There Are No Maximal d.c.e. *wtt*-degrees

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1 Introduction

In this article, we will study the weak-truth-table (*wtt*, for short) degrees of d.c.e. sets and show that there is no maximal d.c.e. *wtt*-degree.

Theorem 1. *For any d.c.e. wtt-degree* $\mathbf{d}_{wtt} < \mathbf{0}'_{wtt}$*, there is a d.c.e. wtt-degree* $\mathbf{c}_{wtt}$ *strictly between* $\mathbf{d}_{wtt}$ *and* $\mathbf{0}'_{wtt}$.

Here $\mathbf{0}'_{wtt}$ is the *wtt*-degree of K, the halting problem. Theorem 1 says that for any d.c.e. set D, if D is *wtt*-incomplete, then we can split K into c.e. sets B and C such that K cannot be *wtt*-reducible to any of $B \oplus D$ and $C \oplus D$. As K is *wtt*-equivalent to $B \sqcup C$, we have that $B \oplus C \oplus D$ is *wtt*-equivalent to K. Thus, $B \oplus D$ and $C \oplus D$ are not *wtt*-reducible to each other, so they are strictly above $\mathbf{d}_{wtt}$. Our current work shows that d.c.e. *wtt*-degrees can always split above less ones (in progress), an analogue of Ladner and Sasso's result for c.e. *wtt*-degrees in [19].

Before giving a proof of the theorem above, we first review some well-known facts of density\nondensity of Turing degrees of c.e. sets and d.c.e. sets. Recall a set $A \subseteq \mathbb{N}$ is computably enumerable (c.e. for short) if A is a domain of some partial computable function, and $D \subseteq \mathbb{N}$ is d.c.e. if D is the difference of two computably enumerable sets, i.e. $D = A - B$ for some c.e. sets A and B. The research on the structures of the c.e. degrees and the d.c.e. degrees has shown many nice properties and also many pathological properties, which are always accompanied with new techniques of constructions.

For the c.e. degrees, Sacks proved that this structure is dense and every nonzero element splits, and Lachlan proved that the density and splitting above

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cannot be combined, where Lachlan developed the $0'''$ argument for the first time, an argument being called "monstrous" construction in the 1980s.

Cooper initiated the study of the structure of d.c.e. degrees in his PhD thesis [3] in 1971. Lachlan observed that the d.c.e. degrees are downwards dense and Cooper [5] proved that each nonzero d.c.e. degree splits, an analogue of Sacks splitting. Recall that a Turing degree is properly d.c.e. if it contains a d.c.e. set, but no c.e. sets. As c.e. degrees are also d.c.e., what Lachlan and Cooper needed to do in their proofs is to consider the case when the given degree is properly d.c.e. As pointed in Downey and Stob [12], and Cooper and Li [8], it is necessary to have the cases separated, as no uniform way working for both cases exists. Cooper [4] even proved that the low_2 d.c.e. degrees are dense.

Cooper and Yi considered the interaction between c.e. degrees and d.c.e degrees in [9] and introduced the notion of isolation. The existence of isolated degrees can be obtained from a result in Kaddah's paper [17], where she proved that low c.e. degrees branch in the d.c.e. degrees. Using this interaction phenomenon, Wu [24] provided another proof of Downey's diamond theorem, where Wu used the isolation to connect the cupping and the capping parts of the diamond embeddings. See Wu and Yamaleev's survey [25] on this topic.

Even though these two degree structures share several algebraic properties, these two structures are not elementarily equivalent. This was first proved in the 1980s by Arslanov in [2] and Downey in [11]. As to the density, Cooper, Harrington, Lachlan, Lempp and Soare proved that the d.c.e. degrees are not dense, where they constructed a maximal d.c.e. degree $\mathbf{d}$ below $\mathbf{0}'$. Obviously, $\mathbf{0}'$ does not split above $\mathbf{d}$.

We will consider d.c.e. *wtt*-degrees in this paper, i.e. the weak-truth-table degrees of d.c.e. sets. For $A, B \subseteq \mathbb{N}$, say that A is *weak-truth-table reducible* to B, denoted as $A \leq_{wtt} B$, if there is a partial computable functional Φ_e and a computable function f such that (i) $A = \Phi_e^B$, and (ii) for every x, $f(x) \geq u(B; e, x)$, where $u(B; e, x)$ is the use of the computation $\Phi_e^B(x)$. We use φ_e to denote the use function of Φ_e. The *wtt*-reduction was proposed by Friedberg and Rogers in 1959 in [15], and is now also called *bounded Turing* reduction. Lachlan proved that the upper semi-lattice of c.e. *wtt*-degrees is distributive, providing a crucial structural difference between c.e. *wtt*-degrees and c.e. Turing degrees. Ladner and Sasso then gave another difference in [19] by showing that the splitting and density can be combined for the c.e. *wtt*-degrees. Technically, weak-truth-table degrees can be handled much easier than Turing degrees. For instance, density of the c.e. *wtt*-degrees can be proved by a finite injury argument, whereas the analogous result for c.e. Turing degrees requires an infinite injury priority proof.

On the other hand, some structural properties of Turing degrees can be obtained from those of *wtt*-degrees via the so-called contiguous degrees. Here, a c.e. Turing degree $\mathbf{c}$ is *contiguous* if $\mathbf{c}$ contains exactly one c.e. *wtt*-degree. That is, any two c.e. sets A, B in a contiguous degree $\mathbf{c}$ are *wtt*-equivalent. Ladner and Sasso proved in [19] that any nonzero *wtt*-degree $\mathbf{c}$ has the anticupping property in the c.e. *wtt*-degrees. Thus, when $\mathbf{c}$ is contiguous, $\mathbf{c}$ also has the anticupping property in the Turing degrees, a result first proved by Yates by direct construction. This kind of "transfer" phenomenon has been further developed by Ambos-Spies in [1], Stob in [23], and Downey in [10].

In this paper, we consider the *wtt*-degrees of d.c.e. sets, and the remainder of this paper will be devoted to the proof of Theorem 1: there are no maximal d.c.e. *wtt*-degrees.

Our notation and terminology are standard and generally follow Soare [22] and Odifreddi [20]. The readers can refer Cooper's paper [6] for d.c.e. Turing degrees and Ambos-Spies' paper [1], Stob's paper [23] and Downey's paper [10] for the general idea on c.e. *wtt*-degrees.

2 Requirements and Construction

Let $K = \{e : \varphi_e(e)\}$, Turing's halting problem, and let $\{K_s : s \in \omega\}$ be a recursive enumeration of K. Note that K is *wtt*-complete among all d.c.e. sets, and we will assume that for each s, $|K_{s+1} \backslash K_s| = 1$. Let D be any d.c.e. set in $\mathbf{d}_{wtt}$, and $\{D_s : s \in \omega\}$ be a d.c.e. approximation of D. An additional addition for this approximation is: for any s, $|(D_{s+1}\backslash D_s) \cup (D_s \backslash D_{s+1})| = 1$.

For the proof of Theorem 1, we will construct c.e. sets B and C satisfying the following requirements:

$\mathcal{S}$**:** $K = B \sqcup C$;
$\mathcal{P}_e^B$**:** $K \neq \Phi_e^{B\oplus D}$, where the use function of $\Phi_e^{B\oplus D}$, i.e. φ_e, is bounded by ψ_e;
$\mathcal{P}_e^C$**:** $K \neq \Phi_e^{C\oplus D}$, where the use function of $\Phi_e^{C\oplus D}$, i.e. φ_e, is bounded by ψ_e.

Here $\{(\Phi_e, \psi_e) : e \in \omega\}$ is a recursive list of all pairs (Φ, ψ), Φ a partial computable functional Φ and ψ a partial computable function. As indicated at the beginning of this paper, the $\mathcal{S}$-requirement ensures that K and $B \oplus C \oplus D$ are *wtt*-equivalent. All the $\mathcal{P}_e^B$ requirements, $e \in \omega$, ensure that K is not *wtt*-reducible to $B \oplus D$, and all the $\mathcal{P}_e^C$ requirements, $e \in \omega$, ensure that K is not *wtt*-reducible to $C \oplus D$. Thus, $B \oplus D$ and $C \oplus D$ are not *wtt*-reducible to each other, which implies that both are strictly above $\mathbf{d}_{wtt}$.

The idea of satisfying the $\mathcal{S}$-requirement is standard. That is, at any stage s, we find a requirement $\mathcal{R}$ with the highest priority with k_s less than the restraint $r(\mathcal{R}, s)$, if exists. If $\mathcal{R}$ is a $\mathcal{R}_e^B$-requirement, then enumerate k_s into C. Otherwise, enumerate k_s into B. Obviously, $B \sqcup C = K$.

Before we describe how to satisfy a $\mathcal{P}$-requirement, a $\mathcal{P}_e^B$-requirement, say, we first review the idea when D is c.e., and then we show the changes we need to make for the case when D is d.c.e.

The main idea for the case *when D is c.e.* is the Sacks preservation strategy, i.e., to find a disagreement between K and $\Phi_e^{B\oplus D}$, we define expansionary stages and extend a *wtt*-reduction Δ_e at expansionary stages such that if there were infinitely many expansionary stages, then we would have $\Delta_e^D = K$, which is impossible. Here we define the length of the agreement between K and $\Phi_e^{B\oplus D}$ at stage s as

$$\ell^B(e, s) = \max\{x : \ (\forall y < x)[\Phi_e^{B\oplus D}(y)[s]\downarrow \text{ with use } \varphi_e(y) < \psi_e(y)$$

$$\text{and } \Phi_e^{B\oplus D}(y)[s] = K_s(y)]\},$$

and say that s is an expansionary stage if for any expansionary stage $t < s$, $\ell^B(e,s) > \ell^B(e,t)$. At an expansionary stage s, for any $y < \ell^B(e,s)$, if $\Delta_e^D(y)$ has no definition at stage s, then define $\Delta_e^D(y)[s] = K_s(y)$ with use $\delta_e(y)[s] = \varphi_e(y)[s]$. So, δ_e is bounded by ψ_e, and if there were infinitely many expansionary stages, then $\Phi_e^{B\oplus D}$ would be total and Δ_e^D would be defined as a total function, showing that $K \leq_{wtt} D$, which is impossible. Thus, there are only finitely many expansionary stages, and we will have some $y \leq \ell^B(e)$, the length of agreement at the last expansionary stage, such that either $\Phi_e^{B\oplus D}(y)\uparrow$ or $\Phi_e^{B\oplus D}(y) \neq K(y)$.

In this strategy, the main point is to protect computations at expansionary stages. Assume that after an expansionary stage s_1 (so a restraint is imposed on the B-part to protect computations), we see that a computation $\Phi_e^{B\oplus D}(y)$ changes, with $y < \ell^B(e,s_1)$, because of the changes of D between stages s_1 and s_2 say. We also assume that s_2 is not an expansionary stage, but we see that $\Phi_e^{B\oplus D}(y)[s_2]$ converges. The strategy says that at the next expansionary stage s_3, we will protect $\Phi_e^{B\oplus D}(y)[s_3]$. This computation $\Phi_e^{B\oplus D}(y)[s_3]$ we see at stage s_3 could be also different from $\Phi_e^{B\oplus D}(y)[s_2]$, and $\Phi_e^{B\oplus D}(y)[s_2]$ is not protected. It is okay, for D c.e., as either there are no more expansionary stages, or the changes between stages s_2 and s_3 will remain forever in D, and no more computation of $\Phi_e^{B\oplus D}(y)$ can be the same as $\Phi_e^{B\oplus D}(y)[s_2]$. Thus, among all the computations of $\Phi_e^{B\oplus D}(y)$ with use $\varphi_e(y)[s] < \psi_e(y)$, we only protect those we see at expansionary stages, and for y above, the change of D undefines $\Delta_e^D(y)$, and we will redefine $\Delta_e^D(y)$ again at the next expansionary stage. Nothing is complicated in this case.

As we are assuming that the uses of $\Phi_e^{B\oplus D}(y)$ are bounded by $\psi_e(y)$, the nature of finite injury allows to protect all computations of $\Phi_e^{B\oplus D}(y)$ we see at all stages. That is, whenever we see a new computation of $\Phi_e^{B\oplus D}(y)$, at stage s_2 above, we can protect it and redefine $\Delta_e^D(y)[s_2] = K_{s_2}(y)$. Of course, if the computation $\Phi_e^{B\oplus D}(y)$ changes between stage s_2 and s_3, then the D-changes undefine $\Delta_e^D(y)[s_2]$ again, allowing us to redefine it at stage s_3.

We adopt this idea of protecting all computations for our purpose when D is d.c.e. It can happen that a computation $\Phi_e^{B\oplus D}(y)[s]$ changes because of some enumeration of z into D, and after many stages, when z leaves D, at stage t say, the D-part of the oracle $B \oplus D$ recovers to the status at stage s, and if we protect $\Phi_e^{B\oplus D}(y)[s]$ at stage s, then we will have $\Phi_e^{B\oplus D}(y)[t] = \Phi_e^{B\oplus D}(y)[s]$. This variation of Sacks preservation strategy allows us to deal with cases when D is any Δ_2^0 set, i.e. $\mathbf{0}'_{wtt}$ splits above any other Δ_2^0 wtt-degrees.

We are now ready to provide a full construction. We first list the requirements as follows:

$$\mathcal{S} < \mathcal{P}_0^B < \mathcal{P}_0^C < \mathcal{P}_1^B < \mathcal{P}_1^C < \cdots < \mathcal{P}_e^B < \mathcal{P}_e^C < \cdots.$$

We say that a requirement $\mathcal{Q}$ has priority higher than $\mathcal{R}$ if $\mathcal{Q} < \mathcal{R}$ in the order defined above. So $\mathcal{S}$ has the highest priority, and at any stage s, we will enumerate k_s into one of B and C, but not both.

For a $\mathcal{P}$-requirement, $\mathcal{P}_e^B$ say, we call stage s a $\mathcal{P}_e^B$-*identical stage* if

1. $\ell^B(e,s) = \ell^B(e,s-1)$,
2. for all $y \leq \ell^B(e,s)$, $\Phi_e^{B\oplus D}(y)[s]$ converges if and only if $\Phi_e^{B\oplus D}(y)[s-1]$ converges,
3. for $\Phi_e^{B\oplus D}(y)[s]$ converges, the computation $\Phi_e^{B\oplus D}(y)[s]$ and $\Phi_e^{B\oplus D}(y)[s-1]$ are the same.

We say that $\mathcal{P}_e^B$ *requires attention at stage* s if s is not a $\mathcal{P}_e^B$-identical stage. Construction at stage 0: For all the requirements, let the corresponding restraint as 0.

Construction at stage $s > 0$:

Step 1. Among requirements $\mathcal{P}_0^B, \mathcal{P}_0^C, \mathcal{P}_1^B, \mathcal{P}_1^C, \cdots, \mathcal{P}_s^B, \mathcal{P}_s^C$, check which one requires attention at stage s. Let it be $\mathcal{Q}[s]$, and set the corresponding restraint as s. For those y less than the length of agreement of $\mathcal{Q}[s]$, define $\Delta_e^D(y) = K_s(y)$ with use $\delta_e(y)[s] = \varphi_e(y)[s]$. Initialize all the requirements with priority lower than $\mathcal{Q}[s]$.

Step 2. Among all the requirements with priority not lower than $\mathcal{Q}[s]$, find the one with higher priority, $\mathcal{R}[s]$ say, whose restraint is larger than k_s. If $\mathcal{R}[s]$ is a $\mathcal{P}^B$-strategy, then enumerate k_s into C. Otherwise, enumerate k_s into B. Initialize all the requirements with priority lower than $\mathcal{R}[s]$.

This completes the construction of stage s.

End of construction

3 Verification

In this section, we verify that the constructed c.e. sets B and C satisfy all the requirements. The actions at step 2 of each stage s ensure that $K = B \sqcup C$, and hence

Lemma 1. *The requirement $\mathcal{S}$ is satisfied.*

Now we verify that all the $\mathcal{P}$-requirements are satisfied. The following lemma is enough to show this.

Lemma 2. *For each $e \in \omega$,*

1. $\mathcal{P}_e^B$ can be initialized at most finitely many times;
2. $\mathcal{P}_e^B$ requires attention at most finitely many times;
3. $\mathcal{P}_e^B$ has finite restraint;
4. The same are true for requirement $\mathcal{P}_e^C$.

Proof. We prove it by induction on e. So we can assume that after a stage s_0 large enough, no more $\mathcal{P}_{e'}$-requirements, with $e' < e$, requires attention, or requires further enumeration of elements of K into B. Thus, after stage s_0, $\mathcal{P}_e^B$ cannot be initialized anymore. (1) holds.

To prove (2), we assume that $\mathcal{P}_e^B$ requires attention infinitely often. Then, the bounding function ψ_e is total. As D is d.c.e., we assume that after stage s_1, D becomes fixed up to $\psi_e(0)$. According to steps 1 and 2 in every stage, after stage s_0, we protect all the computations of $\Phi_e^{B\oplus D}(0)$ whenever we have a new computation, thus, for any computation of $\Phi_e^{B\oplus D}(0)$, if it converges before stage s_1, at stage s' say, and the D-part of the use agrees with $D \upharpoonright \psi_e(0)$, then this computation will converge forever. Of course, no such a computation occurs before stage s_1, then, when $\mathcal{P}_e^B$ requires attention again, we will have a new computation of $\Phi_e^{B\oplus D}(0)$, which will be protected. In both cases, $\Phi_e^{B\oplus D}(0)$ converges. The same idea can be used to prove that $\Phi_e^{B\oplus D}$ converges at any n, by induction.

We now show that Δ_e^D is total and computes K correctly. Again, we first show that $\Delta_e^D(0)$ is defined, with $\Delta_e^D(0) = K(0)$, and the same argument can be applied to show that for any n, $\Delta_e^{D(n)}$ is defined and equals to $K(n)$.

We assume again that D has no more change below $\psi_e(0)$ after stage s_1. Then for $\Delta_e^D(0)$ defined at stage s^* with the D-part of the use agreeing with $D \upharpoonright \varphi_e(0)[s^*]$, i.e.

$$D \upharpoonright \varphi_e(0)[s^*] = D_{s_1} \upharpoonright \varphi_e(0)[s^*],$$

we have $\Delta_e^D(0) = \Delta_e^D(0)[s^*]$. To see this, assume that $s^* < t_1 < t_2 < \cdots < t_n < s_1$ be a list of stages with $D_{t_i} \upharpoonright \varphi_e(0)[s^*] = D_{s^*} \upharpoonright \varphi_e(0)[s^*]$ for each $i \in \{1, \cdots, n\}$, then $\Delta_e^D(0)[t_i] = \Delta_e^D(0)[s^*]$ with use $\varphi_e(0)[s^*]$. This actually shows that for any definition of $\Delta_e^D(0)$, which is defined at other stages, D must have changes at some number z below $\varphi_e(0)[s^*]$, before stage s^* (if $\Delta_e^D(0)$ is defined before stage s^*), or between any two stages in this list (if $\Delta_e^D(0)$ is defined between these two stages). Of course, if there is no such a stage s^*, then at stage s_1, we define $\Delta_e^D(0)$ and by the choice of s_1, D will have no change any more and hence $\Delta_e^{D(0)}$ is defined.

Now we show that $\Delta_e^D(0) = K(0)$. Note that at stage s^* above, we have $\Phi_e^{B\oplus D}(0)[s^*]$ converges, and this computation is protected since s^* onwards and hence after stage s_1, i.e. the computation $\Phi_e^{B\oplus D}(0)$ will be the same as $\Phi_e^{B\oplus D}(0)[s^*]$. Thus, $\Phi_e^{B\oplus D}(0) = \Phi_e^{B\oplus D}(0)[s^*]$, and as we assume that there are infinitely many stages $\mathcal{P}_e^B$ requires attention, we know that after stage s_1, the agreement of $\mathcal{P}_e^B$ will be always larger than 0, and hence $K(0) = \Phi_e^{B\oplus D}(0)[s^*]$ forever, and as a consequence, $\Delta_e^D(0) = K(0)$.

We can then apply the same idea and show that for any n, $\Delta_e^D(n)$ is defined and equals to $K(n)$. This shows that $K \leq_{wtt} D$ via Δ. A contradiction. Thus (2) is true for $\mathcal{P}_e^B$ requirement.

Note that (2) tells us the existence of a stage s_2, after which the $\mathcal{P}_e^B$ requirement never requires attention again, which means that after stage s_2, all stages are $\mathcal{P}_e^B$-identical, and hence no more restraint is imposed. This shows that the last restraint imposed by $\mathcal{P}_e^B$ is before stage s_2, and as a consequence, the $\mathcal{P}_e^B$ requirement has finite restraint. (3) is true.

The same argument can show that (1), (2), (3) above are also true for $\mathcal{P}_e^C$ requirement. (4) is true.

This completes the proof of Lemma 3.2, and hence the proof of Theorem 1.

4 Further Remarks

As pointed out in the introduction, we can improve Theorem 1 and show that the d.c.e. *wtt*-degrees are dense, and hence for a given Turing degree **d**, it can either contain exactly one d.c.e. *wtt*-degree, or contain infinitely many d.c.e. *wtt*-degree. We call a d.c.e. Turing degree **d** contiguous if it contains exactly one d.c.e. *wtt*-degree. A recent work of the authors shows the existence of properly d.c.e. contiguous degrees. We have seen that c.e. contiguous degrees have many unusual applications, like Downey's idea of using c.e. contiguous degrees to show the downwards density of c.e. degrees with strong anti-cupping property, and we are interested in problems of properly d.c.e. contiguous degrees, like the distribution of such degrees and how we can use such degrees to transfer properties of *wtt*-degrees to Turing degrees.

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