Nilpotent elements in the Jacobson-Witt algebra over a finite field

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Let $\mathbb{F}_q$ be the finite field with q elements where q is a power of a prime p. In [4] Kaplansky asked whether the number of elements in the nilpotent cone $\mathcal{N}(L)$ of any simple finite dimensional Lie algebra L over $\mathbb{F}_q$ is given by the formula

$$\#\mathcal{N}(L) = q^{\dim L - \operatorname{rank} L}.$$

Its validity had been known already for the Lie algebras of semisimple algebraic groups in good characteristics [9, III.3.28]. Kaplansky observed that the formula holds also when L is the Witt algebra W_1.

Indirect evidence in favour of a formula of this kind can be seen from general facts about the nilpotent cone. Indeed, in [6, Th. 4.2] Premet proved that for an arbitrary finite dimensional p-Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic p the variety of $[p]$-nilpotent elements is always a set-theoretic complete intersection of codimension s in $\mathcal{L}$ where s stands for the maximum dimension of toral subalgebras of $\mathcal{L}$. This result seems to suggest that the rank of L in the formula for $\#\mathcal{N}(L)$ should probably be replaced by some other quantity.

In this paper we consider the Jacobson-Witt algebra W_n for an arbitrary n. The main result is stated as follows:

Theorem. *The algebra W_n over $\mathbb{F}_q$ has precisely q^{np^n-n} nilpotent elements.*

Note that $\dim W_n = np^n$, while the rank of W_n and the maximum dimension of tori in W_n are both equal to n. Elements of W_n are derivations of the reduced polynomial algebra B_n in n generators. With $D \in W_n$ one associates the maximal D-invariant ideal I of B_n. The factor algebra B_n/I is isomorphic with B_k for some $k \le n$. If D is nilpotent, then D^{p^k} induces a nilpotent linear transformation A of the vector space $I/\mathfrak{m}I$ where $\mathfrak{m}$ denotes the maximal ideal of B_n. This leads to a stratification of the nilpotent cone $\mathcal{N}(W_n)$ by the pairs (I, A). Theorem 3.4 determines the cardinalities of all strata. The total number of nilpotents in W_n is then obtained by summation.

1. Preliminaries

In sections 1 and 2 we work over an arbitrary ground field $\mathbb{F}$ of characteristic $p > 0$. Let B_n be the commutative associative unital algebra defined by a set of generators $x_1, \ldots, x_n$ and a set of relations $x_i^p = 0,\ i = 1, \ldots, n$. It has a basis over $\mathbb{F}$ consisting of all monomials $\prod_{i=1}^n x_i^{a_i}$ with $0 \le a_i < p$. Let $\mathfrak{m}$ denote the ideal of B_n generated by $x_1, \ldots, x_n$. Then $B_n/\mathfrak{m} \cong \mathbb{F}$ and $f^p = 0$ for all $f \in \mathfrak{m}$. So $\mathfrak{m}$ is the unique maximal ideal of B_n.

The Jacobson-Witt algebra W_n is the Lie algebra of all derivations of B_n. It is a free B_n-module with a basis $\partial_1, \ldots, \partial_n$ where

$$\partial_i x_j = \delta_{ij} \quad \text{for } 1 \le i, j \le n.$$

We denote by G the group of all automorphisms of B_n. Each $\sigma \in G$ is determined uniquely by its values on $x_1, \ldots, x_n$. More precisely, the assignment

$$\sigma \mapsto (\sigma x_1, \ldots, \sigma x_n)$$

gives a bijection between G and the set of n-tuples of elements of $\mathfrak{m}$ whose cosets modulo $\mathfrak{m}^2$ form a basis for the vector space $\mathfrak{m}/\mathfrak{m}^2$ over $\mathbb{F}$. Each $\sigma \in G$ induces an automorphism σ_* of W_n by the rule

$$\sigma_*(D) = \sigma \circ D \circ \sigma^{-1}, \qquad D \in W_n.$$

Clearly, $\sigma_*(D^p) = \sigma_*(D)^p$, that is, σ_* commutes with the p-power map on W_n. In fact all automorphisms of W_n are of the form σ_* for $\sigma \in G$, with the only exceptions for $p = 2$, $n \le 2$ and $p = 3$, $n = 1$. For $p > 3$ this was investigated by Jacobson [3]. A general result on isomorphisms of Cartan type Lie algebras which includes also cases of small characteristic is presented in [8, Th. 6.4].

For a proper ideal I of B_n put

$$N(I) = \{D \in W_n \mid D(I) \subset I\}.$$

There is a canonical homomorphism of p-Lie algebras $\pi : N(I) \to \operatorname{Der} B_n/I$. Note that

$$\operatorname{Ker} \pi = IW_n.$$

Indeed, $\operatorname{Ker} \pi = \{D \in W_n \mid D(B_n) \subset I\}$. Writing $D = \sum_{i=1}^n f_i \partial_i$ with $f_i = Dx_i$, we see that $D(B_n) \subset I$ if and only if $f_i \in I$ for all i. Both $N(I)$ and IW_n are p-Lie subalgebras of W_n. If $D \in IW_n$, then

$$D(fg) \equiv fD(g) \pmod{I^2} \quad \text{for all } f \in B_n \text{ and } g \in I;$$

since $I \subset \mathfrak{m}$, it follows that $D(\mathfrak{m}I) \subset \mathfrak{m}I$. Hence we get a canonical homomorphism of p-Lie algebras

$$\lambda : IW_n \to \mathfrak{gl}(I/\mathfrak{m}I).$$

Lemma 1.1. *Given $D \in W_n$, let I be the maximal D-invariant ideal of B_n.*

(i) *I is G-conjugate to the ideal generated by $x_{k+1}, \ldots, x_n$ for some k, $0 \le k \le n$.*

(ii) *D is nilpotent if and only if $D^{p^k} \in IW_n$ and $\lambda(D^{p^k})$ is nilpotent.*

Proof. The factor algebra B_n/I has a maximal ideal $\mathfrak{n} = \mathfrak{m}/I$ whose residue field is isomorphic with $\mathbb{F}$. Since this algebra is D-simple in the sense that it has no nontrivial ideals stable under its derivation induced by D, we get $B_n/I \cong B_k$ for some k by Block's description of differentiably simple algebras [1] (see also [11, Ch. 3]). Note that $\dim \mathfrak{m}/\mathfrak{m}^2 = n$, while $\dim \mathfrak{n}/\mathfrak{n}^2 = k$. There is an exact sequence of vector spaces

$$I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Let $e_1, \dots, e_n$ be any basis for $\mathfrak{m}/\mathfrak{m}^2$ such that $e_{k+1}, \dots, e_n$ span the image of $I/\mathfrak{m}I$. Pick representatives $y_1, \dots, y_n \in \mathfrak{m}$ of these n cosets, taking $y_{k+1}, \dots, y_n$ in I. There exists $\sigma \in G$ such that $\sigma(x_i) = y_i$ for all $i = 1, \dots, n$. Hence the monomials $\prod_{i=1}^n y_i^{a_i}$ with $0 \le a_i < p$ form a basis for B_n over $\mathbb{F}$. It follows that the ideal I' of B_n generated by $y_{k+1}, \dots, y_n$ has codimension p^k. Since $I' \subset I$, and I also has codimension p^k in B_n, we get $I' = I$. Now it is clear that the ideal $\sigma^{-1}(I)$ is generated by the elements $x_{k+1}, \dots, x_n$.

Suppose that D is nilpotent. Then $\pi(D)$ is a nilpotent derivation of B_n/I. Since $\dim B_n/I = p^k$, we must have $\pi(D)^{p^k} = 0$, whence $D^{p^k} \in \operatorname{Ker} \pi = IW_n$. The linear transformation of $I/\mathfrak{m}I$ induced by a nilpotent derivation D^{p^k} must be nilpotent.

Conversely, suppose that $D^{p^k} \in IW_n$ and $\lambda(D^{p^k})$ is nilpotent. We can find an integer $s \ge k$ such that $\lambda(D^{p^s}) = 0$. The derivation $D^{p^s} \in IW_n$ induces a B_n/I-linear endomorphism φ of the B_n/I-module $M = I/I^2$. By our choice of s we have $\varphi(M) \subset \mathfrak{m}M$. Hence $\varphi^j(M) \subset \mathfrak{m}^j M$ for all $j \ge 0$. Since $\mathfrak{m}$ is a nilpotent ideal, there exists an integer $t \ge 0$ such that $\varphi^{p^t} = 0$. Then the derivation $D^{p^{s+t}} \in IW_n$ maps I into I^2. It follows that $D^{p^{s+t}}(I^a) \subset I^{a+1}$ for all integers $a \ge 0$ by induction. But $I^a = 0$ for sufficiently large a. Hence $D^{p^{s+t}}$ is nilpotent, and so is D. □

We will need partial information about powers of certain derivations. Their computation is based on Jacobson's formula for $(a+b)^p$ in any restricted Lie algebra (see [10, Ch. 2]).

Lemma 1.2. *Let J be any ideal of B_n. Suppose that $D_1 \in W_n$ and $D_2 \in J^m W_n$ where $m \ge p-1$. Then*

$$(D_1 + D_2)^p \equiv D_1^p + \operatorname{ad}(D_1)^{p-1} D_2 \pmod{J^{m-p+2} W_n}.$$

Hence $(D_1 + D_2)^p \equiv D_1^p \pmod{J^{m-p+1} W_n}$.

Proof. Note that $[W_n, J^a W_n] \subset J^{a-1} W_n$ and $[JW_n, J^a W_n] \subset J^a W_n$ for all integers $a > 0$. Since $J^m W_n$ is a p-Lie subalgebra of W_n, we have $D_2^p \in J^m W_n$, and the conclusion follows from Jacobson's formula. □

In the next lemma and later in the paper we adopt the convention that any product over the empty set of indices is considered to be 1. So $\prod_{i=1}^0 x_i^{p-1} = 1$.

Lemma 1.3. *For some fixed integer k, $1 \le k \le n$, let J be the ideal of B_n generated by $x_1, \dots, x_k$, and let L be the B_n-submodule of W_n generated by $\partial_{k+1}, \dots, \partial_n$. If*

$$D = D_1 + D_2 + \Big(\prod_{i=1}^k x_i^{p-1}\Big) D_3$$

where $\quad D_1 = \sum_{j=1}^k \Big(\prod_{i=1}^{j-1} x_i^{p-1}\Big)\partial_j, \quad D_2 \in L, \quad D_3 \in W_n, \quad$ *then*

$$D^{p^k} \equiv (D_1 + D_2)^{p^k} + (-1)^k D_3 \pmod{JW_n}.$$

Proof. By Lemma 1.2

$$D^p \equiv (D_1+D_2)^p + \operatorname{ad}(D_1+D_2)^{p-1}\Big(\big(\prod_{i=1}^{k} x_i^{p-1}\big)D_3\Big) \pmod{J^{(k-1)(p-1)+1}W_n}.$$

From the equality $L = \{D' \in W_n \mid D'x_i = 0 \text{ for all } i = 1,\dots,k\}$ it is clear that L is a p-Lie subalgebra of W_n. Since $[D_1, \partial_i] = 0$ for all $i = k+1,\dots,n$, we have $[D_1, L] \subset L$. Now Jacobson's formula yields

$$(D_1+D_2)^p = D_1^p - D_2' \quad \text{for some } D_2' \in L.$$

Since $D_2 x_i = 0$ for all $i = 1,\dots,k$, we have

$$\begin{aligned}\operatorname{ad}(D_1+D_2)^{p-1}\Big(\big(\prod_{i=1}^{k} x_i^{p-1}\big)D_3\Big) &\equiv \operatorname{ad}(\partial_1)^{p-1}\Big(\big(\prod_{i=1}^{k} x_i^{p-1}\big)D_3\Big)\\ &\equiv -\big(\prod_{i=2}^{k} x_i^{p-1}\big)D_3 \pmod{J^{(k-1)(p-1)+1}W_n}.\end{aligned}$$

Hence

$$D^p \equiv D_1^p - D_2' - \big(\prod_{i=2}^{k} x_i^{p-1}\big)D_3 \pmod{J^{(k-1)(p-1)+1}W_n}.$$

Applying Lemma 1.2 again, we get $D^{p^k} \equiv -E^{p^{k-1}} \pmod{JW_n}$ where

$$E = -D_1^p + D_2' + \big(\prod_{i=2}^{k} x_i^{p-1}\big)D_3.$$

By a computation in [5, Lemma 3]

$$D_1^p = -\sum_{j=2}^{k}\big(\prod_{i=2}^{j-1} x_i^{p-1}\big)\partial_j.$$

Thus E has a similar structure as D, but with respect to the $k-1$ elements $x_2,\dots,x_k$ instead of k elements $x_1, x_2,\dots,x_k$. Proceeding by induction on k, we may assume that

$$E^{p^{k-1}} \equiv -(D_1^p - D_2')^{p^{k-1}} + (-1)^{k-1}D_3 \pmod{JW_n}.$$

Then the required formula follows. □

In the case $k = n$ of Lemma 1.3 we have $L = 0$, and the derivation D is a sum of two terms. Lemma 1.4 below describes properties of D. In a slightly different realization such derivations appeared in [5, Lemma 12]. These derivations are characterized by the property that they have trivial stabilizer in G.

Lemma 1.4. *Let* $D = D_1 + \big(\prod_{i=1}^{n} x_i^{p-1}\big)D_2$ *where* $D_1 = \sum_{j=1}^{n}\big(\prod_{i=1}^{j-1} x_i^{p-1}\big)\partial_j$, $D_2 = \sum_{j=1}^{n} \alpha_j\partial_j$ *with* $\alpha_j \in F$. *Then:*

(i) *The derivations* $D^{p^{i-1}}$ *with* $i = 1,\dots,n$ *form a basis for* W_n *over* B_n.

(ii) B_n *has no nontrivial* D*-invariant ideals* (*in other words,* B_n *is* D*-simple*).

(iii) $D^{p^n} = \sum\limits_{i=1}^{n} (-1)^{n-i+1}\alpha_i D^{p^{i-1}}$.

(iv) *D is nilpotent if and only if $\alpha_i = 0$ for all i.*

Proof. Proceeding as in Lemma 1.3, we get

$$D^{p^r} \equiv D_1^{p^r} + (-1)^r \Big(\prod_{i=r+1}^{n} x_i^{p-1}\Big) D_2 \pmod{\mathfrak{m}^{(n-r)(p-1)+1}W_n},$$

$$D_1^{p^r} = (-1)^r \sum_{j=r+1}^{n} \Big(\prod_{i=r+1}^{j-1} x_i^{p-1}\Big)\partial_j$$

for all $r = 0, \ldots, n$. Hence $D^{p^r} \equiv (-1)^r \partial_{r+1} \pmod{\mathfrak{m}W_n}$ for all $r = 0, \ldots, n-1$ and $D^{p^n} \equiv (-1)^n D_2 \pmod{\mathfrak{m}W_n}$. We see that the cosets $D^{p^{i-1}} + \mathfrak{m}W_n$ with $i = 1, \ldots, n$ form a basis for the vector space $W_n/\mathfrak{m}W_n$ over $\mathbb{F}$. Then $\{D^{p^{i-1}} \mid i = 1, \ldots, n\}$ is a generating set for the B_n-module W_n by Nakayama's Lemma; since this module is free, this set is its basis. Thus (i) is proved.

It follows from (i) that each D-invariant ideal of B_n is stable under all derivations; but there are only trivial ideals with this property. Also by (i) we can write

$$D^{p^n} = \sum_{i=1}^{n} f_i D^{p^{i-1}} \quad \text{with } f_i \in B_n.$$

Since D centralizers all powers of D, we get $\sum_{i=1}^{n} D(f_i) D^{p^{i-1}} = [D, D^{p^n}] = 0$, whence $D(f_i) = 0$ for all i. Then $f_1, \ldots, f_n$ are annihilated by all derivations of B_n, and it follows that $f_i \in \mathbb{F}$ for all i. Looking at the cosets modulo $\mathfrak{m}W_n$, we deduce from the earlier congruences that $f_i = (-1)^{n-i+1}\alpha_i$. Since $\dim B_n = p^n$, we have $D^{p^n} = 0$ whenever D is nilpotent. Hence (iv) follows from (i) and (iii). □

Lemma 1.5. *Let $D \in W_n$ be any derivation such that B_n is D-simple. Then:*

(i) *The derivations $D^{p^{i-1}}$ with $i = 1, \ldots, n$ form a basis for W_n over B_n.*

(ii) *The n by n matrix $\Big[D^{p^{i-1}} x_j\Big]_{1 \le i,j \le n}$ with entries in B_n is invertible.*

Proof. Denote by $\mathfrak{a}$ the p-Lie subalgebra of W_n generated by D. If $D' \in \mathfrak{a} \cap \mathfrak{m}W_n$ then the ideal $B_n \cdot D'(B_n)$ of B_n is D-invariant since $[D, D'] = 0$. As this ideal is contained in $\mathfrak{m}$ and B_n is D-simple, we conclude that $D' = 0$. Thus $\mathfrak{a} \cap \mathfrak{m}W_n = 0$. Let $u(\mathfrak{a})$ denote the restricted universal enveloping algebra of $\mathfrak{a}$. The dual space $u(\mathfrak{a})^*$ has a canonical algebra structure, and $\mathfrak{a}$ operates on $u(\mathfrak{a})^*$ via derivations. The natural action of $\mathfrak{a}$ on B_n gives rise to an $\mathfrak{a}$-equivariant homomorphism of algebras $\varphi : B_n \to u(\mathfrak{a})^*$ (see [11, Th. 3.3.1]), which is an isomorphism by [7, Th. 3.2]. For our purposes it suffices to know that φ is injective, but this follows from the facts that $\operatorname{Ker}\varphi$ is a proper D-invariant ideal of B_n and B_n is D-simple. Since $\dim B_n = p^n$ and $\dim u(\mathfrak{a})^* = p^{\dim \mathfrak{a}}$, injectivity of φ entails $\dim \mathfrak{a} \ge n$. Then we must have

$$\dim \mathfrak{a} = n \qquad \text{and} \qquad W_n = \mathfrak{a} \oplus \mathfrak{m}W_n.$$

It follows that the set $\{D^{p^{i-1}} \mid i = 1, \ldots, n\}$ is a basis for $\mathfrak{a}$ over $\mathbb{F}$ and also a basis for W_n over B_n by Nakayama's Lemma.

Since $D^{p^{i-1}} = \sum_{j=1}^{n} (D^{p^{i-1}} x_j)\, \partial_j$, the matrix in (ii) is the transition matrix from one basis $(\partial_i)_{i=1,\ldots,n}$ of the B_n-module W_n to another basis $(D^{p^{i-1}})_{i=1,\ldots,n}$. The invertibility of this matrix is immediate. □

2. The normal form of elements

Lemma 2.1. *Let $D \in W_n$. Suppose that for some k, $1 \le k \le n$, the k by k matrix*

$$\Big[D^{p^{i-1}} x_j\Big]_{1\le i,j\le k}$$

is invertible. Denote by J the ideal of B_n generated by $x_1, \ldots, x_k$. Then there is a unique k-tuple $(y_1, \ldots, y_k)$ such that the set $\{y_1, \ldots, y_k\}$ generates the ideal J and

$$Dy_1 \equiv 1 \pmod{J^{k(p-1)}},$$

$$Dy_j \equiv \prod_{i=1}^{j-1} y_i^{p-1} \pmod{J^{k(p-1)}} \quad \text{for all } j = 2, \ldots, k.$$

Proof. Put $V = J/\mathfrak{m}J$. This vector space has a basis consisting of the cosets of $x_1, \ldots, x_k$. Let U be the vector subspace in W_n spanned by $\{D^{p^{i-1}} \mid i = 1, \ldots, k\}$. There is a bilinear pairing $U \times V \to B_n/\mathfrak{m} \cong \mathbb{F}$ given by the rule

$$\langle D', f + \mathfrak{m}J\rangle = D'(f) + \mathfrak{m} \quad \text{for } D' \in U \text{ and } f \in J.$$

The invertibility of the matrix in the statement of the Lemma can be rephrased in terms of the nondegeneracy of this pairing. Indeed, since $\mathfrak{m}$ is a nilpotent ideal of B_n and the matrix of the pairing coincides with the reduction modulo $\mathfrak{m}$ of the former matrix, one matrix is invertible if and only if so is the other. Note that the vector space V and the pairing $U \times V \to \mathbb{F}$ depend only on the ideal J, but not on the particular set of its generators $x_1, \ldots, x_k$.

By Nakayama's lemma arbitrary k elements $t_1, \ldots, t_k \in J$ generate the ideal J if and only if their cosets modulo $\mathfrak{m}J$ span V, in which case these cosets give another basis for V. If this condition is satisfied, then the n elements $t_1, \ldots, t_k, x_{k+1}, \ldots, x_n$ generate the algebra B_n. Indeed, since V embeds in $\mathfrak{m}/\mathfrak{m}^2$, the cosets of those elements modulo $\mathfrak{m}^2$ form a basis for $\mathfrak{m}/\mathfrak{m}^2$. Moreover, all the assumptions of the lemma remain intact if the original n-tuple of generators $(x_1, \ldots, x_n)$ is replaced with $(t_1, \ldots, t_k, x_{k+1}, \ldots, x_n)$.

Passing to new generators, we may assume from the very beginning that the basis $\{D^{p^{i-1}} \mid i = 1, \ldots, k\}$ for U and the basis $\{x_i + \mathfrak{m}J \mid i = 1, \ldots, k\}$ for V are dual to each other. Thus

$$D^{p^{i-1}} x_j \equiv \delta_{ij} \pmod{\mathfrak{m}}, \qquad 1 \le i, j \le k.$$

Now look at the following statements:

(i) *There exist elements $y_1, \ldots, y_k$ generating the ideal J and invertible elements $u_1, \ldots, u_k$ of B_n such that*

$$Dy_1 = u_1, \qquad Dy_j = u_j \prod_{i=1}^{j-1} y_i^{p-1} \quad \text{for all } j = 2, \ldots, k.$$

(ii) $\quad B_n = D(J) \oplus J^{k(p-1)} \quad \text{and} \quad J \cap \operatorname{Ker} D = 0.$

We claim that (i) implies (ii). Suppose that (i) holds. For each integer r such that $0 \le r < p^k$ let r_i be the coefficients in the p-adic expansion

$$r = \sum_{i=0}^{k-1} r_i p^i, \qquad 0 \le r_i < p,$$

and put $\quad y^{(r)} = \prod_{i=1}^{k} \dfrac{y_i^{r_{i-1}}}{r_{i-1}!}$.

Denote by J_r the ideal of B_n generated by $\{y^{(s)} \mid r \le s < p^k\}$. Then we get a chain of ideals $J_0 \supset J_1 \supset \cdots \supset J_{p^k-1}$ with $J_0 = B_n,\ J_1 = J$. The last ideal J_{p^k-1} is generated by a single element

$$y^{(p^k-1)} = (-1)^k \prod_{i=1}^{k} y_i^{p-1}.$$

Since J is generated by $y_1, \ldots, y_k$ and $y_i^p = 0$ for all i, we have $J^{k(p-1)} = J_{p^k-1}$. Put $J_{p^k} = 0$ for later use.

Let us evaluate the derivation D at $y^{(r)}$ for $r > 0$. Since $y_i^p = 0$ for all i, we have

$$Dy^{(r)} = (Dy_m)\frac{y_m^{r_{m-1}-1}}{(r_{m-1}-1)!} \prod_{i=m+1}^{k} \frac{y_i^{r_{i-1}}}{r_{i-1}!} = (-1)^{m-1} u_m y^{(r-1)}$$

where $m = \min\{i \mid r_{i-1} \ne 0\}$. It follows that $D(J_r) \subset J_{r-1}$, and therefore for each $r,\ 0 < r < p^k$, there is a well-defined map

$$\varphi_r : J_r/J_{r+1} \to J_{r-1}/J_r$$

induced by D. Since $D(fg) \equiv f(Dg) \pmod{J_r}$ for all $f \in B_n$ and $g \in J_r$, the above map is B_n-linear. Note that J_r/J_{r+1} and J_{r-1}/J_r are cyclic free B_n-modules generated by the respective cosets of $y^{(r)}$ and $y^{(r-1)}$. The earlier computation shows that φ_r takes the coset of $y^{(r)}$ to the coset of $(-1)^{m-1}u_m y^{(r-1)}$. Since u_m is an invertible element of B_n, we deduce that φ_r is an isomorphism of B_n-modules.

It follows that $J_{r-1} \subset D(J_r) + J_r$ and $J_r \cap \operatorname{Ker} D \subset J_{r+1}$. These inclusions enable us to prove by induction that $J_0 \subset D(J) + J_r$ and $J \cap \operatorname{Ker} D \subset J_{r+1}$ for all r such that $0 < r < p^k$. Taking $r = p^k - 1$, we get

$$B_n = D(J) + J^{k(p-1)} \qquad \text{and} \qquad J \cap \operatorname{Ker} D = 0.$$

The second equality entails $\dim D(J) = \dim J$. On the other hand, $J^{k(p-1)}$ coincides with the annihilator of J in B_n. Therefore the multiplication by $y^{(p^k-1)}$ induces a vector space isomorphism of B_n/J onto $J^{k(p-1)}$. Hence

$$\dim B_n = \dim J + \dim J^{k(p-1)} = \dim D(J) + \dim J^{k(p-1)}.$$

We see that the sum of $D(J)$ and $J^{k(p-1)}$ must be direct. Thus (ii) follows from (i), as claimed.

Now note that (ii) implies the conclusion of the lemma. Indeed, by (ii) there exists a unique $y_1 \in J$ such that $Dy_1 \equiv 1 \pmod{J^{k(p-1)}}$. Once y_1 is known, we deduce that there exists a unique $y_2 \in J$ such that $Dy_2 \equiv y_1^{p-1} \pmod{J^{k(p-1)}}$, and so on. Since the derivations D, D^p, $D^{p^2}, \ldots$ map J^a to J^{a-1} for all $a > 0$, it follows by induction on j and i that

$$D^{p^{i-1}} y_j \equiv \prod_{s=i}^{j-1} y_s^{p-1} \pmod{J^{(k-i+1)(p-1)}}, \qquad 1 \le i \le j \le k.$$

In particular, $D^{p^{j-1}} y_j \equiv 1 \pmod{J^{(k-j+1)(p-1)}}$, but then $D^{p^{i-1}} y_j \in J^{(k-i+1)(p-1)}$ whenever $j < i \le k$. Hence $D^{p^{i-1}} y_j \equiv \delta_{ij} \pmod{\mathfrak{m}}$ for all $i, j \in \{1, \ldots, k\}$. The nondegeneracy of the pairing $U \times V \to \mathbb{F}$ considered earlier entails $y_i + \mathfrak{m}J = x_i + \mathfrak{m}J$ for all $i = 1, \ldots, k$, and we conclude that $y_1, \ldots, y_k$ generate the ideal J.

So it suffices to prove (i). We shall do this proceeding by induction on k. If $k = 1$, property (i) means just that $Dy_1 \notin \mathfrak{m}$. Hence we may take $y_1 = x_1$. Suppose that $k > 1$ and (i) holds for the ideal J' of B_n generated by $k-1$ elements $x_1, \ldots, x_{k-1}$. Then there exist elements $y_1, \ldots, y_{k-1}$ generating the ideal J' and invertible elements $u_1, \ldots, u_{k-1}$ of B_n such that

$$Dy_1 = u_1, \qquad Dy_j = u_j \prod_{i=1}^{j-1} y_i^{p-1} \quad \text{for all } j = 2, \ldots, k-1.$$

As we have seen, (i) implies that (ii) also holds for J'. In particular,

$$B_n = D(J') \oplus J'^{\,(k-1)(p-1)}.$$

So we can find $v \in J'$ such that $Dx_k - Dv \in J'^{\,(k-1)(p-1)}$. Take $y_k = x_k - v$. The ideal of B_n generated by $y_1, \ldots, y_k$ coincides with J since it contains J' as well as $x_k = y_k + v$. The ideal $J'^{\,(k-1)(p-1)}$ is generated by the element $\prod_{i=1}^{k-1} y_i^{p-1}$. Hence

$$Dy_k = Dx_k - Dv = u_k \prod_{i=1}^{k-1} y_i^{p-1}$$

for some $u_k \in B_n$. It remains only to show that u_k is an invertible element of B_n. Since $D^{p^{k-1}} x_j \in \mathfrak{m}$ for all $j = 1, \ldots, k-1$, we deduce that $D^{p^{k-1}}(J') \subset \mathfrak{m}$. In particular, $D^{p^{k-1}} v \in \mathfrak{m}$. Hence

$$D^{p^{k-1}} y_k = D^{p^{k-1}} x_k - D^{p^{k-1}} v \notin \mathfrak{m}.$$

On the other hand, $D^{p^{k-1}-1}$ induces a B_n-linear map $J'^{\,(k-1)(p-1)} \to B_n/J'$. In fact this map coincides with the composite of all the maps φ_r, $0 < r < p^{k-1}$, introduced earlier, but with J' in place of J. Hence

$$D^{p^{k-1}} y_k = D^{p^{k-1}-1}\big(u_k \prod_{i=1}^{k-1} y_i^{p-1}\big) \equiv u_k\, D^{p^{k-1}-1}\big(\prod_{i=1}^{k-1} y_i^{p-1}\big) \pmod{J'},$$

and it follows that $u_k \notin \mathfrak{m}$, as required. □

Denote by I_k the ideal of B_n generated by $x_{k+1}, \dots, x_n$. In particular, $I_0 = \mathfrak{m}$ and $I_n = 0$. Put

$$\mathcal{D}_k = \Big\{ \sum_{j=1}^{n} f_j \partial_j \mid \; f_j \equiv \prod_{i=1}^{j-1} x_i^{p-1} \pmod{B_n \cdot \prod_{i=1}^{k} x_i^{p-1}} \text{ for all } j = 1, \dots, k \\ \text{and } \; f_j \in I_k \; \text{ for all } j = k+1, \dots, n \Big\},$$

$$\mathcal{D}'_k = \Big\{ \sum_{j=1}^{n} f_j \partial_j \mid \; f_j \equiv \prod_{i=1}^{j-1} x_i^{p-1} \pmod{\mathfrak{m} \cdot \prod_{i=1}^{k} x_i^{p-1}} \text{ for all } j = 1, \dots, k \\ \text{and } \; f_j \in I_k \; \text{ for all } j = k+1, \dots, n \Big\}.$$

Clearly I_k is stable under every derivation in $\mathcal{D}_k$. Hence $\mathcal{D}'_k \subset \mathcal{D}_k \subset N(I_k)$.

Lemma 2.2. *For any $D \in \mathcal{D}_k$ we have:*

(i) *I_k is the maximal D-invariant ideal of B_n.*

(ii) *$D^{p^k} \in I_k W_n$ if and only if $D \in \mathcal{D}'_k$.*

(iii) *D is nilpotent if and only if $D \in \mathcal{D}'_k$ and $\lambda(D^{p^k})$ is nilpotent.*

Proof. Let $D = \sum_{j=1}^{n} f_j \partial_j$ with $f_1, \dots, f_n \in B_n$. For each $j = 1, \dots, k$ we can write

$$f_j = \prod_{i=1}^{j-1} x_i^{p-1} + g_j \prod_{i=1}^{k} x_i^{p-1}$$

where g_j lies in the subalgebra of B_n generated by $x_{k+1}, \dots, x_n$. Let $\alpha_j \in \mathbb{F}$ be such that $g_j - \alpha_j \in \mathfrak{m}$. Then $g_j - \alpha_j \in I_k$ as well. Let $\pi : N(I_k) \to \operatorname{Der} B_n/I_k$ be the canonical map. Recall that $\operatorname{Ker} \pi = I_k W_n$. Identifying B_n/I_k with B_k, we get

$$\pi(D) = \sum_{j=1}^{k} \big(\prod_{i=1}^{j-1} x_i^{p-1} \big) \partial_j + \big(\prod_{i=1}^{k} x_i^{p-1} \big) \sum_{j=1}^{k} \alpha_j \partial_j .$$

Thus $\pi(D)$, regarded as an element of $W_k = \operatorname{Der} B_k$, satisfies the hypotheses of Lemma 1.4. By that lemma the algebra B_n/I_k is $\pi(D)$-simple, but this is equivalent to statement (i) of Lemma 2.2.

By Lemma 1.4 $\pi(D)$ is nilpotent if and only if $\alpha_j = 0$ for all $j = 1, \dots, k$. Furthermore, $\pi(D)$ is nilpotent if and only if $\pi(D)^{p^k} = 0$ or, equivalently, $D^{p^k} \in \operatorname{Ker} \pi$. On the other hand, vanishing of $\alpha_1, \dots, \alpha_k$ means that $g_j \in \mathfrak{m}$ for all $j = 1, \dots, k$, that is, $D \in \mathcal{D}'_k$. This yields (ii). Now (iii) follows from Lemma 1.1. □

We will denote by G_k the subgroup of G consisting of all automorphisms σ of B_n which satisfy the following two conditions:

(1) the ideal of B_n generated by $x_1, \dots, x_k$ is stable under σ,

(2) $\sigma(x_i) = x_i$ for all $i = k+1, \dots, n$.

Proposition 2.3. *Let $D \in W_n$ be any derivation with the maximal D-invariant ideal of B_n equal to I_k. There exists a unique $\sigma \in G_k$ such that $D \in \sigma_*(\mathcal{D}_k)$.*

Proof. Let us check that D satisfies the hypothesis of Lemma 2.1. Consider the algebra $B_n/I_k \cong B_k$ and its derivation $\pi(D)$ induced by D. By the assumption about D this algebra is $\pi(D)$-simple. Put $\overline{x}_i = x_i + I_k$ for $i = 1, \ldots, k$. Lemma 1.5, applied to $\pi(D)$, shows that $\left[\pi(D)^{p^{i-1}}\overline{x}_j\right]_{1\le i,j\le k}$ is an invertible matrix with entries in B_n/I_k. But

$$\pi(D)^{p^{i-1}}\overline{x}_j = D^{p^{i-1}}x_j + I_k.$$

Since I_k is a nilpotent ideal of B_n, it follows that the matrix $\left[D^{p^{i-1}}x_j\right]_{1\le i,j\le k}$ with entries in B_n is also invertible.

Thus Lemma 2.1 applies. Let $(y_1, \ldots, y_k)$ be the k-tuple given by the conclusion of that lemma. Note that $D \in \mathcal{D}_k$ if and only if $y_i = x_i$ for all $i = 1, \ldots, k$. In fact, writing $D = \sum_{j=1}^n f_j\partial_j$ with $f_j = Dx_j$, we have $f_j \in I_k$ for all $j = k+1, \ldots, n$ since $D(I_k) \subset I_k$. At the same time the condition on $f_1, \ldots, f_k$ in the definition of $\mathcal{D}_k$ amounts to the conclusion of Lemma 2.1 for the k-tuple $(x_1, \ldots, x_k)$.

If $\sigma \in G_k$, then $\sigma(I_k) = I_k$. Therefore the maximal $\sigma_*^{-1}(D)$-invariant ideal of B_n coincides with I_k as well, but the k-tuple of Lemma 2.1 defined with respect to the derivation $\sigma_*^{-1}(D)$ changes to $(\sigma^{-1}y_1, \ldots, \sigma^{-1}y_k)$. It follows that $\sigma_*^{-1}(D) \in \mathcal{D}_k$ if and only if $\sigma(x_i) = y_i$ for all $i = 1, \ldots, k$. So it remains to observe that there exists a unique $\sigma \in G_k$ with this property. □

Corollary 2.4. *Suppose that $D \in W_n$ is such that B_n is D-simple. Then there is a unique $\sigma \in G$ such that $D \in \sigma_*(\mathcal{D}_n)$. In particular, D has trivial stabilizer in G.*

Proof. In the special case $k = n$ we have $I_k = 0$ and $G_k = G$ since the ideal $\mathfrak{m}$ generated by $x_1, \ldots, x_n$ is stable under G. □

Corollary 2.5. *The set $\mathcal{N}_{\mathrm{reg}} = \{D \in \mathcal{N}(W_n) \mid B_n$ is D-simple$\}$ is a single G-orbit.*

Proof. By Lemma 2.2 a derivation in $\mathcal{D}_n$ is nilpotent if and only if it lies in $\mathcal{D}'_n$. Hence, by Corollary 2.4, any G-orbit in $\mathcal{N}_{\mathrm{reg}}$ intersects $\mathcal{D}'_n$. However, $\mathcal{D}'_n$ contains only one element. □

In [5] Premet proved, assuming $\mathbb{F}$ to be algebraically closed, that the G-orbit of the derivation in $\mathcal{D}'_n$ is dense in $\mathcal{N}(W_n)$.

3. Counting arguments

In this section we assume that $\mathbb{F} = \mathbb{F}_q$ where q is a power of a prime p. We will denote by $\# X$ the cardinality of a finite set X.

Lemma 3.1. *Let $\mathfrak{g} = \mathfrak{gl}(I_k/\mathfrak{m}I_k)$, and let $\varphi : \mathcal{D}'_k \to \mathfrak{g}$ be the map defined by the rule $\varphi(D) = \lambda(D^{p^k})$ for $D \in \mathcal{D}'_k$. Then*

$$\#\,\varphi^{-1}(A) = q^{k(p^{n-k}-1)+(n-k)(p^n-p^k)-(n-k)^2} \quad \textit{for each } A \in \mathfrak{g}.$$

Proof. Note that each $D \in \mathcal{D}'_k$ can be written as in Lemma 1.3 with

$$D_2 \in \sum_{j=k+1}^{n} I_k\partial_j \qquad \text{and} \qquad D_3 \in I_kW_n.$$

Then $D_1 + D_2 \in \mathcal{D}'_k$ as well, whence both D^{p^k} and $(D_1+D_2)^{p^k}$ lie in I_kW_n. Denote by J the ideal of B_n generated by $x_1, \ldots, x_k$. Applying Lemma 1.3, we get

$$D^{p^k} - (D_1 + D_2)^{p^k} - (-1)^k D_3 \in I_k W_n \cap J W_n = (I_k \cap J) W_n \subset \mathfrak{m} I_k W_n \subset \operatorname{Ker} \lambda.$$

It follows that

$$\varphi(D) = \lambda(D^{p^k}) = \lambda\big((D_1 + D_2)^{p^k}\big) + (-1)^k \lambda(D_3) = \varphi(D_1 + D_2) + (-1)^k \lambda(D_3).$$

If $D' \in I_k W_n$, then

$$D + \big(\prod_{i=1}^{k} x_i^{p-1}\big) D' = D_1 + D_2 + \big(\prod_{i=1}^{k} x_i^{p-1}\big)(D_3 + D') \in \mathcal{D}'_k, \qquad \text{and}$$

$$\varphi\Big(D + \big(\prod_{i=1}^{k} x_i^{p-1}\big) D'\Big) = \varphi(D_1 + D_2) + (-1)^k \lambda(D_3 + D') = \varphi(D) + (-1)^k \lambda(D').$$

The map $\lambda : I_k W_n \to \mathfrak{g}$ is surjective since $\{\lambda(x_i \partial_j) \mid k < i, j \le n\}$ is a basis for $\mathfrak{g}$. Given $A_1, A_2 \in \mathfrak{g}$, we can find $D' \in I_k W_n$ such that $\lambda(D') = (-1)^k (A_2 - A_1)$. We see that $\varphi(D) = A_1$ if and only if $\varphi\big(D + (\prod_{i=1}^k x_i^{p-1}) D'\big) = A_2$, and so there is a bijection between $\varphi^{-1}(A_1)$ and $\varphi^{-1}(A_2)$. Thus any two fibres of φ have the same cardinality. Since $\mathcal{D}'_k$ is an affine translation of the vector subspace

$$V = \sum_{j=1}^{k} \mathfrak{m} t \partial_j + \sum_{j=k+1}^{n} I_k \partial_j \subset W_n \quad \text{where } t = \prod_{i=1}^{k} x_i^{p-1},$$

it follows that

$$\#\,\varphi^{-1}(A) = \#\,\mathcal{D}'_k \,/\, \#\,\mathfrak{g} = q^{\dim V - \dim \mathfrak{g}} = q^{k(\dim \mathfrak{m} t) + (n-k)(\dim I_k) - (n-k)^2}.$$

The isomorphism $B_n / I_k \cong B_k$ shows that I_k has codimension p^k in B_n. Therefore $\dim I_k = p^n - p^k$. Denote by J the ideal of B_n generated by $x_1, \ldots, x_k$. The multiplication by t induces a vector space isomorphism between B_n / J and $B_n t$. Since $B_n / J \cong B_{n-k}$, we get $\dim B_n t = p^{n-k}$. Since $B_n t / \mathfrak{m} t$ is spanned by the coset of t, we deduce that $\dim \mathfrak{m} t = p^{n-k} - 1$. □

Lemma 3.2. *The group G_k has order $q^{k(p^n - p^{n-k} - k)} \prod_{i=1}^{k} (q^k - q^{i-1})$.*

Proof. Denote by J the ideal of B_n generated by $x_1, \ldots, x_k$. Each automorphism $\sigma \in G_k$ is determined uniquely by its values on $x_1, \ldots, x_k$. The condition $\sigma(J) = J$ in the definition of G_k means precisely that $\sigma(x_1), \ldots, \sigma(x_k)$ generate the ideal J. Hence the assignment $\sigma \mapsto \big(\sigma(x_1), \ldots, \sigma(x_k)\big)$ gives a bijection between G_k and the set of k-tuples of elements generating J as an ideal. By Nakayama's lemma arbitrary k elements $t_1, \ldots, t_k \in J$ generate the ideal J if and only if their cosets modulo $\mathfrak{m} J$ form a basis for the vector space $J / \mathfrak{m} J$. As is well known, the number of different bases for a k-dimensional vector space over $\mathbb{F}_q$ equals $\prod_{i=1}^k (q^k - q^{i-1})$. Furthermore, $q^{\dim \mathfrak{m} J}$ is the number of elements in each coset modulo $\mathfrak{m} J$. Hence $q^{k(\dim \mathfrak{m} J)}$ is the number of possible ways to choose representatives of k cosets modulo $\mathfrak{m} J$, and it follows that

$$\#\, G_k = q^{k(\dim \mathfrak{m} J)} \prod_{i=1}^{k} (q^k - q^{i-1}).$$

Since $B_n/J \cong B_{n-k}$, we have $\dim J = p^n - p^{n-k}$. Then $\dim \mathfrak{m}J = p^n - p^{n-k} - k$, and we are done. $\square$

Lemma 3.3. *Denote by $\mathcal{I}_k$ the set of all ideals I of B_n such that $B_n/I \cong B_k$. Then*

$$\#\mathcal{I}_k = q^{(n-k)(p^k-1-k)} \prod_{i=1}^{k} \frac{q^n - q^{i-1}}{q^k - q^{i-1}}.$$

Proof. Each ideal $I \in \mathcal{I}_k$ is generated by a set of $n-k$ elements lying in $\mathfrak{m}$ whose cosets modulo $\mathfrak{m}^2$ are linearly independent over $\mathbb{F}_q$. There are $\prod_{i=1}^{n-k}(q^n - q^{i-1})$ possible ways to choose an $(n-k)$-tuple of linearly independent vectors in the n-dimensional vector space $\mathfrak{m}/\mathfrak{m}^2$. Hence

$$q^{(n-k)(\dim \mathfrak{m}^2)} \prod_{i=1}^{n-k} (q^n - q^{i-1})$$

is the number of $(n-k)$-tuples of elements generating an ideal in $\mathcal{I}_k$. The same ideal I can be generated in

$$q^{(n-k)(\dim \mathfrak{m}I)} \prod_{i=1}^{n-k} (q^{n-k} - q^{i-1})$$

different ways. So it follows that

$$\#\mathcal{I}_k = \frac{q^{(n-k)(\dim \mathfrak{m}^2)}}{q^{(n-k)(\dim \mathfrak{m}I)}} C_{n,\,n-k} \qquad \text{where} \quad C_{n,r} = \prod_{i=1}^{r} \frac{q^n - q^{i-1}}{q^r - q^{i-1}}.$$

Note that $C_{n,r}$ is the q-binomial coefficient equal to the number of r-dimensional subspaces in an n-dimensional vector space over $\mathbb{F}_q$. In the above formula we may replace $C_{n,\,n-k}$ with $C_{n,k}$ since these two numbers are equal. We also have $\dim \mathfrak{m}^2 = p^n - 1 - n$ and $\dim \mathfrak{m}I = p^n - p^k - (n-k)$. Hence $\dim \mathfrak{m}^2 - \dim \mathfrak{m}I = p^k - 1 - k$, yielding the desired equality. $\square$

For an integer k such that $0 \le k \le n$, an ideal I of B_n such that $B_n/I \cong B_k$, and a nilpotent linear transformation $A \in \mathfrak{gl}(I/\mathfrak{m}I)$ put

$$\begin{aligned}
\mathcal{N}_k &= \{D \in \mathcal{N}(W_n) \mid \text{the maximal } D\text{-invariant ideal of } B_n \text{ has codimension } p^k\},\\
\mathcal{N}_I &= \{D \in \mathcal{N}(W_n) \mid \text{the maximal } D\text{-invariant ideal of } B_n \text{ coincides with } I\},\\
\mathcal{N}_{I,\,A} &= \{D \in \mathcal{N}_I \mid \lambda(D^{p^k}) = A\}.
\end{aligned}$$

Theorem 3.4. *Let $0 \le k \le n$, let I be any ideal of B_n such that $B_n/I \cong B_k$, and let A be a nilpotent linear transformation of the vector space $I/\mathfrak{m}I$. Then*

$$\#\mathcal{N}_{I,\,A} = q^{np^n-(n-k)p^k-k(k+1)-(n-k)^2} \prod_{i=1}^{k} (q^k - q^{i-1}),$$

$$\#\mathcal{N}_I = q^{n(p^n-1)-(n-k)p^k-k^2} \prod_{i=1}^{k} (q^k - q^{i-1}),$$

$$\#\mathcal{N}_k = q^{n(p^n-1)-(2n-k)(k+1)/2} \prod_{i=1}^{k} (q^{n-i+1} - 1).$$

Proof. Since I is G-conjugate to I_k, we may assume that $I = I_k$. By Lemma 2.3

$$\mathcal{N}_I = \coprod_{\sigma \in G_k} \sigma_*(\mathcal{N}_I \cap \mathcal{D}_k), \quad \text{a disjoint union.}$$

Since all automorphisms in G_k induce the identity transformation of $I/\mathfrak{m}I$, the derivations D^{p^k} and $\sigma_*(D^{p^k})$ induce the same transformation of $I/\mathfrak{m}I$ for any $\sigma \in G_k$ and $D \in \mathcal{N}_I \cap \mathcal{D}_k$. Hence

$$\mathcal{N}_{I,\,A} = \coprod_{\sigma \in G_k} \sigma_*(\mathcal{N}_{I,\,A} \cap \mathcal{D}_k).$$

By Lemma 2.2 $\mathcal{N}_I \cap \mathcal{D}_k \subset \mathcal{D}'_k$. So it follows that $\mathcal{N}_{I,\,A} \cap \mathcal{D}_k = \varphi^{-1}(A)$ where φ is the map from Lemma 3.1. We see that there is a bijection between $\mathcal{N}_{I,\,A}$ and the cartesian product $G_k \times \varphi^{-1}(A)$. Hence $\#\mathcal{N}_{I,\,A} = \#G_k \cdot \#\varphi^{-1}(A)$.

The set $\mathcal{N}_I$ is a disjoint union of subsets $\mathcal{N}_{I,\,A}$ with A running over the nilpotent cone $\mathcal{N}(\mathfrak{g})$ in the Lie algebra $\mathfrak{g} = \mathfrak{gl}(I/\mathfrak{m}I) \cong \mathfrak{gl}_{n-k}(\mathbb{F}_q)$. Since $\#\mathcal{N}_{I,\,A}$ does not depend on A, we get $\#\mathcal{N}_I = \#\mathcal{N}(\mathfrak{g}) \cdot \#\mathcal{N}_{I,\,A}$. The first factor here is the number of nilpotent $n-k$ by $n-k$ matrices with entries in $\mathbb{F}_q$. It is known from [2]:

$$\#\mathcal{N}(\mathfrak{g}) = q^{\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}} = q^{(n-k)^2-(n-k)} = q^{(n-k)(n-k-1)}.$$

Finally, $\#\mathcal{N}_k = \#\mathcal{I}_k \cdot \#\mathcal{N}_I$ since $\mathcal{N}_k$ is a disjoint union of subsets $\mathcal{N}_{I'}$ with I' running over $\mathcal{I}_k$. Lemmas 3.1, 3.2 and 3.3 provide all values needed. □

The main result stated in the introduction now follows from the next lemma:

Lemma 3.5. *Let* $N_k = \#\mathcal{N}_k$. *Then* $\sum_{k=0}^{n} N_k = q^{n(p^n-1)}$.

Proof. From the explicit formula in Theorem 3.4 we deduce the recurrence relation

$$q^{n-k-1} N_{k+1} = (q^{n-k} - 1) N_k \quad \text{for } k = 0, \ldots, n-1.$$

Now a downward induction on k shows that $\sum_{i=k}^{n} N_i = q^{n-k} N_k$ for all $k = 0, \ldots, n$. Taking $k = 0$ and noting that $N_0 = q^{n(p^n-2)}$, we arrive at the desired conclusion. □

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