Mathematical Analysis of the Guided Modes of Integrated Optical Guides

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Summary. The eigenvalue problem for guided modes of integrated optical guides is formulated as a problem for the set of time-harmonic Maxwell's equations. The original problem is reduced to a strongly-singular domain integral equation, which is often used in practice for computation, and it is proved that the operator of the domain integral equation is a Fredholm operator with zero index. It is also proved that the spectrum of the original problem can only be a set of isolated points.

1 Introduction

In this work we study the natural modes of an optical fiber integrated into a three-layer planar medium, which is representative of typical optical circuits. In the absence of a planar background, the basic properties of optical fibers are described in [1], [2]. More recently, the methods of the theory of unbounded self-adjoint operators have been applied to the analysis of the guided modes of optical fibers [3]. For the guided modes of integrated optical guides, a rigorous mathematical analysis for the scalar case has been presented in [4]–[6]. In [3]–[6] the authors, by using the min-max principle for unbounded self-adjoint operators, proved the existence of guided modes, the number of which is finite and depends on frequency. In [7]–[8] the method of boundary integral equations was applied to the mathematical and numerical study of the guided modes of homogeneous optical fibers.

Due to the complexity of the integrated optical structure, domain integral equations utilizing appropriate Green's functions (to account for the background media) are a popular practical approach for computing the natural fiber modes [9]–[11]. A problem with domain integral equations is that they are strongly-singular, which previously prevented their use in a mathematical study of the spectrum of the eigenvalues, with the exception of [12] for the guided modes of optical fibers in a homogeneous background medium. It was proven in [12] that the operator of the domain integral equation is semi-Fredholm.

In this work a rigorous mathematical analysis of the guided modes of an integrated optical guide is presented based upon a strongly-singular domain integral equation which is useful for practical computation. It is proved that the operator of the domain integral equation is a Fredholm operator with zero index, which, in turn, enables the extensive use of the spectral theory of linear nonselfadjoint operators [13]. By using this result we prove that the spectrum of the operator can only be a set of isolated points. Our results are important for future theoretical investigation of convergence of the known numerical methods [9]–[11].

2 Statement of the problem

We consider the guided modes of the integrated optical guide. Let the three-dimensional space $\{(x_1, x_2, x_3) : -\infty < x_1, x_2, x_3 < \infty\}$ be occupied by an isotropic source-free medium, and let the refractive index be prescribed as a positive real-valued function $n = n(x_1, x_2)$ independent of the longitudinal coordinate x_3. We assume that there exists a bounded domain Ω on the plane $\mathbb{R}^2 = \{(x_1, x_2) : -\infty < x_1, x_2 < \infty\}$ such that $n(x) = n_\infty(x_2)$, $x = (x_1, x_2) \in \Omega_\infty = \mathbb{R}^2 \setminus \overline{\Omega}$, where $n_\infty(x_2)$ depends only on the x_2 variable. It is a piecewise-constant function representing the refractive index of the so-called associated planar waveguide. For simplicity, we take

$$n_\infty(x_2) = \begin{cases} n_1 & \text{if} \quad x_2 > d, \\ n_2 & \text{if} \quad 0 < x_2 < d, \\ n_3 & \text{if} \quad x_2 < 0. \end{cases} \tag{1}$$

We can assume without loss of generality that $n_2 \geq n_3 \geq n_1$. Denote by n_+ the maximum of the function n in the domain Ω, and suppose that $n_+ > n_2$. We assume that $\Omega \subset \Omega_2 = \{(x_1, x_2) : -\infty < x_1 < \infty,\ 0 < x_2 < d\}$ and also that $n(x)$ is a continuous function in Ω_2 that is the guide does not have a sharp boundary. Denote by Γ_1 and Γ_2 the boundaries of the domain Ω_2:

$$\Gamma_1 = \{(x_1, x_2) : -\infty < x_1 < \infty, \quad x_2 = d\}, \tag{2}$$
$$\Gamma_2 = \{(x_1, x_2) : -\infty < x_1 < \infty, \quad x_2 = 0\}. \tag{3}$$

The modal problem can be formulated as a vector eigenvalue problem for the set of differential equations (we use notations [3] for differential operators)

$$\mathrm{Rot}_\beta \boldsymbol{E} = \mathrm{i}\omega\mu_0 \boldsymbol{H}, \tag{4}$$
$$\mathrm{Rot}_\beta \boldsymbol{H} = -\mathrm{i}\omega\varepsilon_0 n^2 \boldsymbol{E}. \tag{5}$$

Here ε_0, μ_0 are the free-space dielectric and magnetic constants, respectively. We consider the propagation constant β as an unknown complex parameter and radian frequency $\omega > 0$ as a given parameter. We seek non-zero solutions $[\boldsymbol{E}, \boldsymbol{H}]$ of set (4), (5) in the space $\left[L_2(\mathbb{R}^2)\right]^6$. Physically this condition means that we are looking for surface modes. In any neighborhood of the boundary Γ_j, $j = 1, 2$, that does not include the domain Ω the vector-functions $\boldsymbol{E}$ and

$\boldsymbol{H}$ are analytic and have to satisfy the usual continuity conditions for their tangential components (see, for example, [14]):

$$(\boldsymbol{\nu} \times \boldsymbol{E})^- = (\boldsymbol{\nu} \times \boldsymbol{E})^+, \quad (\boldsymbol{\nu} \times \boldsymbol{H})^- = (\boldsymbol{\nu} \times \boldsymbol{H})^+, \quad x \in \Gamma_j, \quad j = 1, 2. \tag{6}$$

Here $\boldsymbol{\nu}$ is the unit normal vector on Γ_j, $\cdot \times \cdot$ is the vector product, and $f^+(f^-)$ is the limit of the function f from above (from below) the line Γ_j, $j = 1, 2$.

By $\Lambda_0^{(1)}$ denote the principal (physical) sheet of the Riemann surface of the function $\ln \chi(\beta)$, where $\chi(\beta) = \sqrt{k^2 n_2^2 - \beta^2}$, $k^2 = \omega^2 \varepsilon_0 \mu_0$, which is specified by the conditions $-\pi/2 < \arg \chi(\beta) < 3\pi/2$, $\mathrm{Im}\,(\chi(\beta)) \geq 0$, $\beta \in \Lambda_0^{(1)}$.

Denote by β_j the propagation constants of TE and TM modes of the associated planar waveguide [1]. It is well known [1] that there exist no more than a finite number of values β_j, and that all values β_j belong to the domain

$$\{\beta \in \Lambda_0^{(1)} : \mathrm{Im}\beta = 0, \; kn_3 < |\beta| < kn_2\}. \tag{7}$$

We define $D = \{\beta \in \Lambda_0^{(1)} : \mathrm{Re}\beta = 0\} \bigcup \{\beta \in \Lambda_0^{(1)} : Im\beta = 0, \; |\beta| \leq \gamma\}$, where $\gamma = \max |\beta_j|$. In a similar way to [4] we can see that the domain D corresponds to the continuum of propagation constants of radiation modes that do not belong to $\left(L_2(\mathrm{I\!R}^2)\right)^6$. Therefore we do not investigate the values $\beta \in D$.

Definition 1. *A nonzero vector* $[\boldsymbol{E}, \boldsymbol{H}] \in \left(L_2(\mathrm{I\!R}^2)\right)^6$ *is referred to as an eigenvector of the problem (4)–(6) corresponding to an eigenvalue* $\beta \in \Lambda = \Lambda_0^{(1)} \setminus D$ *if the relations of problem (4)–(6) are valid. The set of all eigenvalues of problem (4)–(6) is called the spectrum of this problem.*

3 Main results

Theorem 1. *The set* $B = \{\beta \in \Lambda_0^{(1)} : \mathrm{Im}\beta = 0, \; |\beta| \geq kn_+\}$ *is free of the eigenvalues of problem (4)–(6).*

This theorem for the case $n_1 = n_2 = n_3$ was proved in [3]. For the general case the proof is analogous.

If $[\boldsymbol{E}, \boldsymbol{H}] \in \left(L_2(\mathrm{I\!R}^2)\right)^6$ is an eigenvector of problem (4)–(6) corresponding to an eigenvalue $\beta \in \Lambda$, then (we use notations [3] for differential operators)

$$\boldsymbol{E}(x) = \left(k^2 n_\infty^2 + \mathrm{Grad}_\beta \mathrm{Div}_\beta\right) \frac{1}{n_\infty^2} \int\limits_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y) \boldsymbol{E}(y) dy, \tag{8}$$

$$\boldsymbol{H}(x) = -\mathrm{i}\omega\varepsilon_0 \mathrm{Rot}_\beta \int\limits_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y) \boldsymbol{E}(y) dy, \tag{9}$$

where $x \in \mathrm{I\!R}^2 \setminus (\Gamma_1 \cup \Gamma_2)$; function $G(\beta; x, y)$ is the well known tensor Green function [15]. Passing the operator $\mathrm{Grad}_\beta \mathrm{Div}_\beta$ under the integral in relation

(8), and using the differentiation rule [16] for weakly singular integrals we obtain a nonlinear spectral problem for a strongly-singular domain integral equation

$$(\mathcal{Q}(\beta)\boldsymbol{E})(x) = 0, \quad x \in \Omega, \tag{10}$$

where

$$(\mathcal{Q}(\beta)\boldsymbol{E})(x) = \boldsymbol{E}(x) + \frac{1}{2}\eta(x)\boldsymbol{E}(x) \tag{11}$$

$$- \int_{\Omega} T(\beta; x, y) \left(\left(\frac{n^2(y)}{n_\infty^2} - 1 \right) \boldsymbol{E}(y) \right) dy \tag{12}$$

$$- v.p. \int_{\Omega} T_1(x, y) \left(\left(\frac{n^2(y)}{n_\infty^2} - 1 \right) \boldsymbol{E}(y) \right) dy \tag{13}$$

$$- \int_{\Omega} L(\beta; x, y) \left(\left(\frac{n^2(y)}{n_\infty^2} - 1 \right) \boldsymbol{E}(y) \right) dy, \tag{14}$$

$$T\boldsymbol{F} = \begin{bmatrix} (K\widehat{\boldsymbol{F}})_1 + \mathrm{i}\beta F_3 \partial\Phi/\partial x_1 \\ (K\widehat{\boldsymbol{F}})_2 + \mathrm{i}\beta F_3 \partial\Phi/\partial x_2 \\ \mathrm{i}\beta F_1 \partial\Phi/\partial x_1 + \mathrm{i}\beta F_2 \partial\Phi/\partial x_2 + \left(k^2 n_\infty^2 - \beta^2\right) F_3 \Phi \end{bmatrix}, \tag{15}$$

$$T_1\boldsymbol{F} = \begin{bmatrix} \left(\widehat{\boldsymbol{F}}(y), \mathrm{grad}_2\right) \mathrm{grad}_2 \Phi_1(x, y) \\ 0 \end{bmatrix}, \tag{16}$$

$$\left(\widehat{\boldsymbol{F}}(y), \mathrm{grad}_2\right) = \partial F_1/\partial y_1 + \partial F_2/\partial y_2, \tag{17}$$

$$\eta(x) = \begin{pmatrix} n^2(x)/n_\infty^2 - 1 & 0 & 0 \\ 0 & n^2(x)/n_\infty^2 - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}$$

$$K(\beta; x, y)\widehat{\boldsymbol{F}}(y) = k^2 n_\infty^2 \widehat{\boldsymbol{F}}(y)\Phi(\beta; x, y) + \left(\widehat{\boldsymbol{F}}(y), \mathrm{grad}_2\right) \mathrm{grad}_2 \Phi_0(\beta; x, y), \tag{19}$$

$$G(\beta; x, y) = \Phi(\beta; x, y) + G^s(\beta; x, y), \quad \Phi(\beta; x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}\left(\chi(\beta)\,|x - y|\right), \tag{20}$$

$$\Phi_1(x, y) = -\frac{1}{2\pi} \ln|x - y|, \quad \Phi_0(\beta; x, y) = \Phi(\beta; x, y) - \Phi_1(x, y), \tag{21}$$

$$L(\beta; x, y)\boldsymbol{F}(y) = \left(k^2 n_\infty^2 + \mathrm{Grad}_\beta \mathrm{Div}_\beta\right) G^s(\beta; x, y)\boldsymbol{F}(y), \quad \widehat{\boldsymbol{F}} = [F_1, F_2]^T. \tag{22}$$

Note here that for any $\beta \in \Lambda$ and any $y \in \Omega$ the functions $G^s(\beta; x, y)$ and $\Phi_0(\beta; x, y)$ are twice continuously differentiable for $x \in \mathbb{R}^2 \setminus (\Gamma_1 \cup \Gamma_2)$. For all $\beta \in \Lambda$ the operator $\mathcal{Q}(\beta)$ that is determined by (10) will be considered as an operator in the space of complex-valued functions $[L_2(\Omega)]^3$. For all $\beta \in \Lambda$ the operator $\mathcal{Q}(\beta)$ has a highly singular kernel.

Theorem 2. *For all $\beta \in \Lambda$ the operator $\mathcal{Q}(\beta)$ is Fredholm with zero index.*

This theorem is proved by general results of the theory of singular integral operators [16].

Definition 2. *A nonzero vector $\boldsymbol{F} \in [L_2(\Omega)]^3$ is called an eigenvector of the operator-valued function $\mathcal{Q}(\beta)$ corresponding to an eigenvalue $\beta \in \Lambda$ if the relation*

$$\mathcal{Q}(\beta)\boldsymbol{F} = 0. \tag{23}$$

is valid.

Theorem 3. *Suppose $[\boldsymbol{E}, \boldsymbol{H}] \in \left[L_2(\mathbb{R}^2)\right]^6$ is an eigenvector of the problem (4)–(6) corresponding to an eigenvalue $\beta_0 \in \Lambda$. Then $\boldsymbol{F} = \boldsymbol{E} \in [L_2(\Omega)]^3$, $x \in \Omega$, is an eigenvector of the operator-valued function $\mathcal{Q}(\beta)$ corresponding to the same eigenvalue β_0. Suppose $\boldsymbol{F} \in [L_2(\Omega)]^3$ is an eigenvector of the operator-valued function $\mathcal{Q}(\beta)$ corresponding to an eigenvalue $\beta_0 \in \Lambda$, and also let*

$$\boldsymbol{E}(x) = \left(k^2 n_\infty^2 + \mathrm{Grad}_\beta \mathrm{Div}_\beta\right) \frac{1}{n_\infty^2} \int\limits_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y)\boldsymbol{F}(y)dy, \tag{24}$$

$$\boldsymbol{H}(x) = -\mathrm{i}\omega\varepsilon_0 \mathrm{Rot}_\beta \int\limits_\Omega \left(n^2(y) - n_\infty^2\right) G(\beta; x, y)\boldsymbol{F}(y)dy, \tag{25}$$

for $x \in \mathbb{R}^2 \setminus (\Gamma_1 \cup \Gamma_2)$. Then $[\boldsymbol{E}, \boldsymbol{H}] \in \left[L_2(\mathbb{R}^2)\right]^6$ and $[\boldsymbol{E}, \boldsymbol{H}]$ is an eigenvector of the problem (4)–(6) corresponding to the same eigenvalue β_0.

This theorem is proved by direct calculations.

Theorem 4. *The spectrum of the problem (4)–(6) can be only a set of isolated points on Λ.*

This theorem is followed from Theorems 1–3 and general results of the theory of operator-valued functions [13].

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