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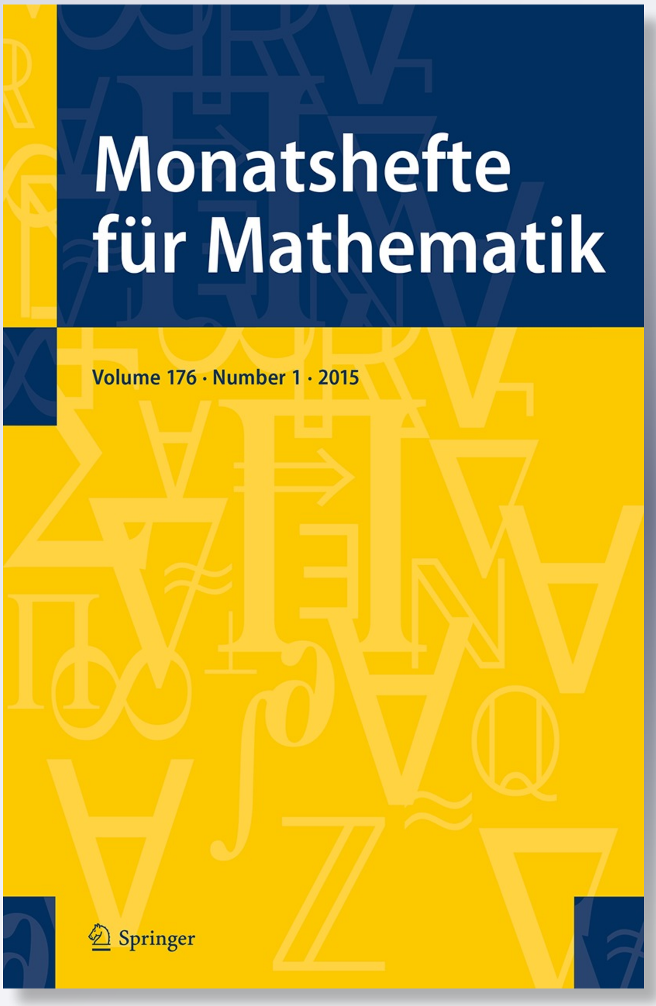

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Cauchy integral and singular integral operator over closed Jordan curves

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Abstract This paper is mostly a review paper. It contains a description of old and recent results concerning the regularity conditions on a Jordan curve in the plane that imply the boundedness of the singular integral operator as well as the boundary behavior of the Cauchy type integral. These results are of significance for boundary value problems in domains with non-smooth and non-rectifiable boundaries.

Keywords Cauchy integral · Singular integral operator · Boundary value problem

Mathematics Subject Classification Primary 30E20 · 30E25; Secondary 45E05

1 Introduction

For several reasons the Cauchy type operators belong to the core of the boundary value problems arising in complex function theory. The properties of the boundary value of

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the Cauchy integral are of traditional interest. The paper provides without proofs the relevant material on the profound connection between the boundary behavior of the Cauchy integral and the boundedness of the associated singular integral operator.

Assume that γ is a rectifiable oriented and closed Jordan curve in the complex plane $\mathbb{C}$ dividing the plane in two domains Ω_+ and $\Omega_- := \mathbb{C} \setminus (\Omega_+ \cup \gamma)$. Given a complex value continuous function f defined on γ, the Cauchy integral is defined, in $\mathbb{C} \setminus \gamma$, by

$$\mathcal{C}_\gamma f(z) = \frac{1}{2\pi i} \int\limits_\gamma \frac{f(\tau)}{\tau - z} d\tau,$$

and the principal value singular integral associated with $\mathcal{C}_\gamma f(z)$ is

$$\mathcal{S}_\gamma f(t) := \lim_{\epsilon \to 0} \frac{1}{\pi i} \int\limits_{\gamma \setminus \{|\tau - t| \leq \epsilon\}} \frac{f(\tau) d\tau}{\tau - t}, \; t \in \gamma.$$

Of course, one must worry about the existence of the limit.

It should be noted that, in the sense of distribution theory, one has $\frac{\partial}{\partial \overline{z}} \frac{1}{z} = \delta$, where δ is the Dirac δ-distribution. Thus the kernel $\frac{1}{\tau - z}$ of the Cauchy integral may be regarded as fundamental solution of the differential equation $\frac{\partial w}{\partial \overline{z}} = 0$ (with the singularity at z). Hence the Cauchy integral is a complex version of the Newtonian potential.

It is reasonably well known that the null solutions of the operators $\frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial z}$ provide us with the two classes of functions, respectively, analytic and anti-analytic. Crucial is the fact that they factorize the two dimensional Laplace operator.

The limit values of $\mathcal{C}_\gamma f$ are closely related with the principal value singular integral. This connection becomes clear from the following classical theorem of Privalov [39]:

Theorem 1.1 *Let γ be a rectifiable closed Jordan curve, f be a continuous function on it.*

(1) If the Cauchy integral $\mathcal{C}_\gamma f(z)$ has non-tangential limit values when z approaches γ almost everywhere, then the principal value integral $\mathcal{S}_\gamma f(t)$ exists almost everywhere in γ and the following formulae

$$(\mathcal{C}_\gamma f)^{\pm}(t) = \frac{1}{2}(\mathcal{S}_\gamma f(t) \pm f(t)) \tag{1}$$

hold.

(2) Conversely, if the principal value integral $\mathcal{S}_\gamma f$ exists almost everywhere in γ, then the Cauchy integral $\mathcal{C}_\gamma f$ has non-tangential limit values $(\mathcal{C}_\gamma f)^+$ and $(\mathcal{C}_\gamma f)^-$ almost everywhere in γ and the formulae (1) hold.

The formulae given above involving the singular integral operator $\mathcal{S}_\gamma f(t)$ are called Plemelj–Sokhotski formulae.

Moreover, it follows from the Calderón et al, results (see [10]) that the principal value singular integral $\mathcal{S}_\gamma f(t)$ exists almost everywhere in γ for any L^p-integrable function f and every rectifiable closed Jordan curve γ.

Notice that, taking into account the rectifiability of γ, for almost every $t \in \gamma$ we get that:

$$\int_\gamma \frac{d\tau}{\tau - t} = \pi i.$$

Hence, the following equality is true almost everywhere on γ

$$\int_\gamma \frac{f(\tau)}{\tau - t} d\tau = \int_\gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau + \pi i f(t).$$

Consequently, the non-tangential limit values of the Cauchy integral are also given almost everywhere on γ, by

$$(\mathcal{C}_\gamma f)^+(t) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t) \tag{2}$$
$$(\mathcal{C}_\gamma f)^-(t) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

2 Singular integral operator

The problem about the L^p boundedness of $\mathcal{S}_\gamma$, for $1 < p < \infty$, has been intensively studied by a prominent group of mathematicians, among others, A. P. Calderon, Y. Meyer, R. Coifman, A. McIntosh, G. David, P. Mattila and S. Semmes.

If γ is the graph of a Lipschitz function, the L^p boundedness of $\mathcal{S}_\gamma$ was first proved in 1977 by Calderón [9] for curves with small Lipschitz constant and extended to the general case in 1982 by Coifman, McIntosh and Meyer, in the celebrated paper [11].

David [13] has shown that the operator $\mathcal{S}_\gamma$ is bounded on L^p if and only if γ is AD-regular, which means that there is a $C > 0$ so that for all $z \in \gamma$ and for all $r > 0$ the arc-length measure (the one-dimensional Hausdorff measure $\mathcal{H}^1$) of $\gamma \cap B(z, r)$ is at most $C r$; here and below $B(z, r)$ stands for the closed disk with center z and radius r. The necessity of the AD-regularity is the easer part of the David's theorem and was also proved by Paatashvili and Khuskivadze in [40] and by Salimov [46] using different techniques. A general set $\mathbf{E} \subset \mathbb{R}^2$ is called AD-regular if it also satisfies a lower bound, i.e., if

$$C^{-1} r \leq \mathcal{H}^1(\mathbf{E} \cap B(z, r)) \leq C r.$$

Note that for a curve the lower bound is automatically valid.

One very important reason for needing to know that the singular integral operator is L^p-bounded for Lipschitz graph is to be able to solve boundary value problems for domains that are bounded by Lipschitz curves.

In [14] David and Semmes introduce the concept of *uniformly rectifiable* set **E**, which means that **E** is contained in an AD-regular curve. They extensively study singular integrals and other type of analysis on uniformly rectifiable sets, also on m-dimensional sets in $\mathbb{R}^n$.

The studies of the L^p-boundedness of $\mathcal{S}_{\mathbf{E}}$ and the analytic capacity of **E** are closely connected. This connection has been widely clarified in the works of Christ [12], Mattila, Melnikov and Verdera [34] and Mattila [32,33]. For example, if **E** is AD-regular and $\mathcal{S}_{\mathbf{E}}$ is bounded in $L^2(\mathbf{E})$, then the analytic capacity of E is positive. Recent results on the L^p-boundedness of the singular integral operator for more general continuous measure have been obtained by Tolsa [49,50].

A great deal of modern complex analysis is of both theoretical and practical significance to ask for what kind of boundaries and functions the Cauchy integral has continuous limit values. The existing literature in this topic is classically known in the case of considering sufficiently smooth curves and Hölder continuous functions. In this context the reader may consult the monographs by Muskhelisvili [35], Gakhov [19] and Lu [30].

In the remainder of this section and in the next one we record the main results concerning the Cauchy integrals on non-smooth curves, done by many mathematicians from the Ex-Soviet Union. On the other hand, some original works of the authors and some of their colleagues are also included here. In Sect. 4, particular emphasis is on the case where the curve is non-rectifiable and even fractal.

It is of interest to look at the more classical result, known as the Plemelj-Privalov Theorem on the Hölder continuous boundedness of the singular integral operator, see [36,37].

Theorem 2.1 *Let γ be a circle. Then, for $0 < \alpha < 1$ the following implication holds:*

$$f \in C^{0,\alpha}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,\alpha}(\gamma), \tag{3}$$

where $C^{0,\nu}(\gamma), 0 < \nu \leq 1$ is the space of Hölder continuous functions in γ with the exponent ν.

This result is also valid for the case of any closed smooth curves, see [35] for instance.

If the curve γ is permitted to have corners, then the singular integral $\mathcal{S}_\gamma 1$ is not a continuous function on γ. Indeed, let $w(t)$ denote the jump in the tangent angle at the corner point $t \in \gamma$, then obviously $w(t) \neq 0$. Using the Cauchy integral theorem it is not difficult to verify that $\mathcal{S}_\gamma 1(t) = 1 - \frac{w(t)}{\pi}$, $t \in \gamma$, which shows the above mentioned discontinuity at each corner point of γ.

The way out of this difficulty is to introduce a new singular Cauchy integral given by

$$f(t) \to \frac{1}{\pi i} \int_\gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau + f(t). \tag{4}$$

Notice however that when γ is smooth curve then this singular integral coincides with the previously defined singular integral $\mathcal{S}_\gamma f$. For this reason in what follows also the symbol $\mathcal{S}_\gamma f$ is used to denoted the new singular integral (4).

In general, the integral $\int_\gamma \frac{d\tau}{\tau - t}$ has no sense for every $t \in \gamma$, hence the formula

$$\int_\gamma \frac{f(\tau) - f(t)}{\tau - t} d\tau = \int_\gamma \frac{f(\tau)}{\tau - t} d\tau - f(t) \int_\gamma \frac{d\tau}{\tau - t}$$

is generally not valid. However, when the singular integral

$$\frac{1}{\pi i} \int_\gamma \frac{d\tau}{\tau - t}$$

has a finite value $\alpha(t)$ for every $t \in \gamma$, then it is possible to express the singular integral operator (4) as

$$\mathcal{S}_\gamma f(t) = \frac{1}{\pi i} \int_\gamma \frac{f(\tau)}{\tau - t} d\tau + (1 - \alpha(t)) f(t).$$

If γ is a piecewise smooth curve, then $\pi\alpha(t)$ is the interior angle of the curve γ at the point t, see [20].

Theorem 2.1 was proved by Privalov [38] for any piecewise smooth curve without cusps. Subsequently Muskhelisvili [35] and Davydov [16] extended the above result for any piecewise smooth and any chord-arc curve, respectively.

Recall that a curve γ is called chord-arc if the arc is comparable to the chord. In more precise terms: if its arc-length parametrization $x \mapsto z(x)$ satisfy

$$\exists C > 0 : \forall (x, y) \in \mathbb{R}^2 : |z(x) - z(y)| \geq C|x - y|.$$

In 1976, Salaev formulated the problem of describing the maximal class Π_α of closed rectifiable Jordan curves for which the assertion of the Plemelj-Privalov theorem [the implication (3)] holds.

Closely related to this problem is the problem of Babaev and Salaev of finding Zygmund type estimate for the Cauchy singular integral in the case of arbitrary closed rectifiable Jordan curve. Using Zygmund type estimates in 1976 Salaev [42] has proved that the Plemelj–Privalov theorem is true for the class of AD-regular closed Jordan curves. But, it is not until the beginning of the 90'th decade that a largest class of curves with purely metrical description for which the Plemelj–Privalov theorem is valid has been found.

In the papers by Salimov [45] important examples of curves were constructed which partly reveled the structure of the class Π_α:

1. Example of a closed rectifiable Jordan curve on which the Plemelj–Privalov theorem does not hold.

2. Example of a non AD-regular closed rectifiable Jordan curve on which the Plemelj–Privalov theorem does hold. In this way, Andrievsky [1] in 1983 constructed another example with the same property.

It should be noted that other interesting and delicate results in this direction were derived by Salimov in [45] where the following assertions are proved:

Theorem 2.2 *Let γ such that $\mathcal{H}^1(\gamma \cap B(t,\delta)) \leq C\,\delta^{\mu}$. If $0<\alpha<1$ and $0<\mu\leq 1$ with $2\alpha+\mu-\alpha\,\mu-1>0$, then*

$$f \in C^{0,\alpha}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,2\alpha+\mu-\alpha\,\mu-1}(\gamma).$$

Theorem 2.3 *For $\frac{1}{2}<\alpha<1$ it follows that*

$$f \in C^{0,\alpha}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,2\alpha-1}(\gamma).$$

In 1990 Salaev, Guseinov and Seifullaev [44] found a geometrical condition which completely characterizes the class Π_α. The condition can be formulated in terms of the plane Lebesgue measures of boundary strips of sets which are contained in squares and in a certain sense essentially cover the curve.

They have also found the condition which completely characterizes the class $\Pi_{\alpha,\beta}$, $0<\beta\leq\alpha<1$, of those closed rectifiable Jordan curves for which the implication

$$f \in C^{0,\alpha}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,\beta}(\gamma), \tag{5}$$

holds.

A generalization of the classical Plemelj–Privalov theorem emerges in a natural way by considering the so-called generalized Hölder spaces $C^{0,\varphi}(\gamma)$ of continuous functions in γ that satisfy the condition $\omega_f(\delta) \leq C\varphi(\delta)$, where φ is a majorant, i.e., $\varphi : (0,d] \to \mathbb{R}_+$ is such that $\varphi(\delta)$ does not decrease , $\delta^{-1}\varphi(\delta)$ does not increase and $\varphi(\delta) \to 0$ as $\delta \to 0$.

In this more general setting, the Plemelj–Privalov theorem looks as

$$f \in C^{0,\varphi}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,\varphi}(\gamma). \tag{6}$$

If the majorant φ satisfies also the condition

$$\int_0^\delta \frac{\varphi(\xi)}{\xi}d\xi + \delta\int_\delta^d \frac{\varphi(\xi)}{\xi^2}d\xi \leq C\,\varphi(\delta), \tag{7}$$

then the assertion (6) may be proved directly using the Zygmund type estimates of the modulus of continuity of the singular integral operator $\mathcal{S}f$ in terms of the modulus of continuity of the density f.

In 1924 Zygmund [51] states, for any continuous function defined on a circle, the estimate:

$$\omega_{\mathcal{S}_\gamma f}(\delta) \le C \left(\int_0^\delta \frac{\omega_f(\xi)}{\xi} d\xi + \delta \int_\delta^d \frac{\omega_f(\xi)}{\xi^2} d\xi \right),$$

where C is a constant that does not depend on δ.

In 1947 Magnaradze [31] extended the previous result for piecewise smooth curves, and then Babaev [3] made the same for chord-arc curves satisfying the additional condition

$$\int_\gamma \frac{d\tau}{\tau - t} = \pi i.$$

Tamrazov [47,48], Babaev and Salaev [5] proved the estimate on arbitrary chord-arc curve.

Later, this estimate on AD-regular curves was proved in the paper [42] in terms of the following continuity modulus:

$$\omega^*_{\mathcal{S}_\gamma f}(\delta) := \sup_{t \ge \delta} \frac{\omega_{\mathcal{S}_\gamma f}(t)}{t}.$$

In [20] Gerus obtained some upper and lower Zygmund type estimates for the so-called contour modulus of smoothness of the Cauchy singular integral as well as for the solid modulus of smoothness of the Cauchy integral on AD-regular curves. Some similar results have been presented by Rakhimov in [41].

The necessity of the condition (7) is proved in the circle case by Bari and Stečkin in [2]. In his Ph.D Thesis, Salimov established the following theorem:

Theorem 2.4 *Let γ be an AD-regular closed Jordan curve. Then*

$$f \in C^{0,\varphi}(\gamma) \Longrightarrow \mathcal{S}_\gamma f \in C^{0,\psi}(\gamma) \tag{8}$$

holds if and only if the majorants φ and ψ are such that

$$\int_0^\delta \frac{\varphi(\xi)}{\xi} d\xi + \delta \int_\delta^d \frac{\varphi(\xi)}{\xi^2} d\xi \le C\, \psi(\delta). \tag{9}$$

In 1992 Guseinov presented in [21] the complete solution of the Salaev problem for generalized Hölder continuous functions spaces: he found a geometrical condition which completely characterize the class $\Pi_{\varphi,\psi}$ of curves satisfying (8). This result generalizes the previous one obtained in [44]. For the sake of completeness recall the definition of the geometric condition $V(\varphi, \psi)$.

Let us introduce, for bounded sets $\mathbf{E}$, partitions of a special form. Let Q be a closed square in the plane with sides parallel to the coordinate axes and $\mathbf{E} \subset Q$. Partitioning Q

into n_0^2 equal squares by dividing its sides into n_0 equal parts and repeat this procedure with each new square. This gives a set of sub-squares of Q, which will be denoted by (Q, n_0). For $P \in (Q, n_0)$ put $[P] = 2^{-1}\log_2 \frac{|Q|}{|P|}$; it is the rank of the sub-square ($|\mathbf{E}|$ denotes the plane measure of the measurable set $\mathbf{E}$).

Let G be a measurable set in Q. Consider the sets ($k = \overline{0, \infty}$)

$$G_k^+ = \bigcup \left\{P,\ P \in (Q, n_0),\ [P] = k,\ |P \cap G| > \frac{|P|}{2}\right\}$$

$$G_k^- = \bigcup \left\{P,\ P \in (Q, n_0),\ [P] = k,\ |P \cap G| \le \frac{|P|}{2}\right\}$$

$$A_k = \bigcup \left\{P,\ P \in (Q, n_0),\ [P] = k,\ |P \cap G_{k+1}^+| > 0,\ |P \cap G_{k+1}^-| > 0\right\}.$$

Thus, for each measurable subset $G \subset Q$ and (Q, n_0) are have associated the system of sets $\{G_k^{\pm}\}_{k=0}^{\infty}$ and $\{A_k\}_{k=0}^{\infty}$. A Jordan curve γ belongs to the class $V(\varphi, \psi)$, if there exists a constant $c > 0$ and a natural number $n_0 > 1$ such that for any square Q

$$\sum_{k=1}^{\infty} |A_{k-1}|(n_0^{-k}|Q|^{\frac{1}{2}})^{-1}\varphi(n_0^{-k}|Q|^{\frac{1}{2}}) \le c|Q|^{\frac{1}{2}}\psi(|Q|^{\frac{1}{2}}),$$

where $\{A_k\}_{k=0}^{\infty}$ is a system of sets constructed from the open in Q set $\Omega_+ \bigcap Q$.

The main result in [21] asserts that the implication (8) holds if and only if the curve γ belongs to the class $V(\varphi, \psi)$

3 Cauchy integral

The existence of continuous limit values of the Cauchy integral

$$\mathcal{C}_\gamma f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\tau)}{\tau - z}d\tau, \tag{10}$$

is well known in the case when the curve γ is piecewise smooth or simply smooth and the function $f \in C^{0,\alpha}(\gamma)$. The classical approach to this kind of study was developed intensively by Muskhelisvili [35] and Gakhov [19].

Thanks to the Plemelj–Privalov theorem the existence of continuous limit values of $\mathcal{C}_\gamma f$ is equivalent to the fact that the function f can be split as the sum $f = f^+ + f^-$, where $f^{\pm}$ are the traces on γ of functions which are analytic inside and outside the curve and vanish at infinity. Consequently, the Cauchy integral of f has continuous limit values if and only if there is a continuous solution of the so-called jump problem across γ with jump f.

The earlier aspects of the study of the continuous extendibility of the function $\mathcal{C}_\gamma f(z)$ to the curve γ from the left (from the domain Ω_+) and from the right (from Ω_-) and the fact that the existence of its continuous limit values implies the existence of the principal value singular integral $\mathcal{S}_\gamma f$ as well as its continuity on γ was developed among others by Babaev and Salaev (see [4]). Therefore, the study of the problem on the existence of continuous limit values of the Cauchy integral must be carried out in an appropriate framework that guaranties the continuity of principal value of the singular integral.

Let us remark that the more natural condition under which one can expect the continuity of $\mathcal{S}_\gamma f$ on γ is the uniform convergence of the truncated singular integrals

$$\mathcal{S}_{\gamma,\epsilon} f(t) := \int\limits_{\gamma\setminus \mathrm{B}(t,\epsilon)} \frac{f(\tau)-f(t)}{\tau-t} d\tau,$$

where ϵ tends to 0.

Davydov was among the first mathematicians who investigated the question on the continuity of the Cauchy integral along a non-smooth curve. In particular, he obtained the following ones (see [15] and [16]).

Theorem 3.1 *Let f be a continuous function defined on a rectifiable closed Jordan curve γ, and $\mathcal{S}_\gamma f(t)$ exist at every $t \in \gamma$ and define a continuous function. If the singular integral*

$$\int\limits_{\gamma} \left|\frac{f(\tau)-f(t)}{\tau-t}\right| |d\tau|,$$

exists and is bounded on γ, then the Cauchy integral (10) has continuous limit values from left and right sides of γ.

Theorem 3.2 *Let f be a Lipschitz continuous function on a rectifiable closed Jordan curve γ ($f \in C^{0,1}(\gamma)$), then the Cauchy type integral (10) has continuous limit values on γ.*

To introduced other possible sufficient conditions one have to restrict the function $\varphi : (0, d] \to \mathbb{R}_+$ in order to guaranty the existence of continuous limit values of the Cauchy type integral for any continuous function f belonging to the generalized Hölder space $C^{0,\varphi}(\gamma)$. Let us enumerate some of that restrictions:

1. φ is a majorant such that

$$\int\limits_0^d \frac{\varphi(\tau)}{\tau} d\vartheta(\tau) < \infty,$$

   where $\vartheta(\tau) = \sup_{z\in\gamma} \mathcal{H}^1(\gamma \cap B(z,\tau))$. (Salaev and Babaev [4]).

2. φ is a majorant such that $\frac{\varphi(\tau)}{\sqrt{\tau}}$ not decreases and

$$\int\limits_0^d \left(\frac{\varphi(\tau)}{\tau}\right)^2 d\tau < \infty.$$

   (Dynkin [17,18]).

3. φ is a majorant such that

$$\int_0^d \frac{\varphi(\tau)}{\tau^{\frac{3}{2}}} d\tau < \infty.$$

(Salimov [45] and Dynkin [17,18]).

According either to Dynkin or to Salimov condition it is easy to see that for every Hölder continuous function f with exponent $\alpha > \frac{1}{2}$ the Cauchy integral has continuous limit values on γ. This result was also obtained by Kats in [22] following different techniques.

In the special case when γ belongs to the class $\Pi_{\varphi,\psi}$, the Cuban authors of this paper have proved in [6] that the Cauchy integral has continuous limit values on γ for any continuous function belonging to the generalized Hölder space $C^{0,\varphi}(\gamma)$.

Notice that all the above mentioned conditions are only sufficient for the existence of continuous limit values of the Cauchy integral. In [43] Salaev and Tokov proved the following very nice result:

Theorem 3.3 *Let γ be an AD-regular curve. Then the Cauchy integral $\mathcal{C}_\gamma f$ has continuous limit values on γ if and only if the truncated integrals $\mathcal{S}_{\gamma,\epsilon}$ converge uniformly in $t \in \gamma$ as ϵ tends to* 0.

Another types of necessary and sufficient conditions have been obtained by Bustamante in [7,8]. In fact,

Theorem 3.4 *Let γ be a closed rectifiable Jordan curve. Let f be a continuous function defined on γ. Then the following conditions are equivalent:*

(a) The Cauchy integral (10) has continuous limit values.
(b) The principal value of singular integral $\mathcal{S}_\gamma f$ is a continuous function on γ and there is a positive number M such that $|\mathcal{C}_\gamma f(z)| \leq M$.
(c) There exists a sequence of Lipschitz continuous functions on γ, $h_n = h_n^+ + h_n^-$, such that $\|h_n^+ - f^+\|_\infty \to 0$, $\|h_n^- - f^-\|_\infty \to 0$.

4 Non-rectifiable curves

At first glance, the Cauchy integral and the singular integral are meaningless for non-rectifiable curves, but it is not so.

The Stieltjes integral $\int_\gamma f(\tau)dg(\tau)$ exists not only for functions of bounded variations. It is enough to assume them to be of bounded generalized variation (see, for instance, [29]). In this connection, the definition of Φ-rectifiable curves given in [24,26] follows

Let $\Phi(x)$ be an increasing function defined for $x \geq 0$, $\Phi(0) = 0$. A curve γ is called Φ-rectifiable, if

$$\sum_{j=1}^{\infty} \Phi(|t_j - t_{j-1}|) < C$$

for any sequence $\{t_0, t_1, \ldots, t_n\}$ of points of the curve γ enumerated in order of its traversal; here positive constant C is independent on the sequence. It is called q-rectifiable if $\Phi(x) = x^q, q > 1$. The class of q-rectifiable curves contains non-rectifiable ones. For appropriate q it contains, for example, the Von Koch snowflake.

Around 2000 Kats started to study the Cauchy–Stieltjes integral

$$\mathcal{SC}_\gamma f(z) = \frac{1}{2\pi i}\int_\gamma f(\tau) d\log(\tau - z), \quad z \notin \gamma,$$

and proved the following

Theorem 4.1 *Let γ be a closed q-rectifiable curve, $f \in C^{0,\alpha}(\gamma)$, $\alpha > q - 1$ and $\alpha > \frac{\overline{dm}(\gamma)}{2}$. Then the Cauchy–Stieltjies integral $\mathcal{SC}_\gamma f(z)$ exists and has continuous limit values on γ satisfying relation*

$$\mathcal{SC}_\gamma^+ f(t) - \mathcal{SC}_\gamma^- f(t) = f(t).$$

Here $\overline{dm}(\gamma)$ is the so called upper metric dimension of the curve γ (see [28]), given by

$$\overline{dm}(\gamma) = \liminf_{\varepsilon \to 0} \frac{\ln N_\varepsilon(\gamma)}{-\ln \varepsilon},$$

where $N_\varepsilon(\gamma)$ is the least number of disks of radius ε necessary to cover γ.

In [25] Kats generalized the Davydov Theorem 3.2 for Φ-rectifiable curves:

Theorem 4.2 *If a curve γ is Φ-rectifiable for a convex function $\Phi(x)$ and $f \in C^{0,1}(\gamma)$, then the Cauchy–Stieltjes integral has continuous limit values under the condition*

$$\sum_{n=1}^{\infty} \phi^2\left(\frac{1}{n}\right) < \infty,$$

where ϕ is the inverse function for Φ.

As was mentioned above, the Cauchy integral $\mathcal{C}_\gamma f$ has continuous limit values if and only if there exists a solution $\Psi^\pm$ of the jump problem

$$\Psi^+(t) - \Psi^-(t) = f(t), \quad t \in \gamma. \tag{11}$$

Kats presented in [22,23] a method for solving the of the jump problem (11), which does not use contour integration and, consequently, can be applied on non-rectifiable and fractal curves. If the function f satisfies a Hölder condition on γ with exponent α, then this method gives a solution of (11) under a single essential restriction on γ

$$\alpha > \frac{\overline{dm}(\gamma)}{2}.$$

It is shown that this condition is sharp over the whole class of curves of fixed upper metric dimension.

Recently in [27] is proposed a new approach to the problem on integral representation of the jump problem on non-rectifiable curves. The scheme of this approach is the following. If ϕ is a distribution with compact support S on the complex plane, then its Cauchy transform is defined by equality

$$\mathcal{C}au\,\phi := \frac{1}{2\pi i}\left\langle \phi, \frac{1}{\zeta - z}\right\rangle,$$

where $z \notin S$, and ϕ is applied to the Cauchy kernel $\dfrac{1}{2\pi i(\zeta - z)}$ with respect to the variable ζ. Obviously, it is holomorphic in $\overline{\mathbb{C}} \setminus S$ and vanishes at infinity point. In particular, the Cauchy integral is the Cauchy transform of distribution

$$\phi : C_0^\infty(\mathbb{C}) \ni \omega \mapsto \int_\gamma f(\tau)\omega(\tau)d\tau,$$

where γ is rectifiable and $f(t)$ is integrable on this curve. Identify a function $F(\zeta)$ on the complex plane with distribution

$$F : C_0^\infty \ni \omega \mapsto \iint_{\mathbb{C}} F(\zeta)\omega(\zeta)d\zeta d\overline{\zeta},$$

if the last integral takes a sense. If F is holomorphic in $\mathbb{C} \setminus \gamma$, then the distributional derivative $\overline{\partial} F$ has support on the curve γ. If the curve would be rectifiable, then the derivative $\overline{\partial} F$ would be representable as follows:

$$\langle \overline{\partial} F, \omega\rangle = \int_\gamma (F^+(\zeta) - F^-(\zeta))\omega(\zeta)d\zeta.$$

Hence, for non-rectifiable γ this distribution is generalized integration over γ with weight $F^+(\zeta) - F^-(\zeta)$. The integration without weight corresponds to functions F with unit jump on Γ. For instance, one can use to this end the characteristic function $\chi^+(z)$ of domain Ω_+, which equals to 1 in Ω_+ and to 0 in Ω_-.

The space $\mathfrak{X}$ of functions defined on γ such that $C^\infty(\mathbb{C})$ is dense in $\mathfrak{X}$ and $\mathfrak{X}C^\infty = \mathfrak{X}$, i.e., $f\omega \in \mathfrak{X}$ for any $\omega \in C^\infty$, $f \in \mathfrak{X}$. If a derivative $\overline{\partial} F$ is continuous in norm of $\mathfrak{X}$, then it is continuable up to functional on $\mathfrak{X}$ and generates a family of distributions

$$\langle f\overline{\partial} F, \omega\rangle := \overline{\partial} F(f\omega). \tag{12}$$

Here the notation $\overline{\partial} F$ still used for the mentioned above functional. Then the Cauchy transforms of these distributions is considered and research their boundary behavior. The scheme is realized for the Hölder space as the space $\mathfrak{X}$.

Theorem 4.3 *Let function $F(z)$ be holomorphic in $\overline{\mathbb{C}} \setminus \gamma$, continuous in $\Omega_+ \cup \gamma$ and $\Omega_- \cup \gamma$, and $F(\infty) = 0$. If f satisfies the Hölder condition with exponent $\alpha > \frac{\overline{dm}(\gamma)}{2}$, then the Cauchy transform $\mathcal{C}au\ f\overline{\partial}F(z)$ is holomorphic in $\overline{\mathbb{C}}\setminus\gamma$, continuous in $\Omega_+ \cup \gamma$ and $\Omega_- \cup \gamma$, vanishes at the infinity point, and has jump $(F^+(t) - F^-(t))f(t)$ on the curve γ.*

The preceding theorem implies the following interesting result:

Let $J(\gamma)$ be set of all continuous functions f such that the jump problem (11) has a solution. Then $fg \in J(\gamma)$ for any $f \in J(\gamma)$ and any $g \in C^{0,\alpha}(\gamma)$ such that $\alpha > \frac{\overline{dm}(\gamma)}{2}$.

Finally, an extension of the singular integral operator for non-rectifiable curve is considered. Let $\Psi(z)$ be a solution of the jump problem (11) vanishing at the infinity point. By virtue of the Sokhotskii-Plemelj formulas the sum $\Psi^+(t) + \Psi^-(t)$ equals to the singular integral $\mathcal{S}_\gamma f(t)$ for smooth γ, and to the singular integral (4) for piecewise-smooth one. In this connection an intrinsic generalization of the singular integral operator for non-rectifiable closed curve is the mapping

$$f \mapsto \mathcal{C}au\,(f\overline{\partial}\chi^+ - f\overline{\partial}\chi^-)|_\gamma.$$

Moreover, it is continuous, under assumptions of Theorem 4.3, in the corresponding spaces.

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