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CONVERGENCE OF THE GALERKIN METHOD FOR NUMERICAL CALCULATION OF THE GUIDED MODES OF AN INTEGRATED OPTICAL GUIDE

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Abstract **– The Galerkin method for numerical calculation of the natural modes of an integrated optical guide is proposed and the convergence of the Galerkin method is proved.**

I. INTRODUCTION

The eigenvalue problems for guided modes of integrated optical guides are attracted much attention. In the paper [1] was proved the existence of the guided modes of integrated optical guide. In the paper [2] was proposed a finite element method for computing guided modes of integrated optical guides under the weak guiding assumption that leads to a scalar model. Due to the complexity of the integrated optical structure, domain integral equation for electric vector field utilizing dyadic Green's function (to account for the background media) is a popular practical approach for computing the natural fiber modes [3]-[5]. A problem with this domain integral equation is that it is strongly-singular, which previously prevented its use in a theoretical investigation of convergence of the known numerical methods. It was recently proved [6] that the operator of the domain integral equation is a Fredholm operator with zero index.

In this work we propose and analytically study the Galerkin method based on domain integral equation for numerical calculation of the natural modes of an optical fiber integrated into a three-layer planar medium, which is representative of typical integrated optical guide. Using the fredholm property of the integral operator we prove the convergence of the Galerkin method.

II. STATEMENT OF THE PROBLEM

We consider the guided modes of an optical fiber integrated into a three-layer planar medium. Let the three-dimensional space be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3. We assume that there exists a bounded domain Ω on the plane $R^2 = \{(x_1, x_2): -\infty < x_1, x_2 < \infty\}$ such that $n = n_\infty(x_2)$, $x = (x_1, x_2) \in \Omega_\infty = R^2 \setminus \overline{\Omega}$, where $n_\infty(x_2)$ depends only on the x_2 variable. It is a piecewise-constant function represents the refractive index of so-called associated planar waveguide. For simplicity, we take $n_\infty(x_2) = \{n_1 \text{ if } x_2 > d,\ n_2 \text{ if } 0 < x_2 < d,\ n_3 \text{ if } x_2 < 0\}$. We assume without loss of generality that $n_2 \geq n_3 \geq n_1$. Denote by n_+ the maximum of the function n in the domain Ω. We assume that $\Omega \subset \Omega_2 = \{(x_1, x_2): -\infty < x_1 < \infty, 0 < x_2 < d\}$, $n_+ > n_2$, and also that function n is a continuous function in Ω_2, i.e., that the guide does not have a sharp boundary.

The modal problem can be formulated as a vector eigenvalue problem for the set of differential equations (we use notations from [1] for differential operators)

$$\mathrm{Rot}_\beta\, \mathbf{E} = i\omega\mu_0 \mathbf{H}, \quad \mathrm{Rot}_\beta\, \mathbf{H} = -i\omega\varepsilon_0 n^2 \mathbf{E}. \tag{1}$$

Here ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as an unknown complex parameter and radian frequency $\omega > 0$ as a given parameter. We seek non-zero solutions $[\mathbf{E}, \mathbf{H}]$ of set (1) in the space $\left(L_2(R^2)\right)^6$.

Denote by $\Lambda^{(1)}$ the sheet of the Riemann surface of the function $\sqrt{k^2 n_2^2 - \beta^2}$, where $k^2 = \omega^2 \varepsilon_0 \mu_0$, which is specified by the condition $\operatorname{Im}\sqrt{k^2 n_2^2 - \beta^2} \geq 0$. Denote by β_j the propagation constants of TE and TM modes of the associated planar waveguide [7]. It is well known that there exist no more than a finite number of values β_j. All of the values β_j belong to domain $\{\beta \in \Lambda^{(1)} : \operatorname{Im}\beta = 0, kn_3 < |\beta| < kn_2\}$. In a similar way to [1] we can see that the domain $D = \{\beta \in \Lambda^{(1)} : \operatorname{Re}\beta = 0\} \cup \{\beta \in \Lambda^{(1)} : \operatorname{Im}\beta = 0, |\beta| < \gamma\}$, where $\gamma = \max_j \beta_j$, corresponds to the continuum of propagation constants of radiation modes that do not belong to $(L_2(R^2))^6$. Therefore we do not investigate the values $\beta \in D$.

Definition 1. *A nonzero vector* $[\mathbf{E}, \mathbf{H}] \in (L_2(R^2))^6$ *is referred to as an eigenvector of problem (1) corresponding to an eigenvalue* $\beta \in \Lambda = \Lambda^{(1)} \setminus D$ *if relation (1) is valid. The set of all eigenvalues of problem (1) is called the spectrum of this problem.*

II. Galerkin Method

If $[\mathbf{E}, \mathbf{H}]$ is an eigenvector of problem (1) corresponding to an eigenvalue $\beta \in \Lambda$, then

$$\mathbf{E}(x) = \left(k^2 n_\infty^2 + \operatorname{Grad}_\beta \operatorname{Div}_\beta\right) \frac{1}{n_\infty^2} \int_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y) \mathbf{E}(y) dy, \quad (2)$$

$$\mathbf{H}(x) = -i\omega\varepsilon_0 \operatorname{Rot}_\beta \int_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y) \mathbf{E}(y) dy, \quad x \notin \partial\Omega_2, \quad (3)$$

where function G is the well known tensor Green function [4]. For any $(x, y) \in \Omega^2$ the function G is analytic for $\beta \in \Lambda$. Passing the operator $\operatorname{Grad}_\beta \operatorname{Div}_\beta$ under the integral in relation (2), and using the differentiation rule [11] for weakly singular integrals we obtain a nonlinear spectral problem for a strongly-singular domain integral equation

$$A(\beta)\mathbf{E} = 0, x \in \Omega; \quad A : (L_2(\Omega))^3 \to (L_2(\Omega))^3. \quad (4)$$

Definition 2. *A nonzero vector* $\mathbf{E} \in (L_2(\Omega))^3$ *is called an eigenvector of the operator-valued function* $A(\beta)$ *corresponding to an eigenvalue* $\beta \in \Lambda$ *if relation (4) is valid. Denote by* $s(A) \subset \mathrm{L}$ *the spectrum of operator-valued function* $A(\beta)$.

Theorem 1. *For all* $\beta \in \Lambda$ *the operator* $A(\beta)$ *is Fredholm with zero index. The set* $\{\beta \in \Lambda^{(1)} : \operatorname{Im}\beta = 0, |\beta| \geq kn_+\}$ *is free of the eigenvalues of problem (1). The spectrum of problem (1) is equivalent to the spectrum of operator-valued function* $A(\beta)$ *and can be only a set of isolated points on* Λ.

This theorem was proved in [6]. The eigenvectors of problem (1) is equivalent to the eigenvectors of the operator-valued function $A(\beta)$ corresponding to the same eigenvalues b in the sense of results [6].

Consider the Galerkin method for numerical approximation of integral equation (4). We cover W with small squares D_i and denote by W_n the sub-domain $\mathrm{W}_n = \bigcup_{i=1}^{n} \mathrm{D}_i \subseteq \mathrm{W}$. We seek the approximate solution $\mathbf{E}_n$ of equation (4) in the form of linear combination $\mathbf{E}_n(x) = \sum_{i=1}^{n} a_i \mathbf{F}_i(x)$, $x \in \mathrm{W}_n$, where $\mathbf{F}_i$ are basis functions, $\mathbf{F}_i(x) = 1$, if $x \in \mathrm{D}_i$, $\mathbf{F}_i(x) = 0$, if $x \notin \mathrm{D}_i$. We seek the non-zero approximate solution $\mathbf{E}_n$ in the space $H_n = \operatorname{span}\{\mathbf{F}_1, K, \mathbf{F}_n\}$. The unknown coefficients a_i can be determined from the set of linear algebraic equations:

$$\sum_{i=1}^{n} a_i \left(A(b)\mathbf{F}_i, \mathbf{F}_j \right) = 0, \; j = 1, K, n, \tag{5}$$

where $(\cdot,\cdot)$ denotes inner product in $\left(L_2(\Omega)\right)^3$. The singular Galerkin elements $\left(A(b)\mathbf{F}_i, \mathbf{F}_i\right)$ are calculated analytically by formula [8]:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\frac{1}{2p}\int_{D_i} \ln|x-y|dy = -\frac{1}{2}, \tag{6}$$

that is true if point x is at center of the square D_i.

Therefore, using Galerkin method for solving nonlinear spectral problem for strongly-singular domain integral equation (4), we obtain finite-dimensional nonlinear spectral problem (5), that we can rewrite in the operator form:

$$A_n(\beta)\mathbf{E}_n = 0, x \in \Omega_n; \quad A_n : H_n \to H_n, \tag{7}$$

where the operator-valued function $A_n(\beta)$ is determined by (5).

Convergence of the presented numerical algorithm is governed by the theorem, which follows from theorem 1 and results of paper [9]. Following [9], we denote by N' the infinite subset of the set of integers N. Denote by $\mathbf{E}_n \to \mathbf{E}$, $n \in N'$, the convergence $\mathbf{E}_n \to \mathbf{E}$ for $n \to \infty$, $n \in N'$.

Theorem 2. *If* $b_n \in s(A_n)$, $A_n(\beta_n)\mathbf{E}_n = 0$, $\|\mathbf{E}_n\| = 1$, *and* $b_n \to b_0 \in L$, $\mathbf{E}_n \to \mathbf{E}_0$, $n \in N' \subseteq N$, *then* $b_0 \in s(A)$ *and* $A(\beta_0)\mathbf{E}_0 = 0$, $\|\mathbf{E}_0\| = 1$.

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