some (and, hence, for all) $n \in \mathbb{N}$.*

Corollary 3.5 *Let an operator $T \in S(\mathcal{M}, \tau)$ be p-hyponormal for some* $0 < p \le 1$.

(i) *If $T^n \in S_0(\mathcal{M}, \tau)$ for some $n \in \mathbb{N}$ then $T \in S_0(\mathcal{M}, \tau)$;*
(ii) *if $T^n = 0$ for some $n \in \mathbb{N}$ then $T = 0$;*
(iii) *if $T^3 = T$ then $T = T^*$;*
(iv) *if $T^2 \in L_1(\mathcal{M}, \tau)$ then $T \in L_2(\mathcal{M}, \tau)$ and $\|T\|_2^2 = \|T^2\|_1$.*

Proof Every p-hyponormal operator $T \in S(\mathcal{M}, \tau)$ is paranormal [11, Theorem 4.4]. □

Lemma 3.6 *If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and $T^{*p}T^q \in S_0(\mathcal{M}, \tau)$ for some $p, q \in \mathbb{N} \cup \{0\}$, $p + q \ge 1$ then $T \in S_0(\mathcal{M}, \tau)$.*

Proof Without loss of generality assume that $p = q$ (if $p < q$ then $(T^*)^{q-p} \cdot T^{*p}T^q \in S_0(\mathcal{M}, \tau)$; if $q < p$ then $T^{*p}T^q \cdot T^{p-q} = T^{*p}T^p \in S_0(\mathcal{M}, \tau)$). We apply mathematical induction on $p \in \mathbb{N}$. If $p = 1$ then by items (i) and (iii) of Lemma 2.1 we have

$$|T|^2 = T^*T \in S_0(\mathcal{M}, \tau) \Leftrightarrow |T| \in S_0(\mathcal{M}, \tau) \Leftrightarrow T \in S_0(\mathcal{M}, \tau).$$

Suppose that the assertion holds for all $p = 1, 2, \dots, n$. Then for the operator

$$(T^*)^{n+1}T^{n+1} = T^{*n} \cdot T^*T \cdot T^n \in S_0(\mathcal{M}, \tau)$$

we have

$$0 \le T^{*n} \cdot TT^* \cdot T^n \le T^{*n} \cdot T^*T \cdot T^n \in S_0(\mathcal{M}, \tau),$$

hence $T^{*n} \cdot TT^* \cdot T^n = |T^*T^n|^2 \in S_0(\mathcal{M}, \tau)$ by item (ii) of Lemma 2.1 and

$$|T^*T^n|^2 \in S_0(\mathcal{M}, \tau) \Leftrightarrow |T^*T^n| \in S_0(\mathcal{M}, \tau) \Leftrightarrow T^*T^n \in S_0(\mathcal{M}, \tau)$$

by items (i) and (v) of Lemma 2.1. Now $T^{*n}T^n = (T^*)^{n-1} \cdot T^*T^n \in S_0(\mathcal{M}, \tau)$ and $T \in S_0(\mathcal{M}, \tau)$ by the induction hypothesis. □

Theorem 3.7 *If an operator $T \in S(\mathcal{M}, \tau)$ is hyponormal and $T^{*p}T^q \in S_0(\mathcal{M}, \tau)$ for some $p, q \in \mathbb{N} \cup \{0\}$, $p + q \ge 1$ then T is normal.*

Proof Follows from Lemma 3.6 and Theorem 2.2 of [6]. □

Corollary 3.8 [32, Theorem 1.2] *If an operator $T \in \mathcal{B}(\mathcal{H})$ is hyponormal and $T^{*p}T^q$ is completely continuous for some $p, q \in \mathbb{N}$ then T is normal.*

Above we also showed that this Istrăţescu Theorem may be deduced from Ando–Berberian–Stampfli Theorem, see [1, 4, 39] and [29, Problem 206].

Theorem 3.9 *If an operator $T \in S(\mathcal{M}, \tau)$ is p-hyponormal for some $0 < p \le 1$ then the operator $(T^*T)^p - (TT^*)^p$ cannot have the inverse in $\mathcal{M}$.*

Proof Let, on the contrary, the operator $(T^*T)^p - (TT^*)^p$ possess the inverse in $\mathcal{M}$, i.e. $(T^*T)^p - (TT^*)^p \ge \varepsilon I$ for some number $\varepsilon > 0$. Then $(T^*T)^p \ge (TT^*)^p + \varepsilon I$ and for arbitrary $0 < t < \tau(I)$ we have

$$\begin{aligned}\mu(t; T^*T)^p &= \mu(t; (T^*T)^p) \ge \mu(t; (TT^*)^p + \varepsilon I) = \varepsilon + \mu(t; (TT^*)^p)\\ &= \varepsilon + \mu(t; TT^*)^p \end{aligned} \tag{3.5}$$

by items (iii) and (v) of Lemma 2.1 and by the well-known representation

$$\mu(t; X) = \inf\{s \ge 0 : \ d_X(s) \le t\}, \quad t > 0, \tag{3.6}$$

see [26, Proposition 2.2]. Here $d_X(s) = \tau(E^{|X|}(s, +\infty))$, $s > 0$, is the distribution function of an operator $X \in S(\mathcal{M}, \tau)$ and $E^{|X|}(s, +\infty)$ is the spectral projection of the operator $|X|$, corresponding to the interval $(s, +\infty)$.

On the other hand, by items (i) and (v) of Lemma 2.1 we have

$$\mu(t; T^*T) = \mu(t; |T|)^2 = \mu(t; T^*)^2 = \mu(t; |T^*|)^2 = \mu(t; TT^*)$$

for all $0 < t < \tau(I)$. We obtain a contradiction with (3.5). Theorem is proved. □

In particular, a positive self-commutator cannot have the inverse in $\mathcal{M}$. Recall that Theorem 3.9 for $p = 1$ was established by the author via different method in [14, Theorem 3] (see also [15, Theorem 2]). For $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\tau = \mathrm{tr}$ and $p = 1$ Theorem 3.9 was proved by Putnam [38]; see also [29, Problem 236].

Lemma 3.10 [20, Theorem 17] *We have $\tau(ST) = \tau(TS)$ for all $S, T \in S(\mathcal{M}, \tau)$ with $ST, TS \in L_1(\mathcal{M}, \tau)$.*

Theorem 3.11 *If $A, B \in L_2(\mathcal{M}, \tau)$, the operator A is hyponormal and B is cohyponormal then $\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2$ for all $X \in \mathcal{M}$.*

Proof By Lemma 3.1 the operators A^*XB, B^*X^*A lie in $L_1(\mathcal{M}, \tau)$, hence by linearity of the extension of the trace τ to $L_1(\mathcal{M}, \tau)$ and by Lemma 3.10 we obtain

$$\begin{aligned}\|AX - XB\|_2^2 &= \tau((AX - XB)^*(AX - XB)) \\ &= \tau(X^*A^*AX) + \tau(B^*X^* \cdot XB) - \\ &\quad - \tau(X^* \cdot A^*XB) - \tau(B^*X^* \cdot AX) \\ &= \tau(X^*A^*AX + XBB^*X^*) - \tau(A^*XBX^* + AXB^*X^*).\end{aligned}$$

The operator $A^*XBX^* + AXB^*X^*$ does not change when we replace A and B with A^* and B^*, respectively. If A is hyponormal and B is cohyponormal then $X^*A^*AX + XBB^*X^* \geq X^*AA^*X + XB^*BX^*$ and we apply monotonicity of the trace τ on the positive cone $L_1(\mathcal{M}, \tau)^+$. □

Corollary 3.12 *If operators $A, B \in L_2(\mathcal{M}, \tau)$ are normal then $\|AX - XB\|_2 = \|A^*X - XB^*\|_2$ for all $X \in \mathcal{M}$.*

Theorem 3.13 *Let an operator $A \in S(\mathcal{M}, \tau)$ be hyponormal and $X \in S(\mathcal{M}, \tau)$ with $AX \in L_p(\mathcal{M}, \tau)$ for some $< p < +\infty$. Then $A^*X \in L_p(\mathcal{M}, \tau)$ with $\|A^*X\|_p \leq \|AX\|_p$. For $0 < p \leq 2$ the following conditions are equivalent:*

(i) $\|AX\|_p = \|A^*X\|_p$;
(ii) $A^*AX = AA^*X$;
(iii) $|AX| = |A^*X|$.

Proof We have

$$|A^*X|^2 = X^*AA^*X \leq X^*A^*AX = |AX|^2, \tag{3.7}$$

and $\mu(t; A^*X) = \mu(t; |A^*X|^2)^{1/2} \leq \mu(t; |AX|^2)^{1/2} = \mu(t; |AX|)$ for all $t > 0$ by items (i), (iii) and (v) of Lemma 2.1. Thus, $A^*X \in L_p(\mathcal{M}, \tau)$ and $\|A^*X\|_p \leq \|AX\|_p$ for every $0 < p < +\infty$.

(i)⇒(ii). If $0 < p \leq 2$ then by (3.7) and by the operator monotonicity of the function $f(t) = t^{p/2}$ on the semiaxis $[0, +\infty)$ we have $|AX|^p - |A^*X|^p \geq 0$. By linearity of the extension of the trace τ to $L_1(\mathcal{M}, \tau)$ we obtain

$$0 = \|AX\|_p^p - \|A^*X\|_p^p = \tau(|AX|^p) - \tau(|A^*X|^p) = \tau(|AX|^p - |A^*X|^p).$$

Hence $|AX|^p - |A^*X|^p = 0$ by the faithfulness of the trace τ on the cone $L_1(\mathcal{M}, \tau)^+$, i.e., $|AX|^p = |A^*X|^p$. Thus,

$$|AX|^2 = (|AX|^p)^{2/p} = (|A^*X|^p)^{2/p} = |A^*X|^2$$

and $0 = X^*A^*AX - X^*AA^*X = X^*(A^*A - AA^*)X = |\sqrt{A^*A - AA^*}X|^2$, i.e., $\sqrt{A^*A - AA^*}X = 0$ and $A^*AX - AA^*X = \sqrt{A^*A - AA^*}\sqrt{A^*A - AA^*}X = 0$.

(ii)⇒(iii). If $A^*AX = AA^*X$ then

$$|AX|^2 = X^* \cdot A^*AX = X^* \cdot AA^*X = |A^*X|^2.$$

(iii)⇒(i). We have $|AX|^p = |A^*X|^p$ and

$$\|AX\|_p^p = \tau(|AX|^p) = \tau(|A^*X|^p) = \|A^*X\|_p^p.$$

Theorem is proved. □

Theorem 3.14 *Let an operator $T \in S(\mathcal{M}, \tau)$ be p-hyponormal for some $0 < p \le 1$. Then* $\mathrm{Ker}(T^*) \subseteq \mathrm{Ker}(T)$ *and if $T = U|T|$ is the polar decomposition of T then for every $0 < q \le \min\{2p, 1\}$ we have $U^*|T|^qU \ge |T|^q \ge U|T|^qU^*$.*

Proof If $A \in S(\mathcal{M}, \tau)^+$ then $\mathrm{Ker}(A) = \mathrm{Ker}(A^r)$ for all $r > 0$. It follows from the following arguments: if $0 < \alpha < \beta, \xi \in \mathcal{H}$ and $A^\alpha\xi = 0$ then $A^\beta\xi = A^{\beta-\alpha}A^\alpha\xi = 0$; if $A\xi = 0$ then

$$0 = \langle A\xi, \xi\rangle = \langle A^{1/2}\xi, A^{1/2}\xi\rangle = \|A^{1/2}\xi\|^2$$

and $A^{1/2}\xi = 0$.

Since $|T|^{2p} \ge |T^*|^{2p}$ we have $|T^*|^p = V|T|^p$ for some $V \in \mathcal{M}$ with $\|V\| \le 1$ [45, p. 261]. Hence $\mathrm{Ker}(|T|^p) \subseteq \mathrm{Ker}(|T^*|^p)$ and

$$\mathrm{Ker}(T) = \mathrm{Ker}(|T|) = \mathrm{Ker}(|T|^P) \subseteq \mathrm{Ker}(|T^*|^p) = \mathrm{Ker}(|T^*|) = \mathrm{Ker}(T^*).$$

Since $|T|^{2p} \ge |T^*|^{2p}$, by the operator monotonicity of the function $f(t) = t^{\frac{q}{2p}}$ on the semiaxis $[0, +\infty)$ we obtain

$$|T|^q \ge |T^*|^q. \tag{3.8}$$

Therefore, $U^*|T|^qU \ge U^*|T^*|^qU$. By Hansen inequality [30] for the operator monotone function $g(t) = t^q \quad (t \ge 0)$ and via the equalities $U^*U|T| = |T|$, $|T^*| = U|T|U^*$ we obtain

$$U^*|T|^qU \ge |T|^q \ge U^*|T^*|U = U^*(U|T|U^*)^qU \ge U^*U|T|^qU^*U = |T|^q.$$

On the other hand, by Hansen inequality [30] for the operator monotone function g and inequality (3.8) we have $U|T|^qU^* \le (U|T|U^*)^q = |T^*|^q \le |T|^q$. □

Corollary 3.15 *In conditions of Theorem 3.14 we have*

$$\mu(t; U^*|T|^q U) = \mu(t; U|T|^q U^*) = \mu(t; |T|^q) = \mu(t; T)^q \quad \textit{for all} \quad t > 0.$$

Proof By items (i) and (v) of Lemma 2.1 we have

$$\mu(t; A^*A) = \mu(t; AA^*) \quad \text{for all} \quad A \in S(\mathcal{M}, \tau) \quad \text{and} \quad t > 0. \tag{3.9}$$

By item (iii) of Lemma 2.1 we have

$$\mu(t; U^*|T|^q U) \geq \mu(t; |T|^q) \geq \mu(t; U|T|^q U^*) \quad \text{for all} \quad t > 0.$$

Since $UU^* \leq I$, by (3.9) with $A = |T|^{q/2}U$ and items (i), (iii) and (v) of Lemma 2.1 we obtain

$$\mu(t; U^*|T|^q U) = \mu(t; |T|^{q/2}UU^*|T|^{q/2}) \leq \mu(t; |T|^q) = \mu(t; T)^q \quad \text{for all} \quad t > 0.$$

Analogously, with $A = |T|^{q/2}U^*$ we have $\mu(t; U|T|^q U^*) = \mu(t; |T|^q)$ for all $t > 0$. By items (i) and (v) of Lemma 2.1 we obtain $\mu(t; |T|^q) = \mu(t; |T|)^q = \mu(t; T)^q$ for all $t > 0$. □

Corollary 3.16 *In conditions of Theorem 3.14, if the operator $Y := |T|^q - U|T|^q U^*$ lies in $L_1(\mathcal{M}, \tau)$ then $X := U^*|T|^q U - |T|^q \in L_1(\mathcal{M}, \tau)$ and $\tau(X) = \tau(YP) \leq \tau(Y)$ for the projection $P = UU^*$.*

Proof Since $Y \in L_1(\mathcal{M}, \tau)$, we have $X = U^*YU \in L_1(\mathcal{M}, \tau)$. Also $U = UU^*U$ and $P := UU^*$, $Q := U^*U \in \mathcal{M}^{\text{pr}}$ [29, Problem 127]. Then $|T|^p = Q|T|^p$ and

$$X = U^*|T|^q U - Q|T|^q = U^*\,[|T|^p, U] \in L_1(\mathcal{M}, \tau).$$

Since the operator $[|T|^p, U]\,U^* = |T|^p P - U|T|^q U^* = |T|^p P - U|T|^q U^* P = YP$ also lies in $L_1(\mathcal{M}, \tau)$, by Lemma 3.10 and the inequality $P^{\perp}YP^{\perp} \geq 0$ we obtain

$$\tau(X) = \tau(YP) = \tau(Y - YP^{\perp}) = \tau(Y) - \tau(YP^{\perp}) = \tau(Y) - \tau(P^{\perp}YP^{\perp}) \leq \tau(Y).$$

The assertion is proved. □

Theorem 3.17 *Let an operator $U \in \mathcal{M}$ be a isometry and let a number $0 < p \leq 1$.*

(i) *If $T \in S(\mathcal{M}, \tau)$ is paranormal then UTU^* is paranormal;*
(ii) *if $A \in S_0(\mathcal{M}, \tau)^+$ and $A^p \geq (UAU^*)^p$ then $AU = UA$;*
(iii) *if $T \in S(\mathcal{M}, \tau)$ is p-hyponormal then UTU^* p-hyponormal.*

Proof (i). We have $P := UU^* \in \mathcal{M}^{\text{pr}}$, hence $0 \leq P \leq I$. Multiply both sides of inequality (1.1) from the left by the operator U and from the right by the operator U^*, for all $\lambda > 0$ we obtain

$$\begin{aligned} 2(UTU^*)^*UTU^* = 2UT^*U^* \cdot UTU^* &\leq \lambda^{-1}UT^*U^* \cdot UTU^* \cdot UTU^* \cdot UTU^* + \lambda P \\ &\leq \lambda^{-1}(UTU^*)^{*2} \cdot (UTU^*)^2 + \lambda I. \end{aligned}$$

Thus, the operator UTU^* satisfies inequality (1.1). (ii). We have $A^{1/2} \in S_0(\mathcal{M}, \tau)^+$ by item (v) of Lemma 2.1 and by the definition of the ideal $S_0(\mathcal{M}, \tau)$. Hence the operator $B := UA^{1/2}$ lies in $S_0(\mathcal{M}, \tau)$ and

$$(B^*B)^p \geq (BB^*)^p,$$

i.e. the operator B is p-hyponormal. By Theorem 2.2 of [6] the operator B is normal and

$$A = UAU^*.$$

Multiply both sides of the last equality from the right by the operator U and obtain $AU = UA$.
(iii). We have $(UT^*U^* \cdot UTU^*)^n = U(T^*T)^n U^*$ for all $n \in \mathbb{N}$.

Step 1. Let $T \in \mathcal{M}$. For every number $\varepsilon > 0$ by Weierstrass Theorem on uniform approximation of continuous functions on (closed) interval we choose a polynomial

$$\mathcal{P}(t) = a_0 + a_1 t + \cdots + a_k t^k, \quad a_0, a_1, \ldots, a_k \in \mathbb{R},$$

such that $|\mathcal{P}(t) - t^p| < \varepsilon$ for all $0 \leq t \leq \|T\|^2$. Then by Functional Calculus we have

$$\|\mathcal{P}(T^*T) - (T^*T)^p\| < \varepsilon, \quad \|\mathcal{P}(UT^*TU^*) - (UT^*TU^*)^p\| < \varepsilon$$

and $\|U\mathcal{P}(T^*T)U^* - U(T^*T)^pU^*\| \leq \|U\| \cdot \|U^*\| \cdot \|\mathcal{P}(T^*T) - (T^*T)^p\| < \varepsilon$. Since

$$\mathcal{P}(UT^*TU^*) = U\mathcal{P}(T^*T)U^*,$$

by the triangle inequality for the C^*-norm $\|\cdot\|$ we achieve the estimate

$$\begin{aligned}\|U(T^*T)^pU^* - (UT^*TU^*)^p\| = \|&(U(T^*T)^pU^* - U\mathcal{P}(T^*T)U^*) \\ &+ (\mathcal{P}(UT^*TU^*) - (UT^*TU^*)^p)\| < 2\varepsilon.\end{aligned}$$

By arbitrariness of $\varepsilon > 0$ we obtain

$$U(T^*T)^pU^* = (UT^*TU^*)^p. \tag{3.10}$$

Step 2. Let $T \in S(\mathcal{M}, \tau)$ and $P_n \in \mathcal{M}^{\text{pr}}$ be the spectral projection of the operator T^*T, corresponding to the interval $[0, n]$, $n \in \mathbb{N}$. Then $P_n \xrightarrow{\tau} I$ as $n \to +\infty$ and

$$n^p P_n \geq P_n(T^*T)^p P_n = (P_n T^*T P_n)^p, \quad n \in \mathbb{N}.$$

By Step 1 (see (3.10)) we have $U(P_nT^*TP_n)^pU^* = (UP_nT^*TP_nU^*)^p$, $n \in \mathbb{N}$. Passing in these equalities to limits in the topology t_τ as $n \to +\infty$, taking into account joint t_τ-continuity of multiplication in $S(\mathcal{M}, \tau)$ and t_τ-continuity of operator functions [42, Theorem 2.6], we obtain (3.10).

Step 3. Analogously (see Steps 1, 2) we have $U(TT^*)^pU^* = (UTT^*U^*)^p$. Therefore,

$$(UT^*U^* \cdot UTU^*)^p = U(T^*T)^pU^* \geq U(TT^*)^pU^* = (UTU^* \cdot UT^*U^*)^p$$

and Theorem is proved. □

Note that for every number $0 < p \leq 1$ the set of all p-hyponormal operators $T \in S(\mathcal{M}, \tau)$ is t_τ-closed in $S(\mathcal{M}, \tau)$ (it follows from t_τ-continuity of the involution and the multiplication in $S(\mathcal{M}, \tau)$ and [42, Theorem 2.6]).

Corollary 3.18 *Let $A \in S_0(\mathcal{M}, \tau)^+$ and an operator $U \in \mathcal{M}$ be an isometry. If $A^p \leq (UAU^*)^p$ for some $0 < p \leq 1$ then $AU = UA$. If $T \in S(\mathcal{M}, \tau)$ is p-cohyponormal then UTU^* is p-cohyponormal.*

Proof An operator $B := UA^{1/2} \in S_0(\mathcal{M}, \tau)$ is p-cohyponormal. By Corollary 2.3 of [6] the operator B is normal. □

Theorem 3.19 *For $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ the following conditions are equivalent:*

(i) *an operator $A + iB$ is hyponormal;*
(ii) *an operator $aA + ibB$ is hyponormal for some numbers $a, b > 0$. If an operator A is invertible in $S(\mathcal{M}, \tau)$ then (i) and (ii) are equivalent to the condition:*
(iii) *an operator $A^{-1} - iB$ is hyponormal.*

Proof An operator $A + iB$ is hyponormal if and only if

$$i(AB - BA) \geq 0, \tag{3.11}$$

i. e., $i[A, B] \geq 0$. Hence (i)⇔(ii). Let us show that (i)⇒(iii). Multiply both sides of inequality (3.11) from the left and the right by the operator $A^{-1} \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ and obtain

$$i(BA^{-1} - A^{-1}B) \geq 0.$$

This condition is necessary and sufficient for hyponormality of the operator $A^{-1} - iB$, see (3.11). The rest is clear. □

Clearly, an operator $(A + iB)^2$ is hyponormal if and only if

$$i[A^2 - B^2, AB + BA] \geq 0.$$

Recall that there exists a hyponormal operator, whose square is not hyponormal [29, Problem 209]. On invertibility in $S(\mathcal{M}, \tau)$ see [12, 17, 41].

Corollary 3.20 *Let operators $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ be invertible in $S(\mathcal{M}, \tau)$. Then the following conditions are equivalent:*

(i) *an operator $A + iB$ is hyponormal;*

(ii) *an operator* $A^{-1} + iB^{-1}$ *is hyponormal.*

Theorem 3.21 *If an operator* $T \in S(\mathcal{M}, \tau)$ *is hyponormal and* $T^2 \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ *then* T *is normal.*

Proof Let $T = A + iB$ be the Cartesian representation of the operator $T \in S(\mathcal{M}, \tau)$ with $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$. If $T^2 \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ then the operators A and B anticommute, i. e., $AB = -BA$. Since T is hyponormal, from (3.11) we obtain $iAB \geq 0$. Therefore, $-iBA = (iAB)^* \geq 0$ and

$$(\mathbb{R}^+ \supset)\sigma(iAB) \cup \{0\} = \sigma(-iBA) \cup \{0\} = -i\sigma(BA) \cup \{0\} = -i\sigma(AB) \cup \{0\}$$

by the equality $\sigma(XY) \cup \{0\} = \sigma(YX) \cup \{0\}$ for all $X, Y \in S(\mathcal{M}, \tau)$ [40, Chap. I, Proposition 2.1]. Hence

$$i\sigma(AB) \cup \{0\} = -i\sigma(AB) \cup \{0\} \subset \mathbb{R}^+$$

and $\sigma(AB) = \{0\} = \sigma(iAB)$. Consider the Abelian von Neumann subalgebra $\mathcal{A}$ in $\mathcal{M}$, generated by I and by all spectral projections of the positive operator iAB. With regard to a *-isomorphism $\mathcal{A} \simeq L_\infty(\Omega, \mathfrak{A}, \nu)$ for some localizable measure space $(\Omega, \mathfrak{A}, \nu)$ and by nonnegativity of the function $iAB \in L_0(\Omega, \mathfrak{A}, \nu)$, we obtain $iAB = 0 = AB$, since the spectrum of a multiplicator $M_f : L_2(\Omega, \mathfrak{A}, \nu) \to L_2(\Omega, \mathfrak{A}, \nu)$ by a measurable function $f \in L_0(\Omega, \mathfrak{A}, \nu)$ coincides with its set of essential values

$$\mathcal{E}_f = \{\lambda \in \mathbb{C} : \ \forall \varepsilon > 0 \ (\nu\{\omega \in \Omega : \ |f(\omega) - \lambda| < \varepsilon\} \neq 0)\},$$

see [43, Theorem I.7.11]. Therefore, $0 = AB = (AB)^* = BA$ and the operator T is normal. □

Corollary 3.22 *If an operator* $T \in S(\mathcal{M}, \tau)$ *is cohyponormal and the operator* T^2 *is Hermitian then* T *is normal.*

Theorem 3.23 *Let operators* $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ *and* $\{A, B\} \cap \mathcal{M}^{\mathrm{sym}} \neq \emptyset$. *If the operator* $T := A + iB$ *is hyponormal (or cohyponormal), then* T *is normal.*

Proof Let, for definiteness, $A^2 = I$. An operator T is hyponormal if and only if (3.11) holds. Multiply both sides of inequality (3.11) from the left and right by the Hermitian symmetry A, and achieve $i(BA - AB) \geq 0$. From this relation and (3.11) we have $AB = BA$, i.e., the operator T is normal. □

Corollary 3.24 *Let operators* $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$ *and* $\{A, B\} \cap \mathcal{M}^{\mathrm{pr}} \neq \emptyset$. *If the operator* $T := A + iB$ *is hyponormal (or cohyponormal), then* T *is normal.*

Proof Let, for definiteness, $A \in \mathcal{M}^{\mathrm{pr}}$ and (3.11) holds. For the Hermitian symmetry $S := 2A - I$ we have

$$i(SB - BS) = \frac{i}{2}(AB - BA) \geq 0.$$

Hence the operator $S + iB$ is hyponormal via (3.11). By Theorem 3.23 the operator $S+iB$ is normal, i.e., $SB = BS$. Therefore, $AB = BA$ and the operator T is normal. □

Corollary 3.25 ([5, Proposition 4.2]) *Let operators $A \in \mathcal{B}(\mathcal{H})^{\mathrm{h}}$ and $P \in \mathcal{B}(\mathcal{H})^{\mathrm{pr}}$. If $i[A, P] \geq 0$ then $AP = PA$.*

Proposition 3.26 *Let $A, B \in S(\mathcal{M}, \tau)^{\mathrm{h}}$, $C \in S(\mathcal{M}, \tau)$ and the operator $A + iB$ be hyponormal.*

(i) *If $AC = CA$ then the operator $A + iCBC^*$ is hyponormal.*
(ii) *If $BC = CB$ then the operator $CAC^* + iB$ is hyponormal. If C is invertible in $S(\mathcal{M}, \tau)$ then the inverse assertions hold.*

Proof (i). Note that the operator CBC^* is Hermitian. Multiply both sides of inequality (3.11) from the left by the operator C and from the right by the operator C^*, then by taking into account the equalities $AC = CA$ and $AC^* = C^*A$ we obtain

$$0 \leq i(CABC^* - CBAC^*) = i(A \cdot CBC^* - CBC^* \cdot A) = i[A, CBC^*].$$

Via the Cartesian criterion of hyponormality (3.11) the operator $A + iCBC^*$ is hyponormal.

(ii). Multiply both sides of inequality (3.11) from the left by the operator C and from the right by the operator C^*, take into account the equalities $BC = CB$ and $BC^* = C^*B$ and obtain $i[CAC^*, B] \geq 0$. The rest is clear. □

If $A, B \in S(\mathcal{M}, \tau)$ with $AB = BA$ and the operator B is normal then by Fuglede–Putnam Theorem for τ-measurable operators [3, Theorem 6] we have $AB^* = B^*A$, hence $A^*B = (B^*A)^* = (AB^*)^* = BA^*$.

Proposition 3.27 *Let operators $A, B \in S(\mathcal{M}, \tau)$ with $A^*B = BA^*$. Then the following conditions are equivalent:*

(i) *A and B are hyponormal (respectively, cohyponormal; normal);*
(ii) *$aA + bB$ is hyponormal (respectively, cohyponormal; normal) for all $a, b \in \mathbb{C}$.*

Proof (i)⇒(ii). It is clear that $B^*A = (A^*B)^* = (BA^*)^* = AB^*$. If A and B are hyponormal then for all $a, b \in \mathbb{C}$ we have

$$\begin{aligned}(aA + bB)^*(aA + bB) &= |a|^2 A^*A + \overline{a}bA^*B + a\overline{b}B^*A + |b|^2 B^*B \\ &\geq |a|^2 AA^* + \overline{a}bBA^* + a\overline{b}AB^* + |b|^2 BB^* \\ &= (aA + bB)(aA + bB)^*.\end{aligned}$$

(ii)⇒(i). For $a = 1, b = 0$ (respectively, for $a = 0, b = 1$) the operator A is hyponormal (respectively, B is hyponormal). The rest is clear. □

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