OPEN QUESTIONS

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In [7] Kac and Weisfeiler conjectured that the maximum dimension of irreducible representations of a finite dimensional restricted Lie algebra L over an algebraically closed field of characteristic $p > 0$ is equal to

$$p^{\frac{1}{2}(\dim(L)-r(L))}$$

where $r(L)$ is the so-called index of L, defined as the minimum dimension of stabilizers of linear functions on L with respect to the coadjoint action of L. This conjecture is known to be true for solvable Lie algebras when $p > 2$ [5] and for Lie algebras which admit a linear function with a toral stabilizer [4]. The Jacobson-Witt algebra W_n has such linear functions only when $n \leq 2$. Therefore the above conjecture is true for W_1 and W_2 [3]. The indices $r(W_n)$ were computed for all n in [2]. In particular, $r(W_3) = p$. However, it is extremely difficult to understand generic irreducible representations of W_3.

Question 1. *Does the Kac-Weisfeiler conjecture hold for the Jacobson-Witt algebra W_3? Are the reduced enveloping algebras of W_3 generically semisimple or not?*

Question 2. *Can verification of the Kac-Weisfeiler conjecture be reduced to the case of semisimple Lie algebras and their central extensions or even to the case of simple Lie algebras and their central extensions?*

A natural way to approach this question would be to argue by induction on $\dim L$ using suitable reductions of irreducible representations with respect to ideals of L. If V is a simple L-module and I is a minimal solvable ideal of L such that the image I_V of I in $gl(V)$ contains nonscalar transformations, then two cases may occur. When I_V is abelian, V is induced from a simple module of a proper restricted subalgebra of L. In the second case I_V is a Heisenberg algebra, and a different type of reduction is needed. To reduce further to central extensions of simple Lie algebras one has to consider also nonsolvable ideals. Actually it was claimed in [1] that the Kac-Weisfeiler conjecture reduces to the case of semisimple Lie algebras, but the required reduction of representations was done only in the first of the aforementioned cases, and the possibility of nontrivial central extensions was not taken into account.

Question 3. *Suppose that a restricted Lie algebra L has a linear function with a toral stabilizer in L. What further conditions on L are needed to guarantee that for each $\xi \in L^*$ the corresponding reduced enveloping algebra of L is semisimple if and only if the stabilizer of ξ in L is a torus?*

Further questions are related to simple Lie algebras. Let $L_{[p]}$ stand for the minimal p-envelope of L (see [6]). Thus L is restricted if $L_{[p]} = L$.

Question 4. *Let L be a simple finite dimensional Lie algebra over an algebraically closed field of characteristic $p > 0$. Is it always true that $L_{[p]} = T + L$ for a suitable torus T? Does this equality holds for each torus of maximal dimension in $L_{[p]}$?*

When $p > 3$ one can use the classification of simple Lie algebras, although the case of Hamiltonian Lie algebras is not quite straightforward. However, the question is meaningful also in characteristics 2 and 3.

Question 5. *Let L be a simple finite dimensional Lie algebra over an algebraically closed field of characteristic $p > 2$. Suppose that L is not restricted and T is any torus in $L_{[p]}$. Is it always true that all nonzero elements of the group*

$$\{\alpha \in T^* \mid \alpha(t^{[p]}) = \alpha(t)^p \text{ for all } t \in T\}$$

are roots of L, that is, L has precisely $p^{\dim T} - 1$ nonzero roots with respect to T?

When $p = 2$ there are counterexamples given by the contragredient Lie algebras $\mathfrak{g}(A)$ with some of the matrices listed in [8]. Such a matrix of smallest size is

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Question 6. *Does there exist a restricted simple finite dimensional Lie algebra over an algebraically closed field of characteristic 2 which has toral rank 2 and is not isomorphic to the classical Lie algebras of type A_2, G_2?*

This question is justified by the following observations. Over a field of characteristic 2 the classical Lie algebra of type B_2 is not simple since root vectors corresponding to short roots generate a proper ideal. After properly rescaling basis elements in the Chevalley $\mathbb{Z}$-form of B_2, the reduction modulo 2 produces a simple contragredient Lie algebra with the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which is not restricted, however. The restricted Cartan type Lie algebras W_2 and K_3 (the contact algebra defined with respect to the contact form $dx_1 + x_2\,dx_3$) are isomorphic to the Lie algebra of type A_2, while $S_3^{(1)}$ and $H_4^{(2)}$ to that of type G_2.

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