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REFINED INEQUALITIES FOR EUCLIDEAN MOMENTS OF A DOMAIN WITH RESPECT TO ITS BOUNDARY*

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Abstract. Euclidean moments of simply connected plane domains are investigated. The moments are defined as the pth power of the L^p-norms of the distance function to the boundary of the domain. As was shown by Avkhadiev (1998) the Euclidean moment of inertia ($p = 2$) gives two-sided estimates for the torsional rigidity of the domain. The estimate of the torsional rigidity connected with the domain area is the famous Saint-Venant–Pólya inequality, which was refined by Payne (1962). In this paper we obtain Payne-type inequalities for the Euclidean moments. A surprising fact is that new extremal domains, different from a disk, are found.

Key words. isoperimetric inequality, Bonnesen's inequality, Euclidean moments of a domain with respect to its boundary, torsional rigidity

AMS subject classifications. 30A10, 30C75

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1. Introduction. In this paper we are concerned with the torsional problem of mathematical physics. Let G be a simply connected plane domain, and let $\mathrm{A}(G)$ be the area of G. Denote by $\mathrm{P}(G)$ the torsional rigidity,

$$\mathrm{P}(G) := 2\int_G u(z,G)\,\mathrm{dA},$$

where $u(z,G)$ is the solution to the boundary value problem $\triangle u = -2$ in G, $u = 0$ on ∂G. In 1948 Pólya proved the important property of the torsional rigidity [5]

$$\mathrm{P}(G) \le \frac{\mathrm{A}(G)^2}{2\pi},$$

which had been known as the Saint-Venant hypothesis. In 1963 Payne proved that, in fact, the Saint-Venant–Pólya inequality follows from the sharper result [4]

$$\mathrm{A}(G)^2 - 2\pi\mathrm{P}(G) \ge \left(\mathrm{A}(G) - 2\pi u(G)\right)^2, \tag{1}$$

where $u(G) := \sup\{u(z,G) : z \in G\}$. In both cases a disk is the unique extremal domain.

The Saint-Venant–Pólya and Payne inequalities give only the upper bounds for the torsional rigidity. In 1995 Avkhadiev [1] proved the two-sided estimates for $\mathrm{P}(G)$ of simply connected domains in terms of the Euclidean moment of inertia $\mathrm{I}_2(G)$,

$$\mathrm{I}_2(G) \le \mathrm{P}(G) \le 64\,\mathrm{I}_2(G), \quad \mathrm{I}_2(G) := \int_G \rho(z,G)^2\,\mathrm{dA},$$

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