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# An $\mathbb{R}$-Linear Conjugation Problem for Two Concentric Annuli

**A. Yu. Kazarin*** **and Yu. V. Obnosov****

(Submitted by A. M. Elizarov)

*Kazan Federal University, Kremlevskaya ul. 18, Kazan, Tatarstan, 420008 Russia*

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**Abstract**—We consider an infinite planar four-phase heterogeneous medium with three concentric circles as a boundary between isotropic medium's components of distinct resistivities/conductivities. It is supposed that the velocity field in this structure is generated by a finite set of arbitrary multipoles. We distinguish two cases when multipoles are inside of medium's components or at the interface. An exact analytical solution of the corresponding $\mathbb{R}$-linear conjugation boundary value problem is derived for both cases. Examples of flow nets (isobars and streamlines) are presented.

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## 1. INTRODUCTION

One of the most common heterogeneous structures encountered in nature are circular and annulus-shaped structures. Because of their simplicity, these structures are studied in many applications ([1, 7–9, 15]). Such media are the most easiest for investigations in both analytical and numerical ways and also they are a good starting point to apply different methods and algorithms.

A remarkable fact of the theory of function of complex variables is that every analytical (holomorphic) function in its analyticity domain can be interpreted as a complex potential of some steady two-dimensional flow [5].

It is well known that for the case of one circular inclusion the corresponding complex potential of a flow generated by a single dipole at infinity is, up to a multiplicative constant, Zhukovsky's function (the Miln-Thomson theorem [6], p. 153). For this structure a more general problem of determination of a complex potential for a flow generated by a set of arbitrary multipoles can be solved. In the monography [11], pp. 90–97 the solution is given in terms of Cauchy type integrals. The generalization of Miln-Thomson theorem was obtained in the monograph [14], pp. 26–34. Also a solution for a three-phase structure with two concentric circles as an interface was given there. The solution method, used in this monograph , can be applied for investigation of multiphase circular structures.

It is well known that for an arbitrary heterogeneous medium corresponding $\mathbb{R}$-linear conjugation boundary-value problem ([3], p. 53) can not be solved analytically. Only for some specific structures it is possible to do. For example, the problem of the perturbation of a given complex potential by inserting distinct inclusions into an isotropic medium was solved for circular [12], elliptical [17], parabolic [13], hyperbolic [10], circular and elliptical annuli inclusions [16] and [4]. Much more progress can be made if all inclusions are perfectly resisting [2].

The objective of the present work is to determine a complex potential generated by a set of arbitrary multipoles in a four-phase structure consisting of two adjoined concentric annuli, theirs interior and

*E-mail: alkidkaz@mail.ru
**E-mail: yobnosov@kpfu.ru

exterior. From mathematical point of view, we have to solve a boundary-value problem of $\mathbb{R}$-linear conjugation in the class of piece-wise meromorphic functions with principal parts fixed in advance. We divide our solution into two parts. At first we consider the case when there are no multipoles at the structures interface and boundary singularities are admitted in the second part.

Let us turn to a strict statement of the problem.

## 2. STATEMENT OF THE PROBLEM

We consider a four—phase continuous isotropic linear medium consisting of the exterior of the circle $S_1 = \{z : |z| > r_1\}$, the circle $S_4 = \{z : |z| < r_3\}$ and two annuli $S_2 = \{z : r_2 < |z| < r_1\}$, $S_3 = \{z : r_3 < |z| < r_2\}$.

It is required to define a stationary power field $\mathrm{v}(x, y) = (v_x, v_y) = \mathrm{v}_k(x, y)$, $(x, y) \in S_k$, $k = \overline{1, 4}$, such that

$$\operatorname{div} \mathrm{v}_k = 0, \quad \operatorname{curl} \mathrm{v}_k = 0 \tag{1}$$

in all uniform components $S_k$. It is supposed that the principal part $f(z)$ of the corresponding complex potential

$$w(z) = (\varphi(x, y), \psi(x, y)), \quad \varphi'_x = \psi'_y = v_x, \quad \varphi'_y = -\psi'_x = v_y,$$

has a finite set of singular points $T = T_1 \cup T_2 \cup T_3 \cup T_4$, $T_k \subset S_k$.

Along the interface lines $l_k = \{t : |t| = r_k\}$ usual boundary conditions hold: continuity of the stream functions and linear proportionality of the potential functions, i.e.

$$\psi_k(t) = \psi_{k+1}(t), \quad \rho_k \varphi_k(t) = \rho_{k+1} \varphi_{k+1}(t), \quad k = \overline{1, 3}, \tag{2}$$

where constant coefficient $\rho_k$ characterizes physical properties of the phase $S_k$.

Henceforth, the plane $(x, y)$ is understood as a plane of the complex variable $z = x + \mathrm{i}y$. A vector-function $\mathrm{v}(x, y)$ is interpreted as an anti-holomorphic, due to the conditions (1), complex-valued function $\mathrm{v}(z) = v_x(z) + \mathrm{i}v_y(z)$, which is complex conjugated with the derivative of the complex potential function $w'(z) = v(z) = v_x(z) - \mathrm{i}v_y(z)$.

As is well known ([3], p. 53), the real boundary conditions (2) are equivalent to complex ones, which in our case take the form:

$$\begin{cases} v_1(t) = A_1 v_2(t) + B_1 r_1^2 t^{-2} \overline{v_2(t)}, & t \in l_1 = \{t : |t| = r_1\}, \\ v_2(t) = A_2 v_3(t) + B_2 r_2^2 t^{-2} \overline{v_3(t)}, & t \in l_2 = \{t : |t| = r_2\}, \\ v_3(t) = A_3 v_4(t) + B_3 r_3^2 t^{-2} \overline{v_4(t)}, & t \in l_3 = \{t : |t| = r_3\}. \end{cases} \tag{3}$$

The coefficients $A_k$, $B_k$ are determined via the formulae:

$$A_k = (\rho_k + \rho_{k+1})/2\rho_k, \quad B_k = (\rho_k - \rho_{k+1})/2\rho_k, \ k = 1, 2, 3.$$

We introduce also the notations

$$\Delta_k = \frac{B_k}{A_k} = \frac{\rho_k - \rho_{k+1}}{\rho_k + \rho_{k+1}}, \quad A_k = \frac{1}{1 + \Delta_k}, \quad B_k = \frac{\Delta_k}{1 + \Delta_k}, \quad k = 1, 2, 3,$$

which will be used below.

Thus, it is required to find a piecewise meromorphic solution $v(z)$ of the boundary value problem (3). The principal part $F(z) = f'(z)$ of $v(z)$ is a fixed rational function with a finite number of poles. We start with the case when poles of $F(z)$ do not belong to the interface components lines $l_k$.

## 3. SOLUTION OF THE BOUNDARY VALUE PROBLEM (3) IN THE CASE OF INNER MULTIPOLES

Piecewise meromorphic solution $v(z)$ of the problem (3) with a given principal part $F(z) = f'(z)$ can be written as:

$$v(z) = v_k(z) = F_k(z) + V_k(z), \quad z \in S_k, \quad p = \overline{1,4}, \tag{4}$$

where $F_k(z)$ is the sum of all simple fractions, the summands of rational function $F(z)$, with their poles in the domain $S_k$ and $V_k(z)$ is an unknown holomorphic in $S_k$ function. For a function $F_1(z)$ is admissible a polynomial term and holomorphic summand $V_1(z)$ vanishes at infinity.

Let $S_k^+$ and $S_k^-$ are the interior and the exterior of the circle $l_k$ respectively. Due to the Laurent theorem analytic functions in the annuli $S_2$, $S_3$ can be represented as a sum:

$$V_k(z) = V_k^+(z) + V_k^-(z), \quad V_k^-(\infty) = 0, \tag{5}$$

where $V_k^+(z)$ and $V_k^-(z)$ are holomorphic functions in the domains $S_{k-1}^+$ and $S_k^-$ correspondingly.

Let us introduce now the following functions:

$$\Phi_1 = \begin{cases} -V_1(z) + A_1[F_2(z) + V_2^-(z)] + B_1 r_1^2 z^{-2}\overline{V_2^+(z_1^*)}, & z \in S_1^-, \\ F_1(z) - A_1 V_2^+(z) - B_1 r_1^2 z^{-2}[\overline{F_2(z_1^*)} + \overline{V_2^-(z_1^*)}], & z \in S_1^+, \end{cases}$$

$$\Phi_2 = \begin{cases} -V_2^-(z) + A_2[F_3(z) + V_3^-(z)] + B_2 r_2^2 z^{-2}\overline{V_3^+(z_2^*)}, & z \in S_2^-, \\ F_2(z) + V_2^+(z) - A_2 V_3^+(z) - B_2 r_2^2 z^{-2}[\overline{F_3(z_2^*)} + \overline{V_3^-(z_2^*)}], & z \in S_2^+, \end{cases}$$

$$\Phi_3 = \begin{cases} -V_3^-(z) + A_3 F_4(z) + B_3 r_3^2 z^{-2}\overline{V_4(z_3^*)}, & z \in S_3^-, \\ F_3(z) + V_3^+(z) - A_3 V_4(z) - B_3 r_3^2 z^{-2}\overline{F_4(z_3^*)}, & z \in S_3^+, \end{cases}$$

where $z_k^* = r_k^2/\overline{z}$ is the point symmetrical with $z$ about the circle $l_k$.

Each function $\Phi_k(z)$ $(k = \overline{1,3})$ is holomorphic in the domains $S_k^+/\{0\}$ and $S_k^-$ and due to the corresponding boundary condition (3) continuous across the line $l_k$. At the origin this function has a simple pole and it vanishes at infinity as $V_k^-(\infty) = F_k(\infty) = 0$. By the generalized Liouville theorem $\Phi_k(z) = C_k/z$, where $C_k$ is a constant to be determined. Thus, we get the following system for definition of unknown functions $V_1(z)$, $V_2^\pm(z)$, $V_3^\pm(z)$, $V_4(z)$:

$$\begin{cases} -V_1(z) + A_1[F_2(z) + V_2^-(z)] + B_1 r_1^2 z^{-2}\overline{V_2^+(z_1^*)} = C_1/z, & z \in S_1^-, \\ F_1(z) - A_1 V_2^+(z) - B_1 r_1^2 z^{-2}[\overline{F_2(z_1^*)} + \overline{V_2^-(z_1^*)}] = C_1/z, & z \in S_1^+, \\ -V_2^-(z) + A_2[F_3(z) + V_3^-(z)] + B_2 r_2^2 z^{-2}\overline{V_3^+(z_2^*)} = C_2/z, & z \in S_2^-, \\ F_2(z) + V_2^+(z) - A_2 V_3^+(z) - B_2 r_2^2 z^{-2}[\overline{F_3(z_2^*)} + \overline{V_3^-(z_2^*)}] = C_2/z, & z \in S_2^+, \\ -V_3^-(z) + A_3 F_4(z) + B_3 r_3^2 z^{-2}\overline{V_4(z_3^*)} = C_3/z, & z \in S_3^-, \\ F_3(z) + V_3^+(z) - A_3 V_4(z) - B_3 r_3^2 z^{-2}\overline{F_4(z_3^*)} = C_3/z, & z \in S_3^+. \end{cases} \tag{6}$$

We rewrite the last equality of the system (6) as follows

$$F_3(z) + V_3^+(z) - A_3 V_4(z) = \frac{1}{z}\left(B_3 r_3^2 \overline{F_4(z_3^*)}/z + C_3\right), \quad z \in S_3^+.$$

All summands on the left hand-side of the last equality are holomorphic everywhere in the circle $S_3^+$ and, in particular, at the point $z = 0$, consequently at the origin should vanish coefficient at the factor $1/z$ on the right-hand side. It is not difficult to prove that the last demand takes place if

$$C_3 = -B_3 \lim_{z\to 0} r_3^2 \overline{F_4(z_3^*)}/z = B_3 \overline{a_4}, \quad a_4 = \operatorname{res}_\infty F_4(z). \tag{7}$$

Indeed, every vanishing at infinity rational function $P(z)$, and $F_4(z)$ in particular, can be represented as a finite sum of summands of view $P_k(z) = c_k/(z - z_0)^k$. Obviously that $\lim_{z\to 0} r^2 z^{-1}\overline{P_k(r^2/\overline{z})} =$

$\{\overline{c_1}, k=1; 0, k>1\}$, and $c_1 = \mathrm{res}_{z_0} P(z)$. Wherefrom follows our assertion due to the Cauchy's theorem about the total sum of residues.

Next, we find $C_2$ from the fourth equation (6)

$$C_2 = -\mathrm{res}_0 \left( B_2 r_2^2 z^{-2} \left[ \overline{F_3(z_2^*)} + \overline{V_3^-(z_2^*)} \right] \right) = B_2 \left( \overline{a_3} - \lim_{z\to 0} r_2^2 z^{-1} \overline{V_3^-(z_2^*)} \right),$$

where $a_3 = \mathrm{res}_\infty F_3(z)$. The last limit equals $-\overline{C}_3 - C_3/\Delta_3$, as from the fifth equation (6) follows $\mathrm{res}_\infty V_3^-(z) = C_3 + \overline{C}_3/\Delta_3$. So,

$$C_2 = B_2 \left( \overline{a}_3 + \overline{C}_3 + C_3/\Delta_3 \right). \tag{8}$$

Analogously, from the second and the third equations (6) we find

$$C_1 = B_1 \left( \overline{a}_2 + \overline{C_2} + C_2/\Delta_2 \right), \quad a_2 = \mathrm{res}_\infty F_2(z). \tag{9}$$

We start to solve the system (6) by excluding $V_4(z)$ from its two last equations

$$V_3^-(z) = (1-\Delta_3)F_4(z) + \Delta_3 r_3^2 z^{-2} [\overline{F_3(z_3^*)} + \overline{V_3^+(z_3^*)}] - (C_3 + \Delta_3 \overline{C_3})/z.$$

Substitution of this result into the third equation (6) gives

$$\begin{aligned} V_2^-(z) &= A_2 F_3(z) + A_2(1-\Delta_3)F_4(z) + B_2 r_2^2 z^{-2} \overline{V_3^+(z_2^*)} \\ &+ A_2 \Delta_3 r_3^2 z^{-2} \left[ \overline{F_3(z_3^*)} + \overline{V_3^+(z_3^*)} \right] - \left[ A_2(C_3 + \Delta_3 \overline{C_3}) + C_2 \right]/z. \end{aligned}$$

From now on, for the sake of brevity, we denote $\delta_j = \Delta_j r_j^2$, $\delta_{ij} = \Delta_i \Delta_j (r_j/r_i)^2$, and $z_{ij}^* = (r_j/r_i)^2 z$, i.e. $z_{ij}^*$ is the successive symmetry $z$ about $l_i$ and $l_j$. Excluding $V_2^-(z)$ from the second equation (6) and using (7)–(9) we get

$$\begin{aligned} V_2^+(z) &= \frac{F_1(z)}{A_1} - \frac{\delta_1}{z^2} \left( \overline{F_2(z_1^*)} + \frac{\overline{F_3(z_1^*)}}{1+\Delta_2} + \frac{(1-\Delta_3)}{1+\Delta_2} \overline{F_4(z_1^*)} \right) - \frac{\delta_{13}}{1+\Delta_2} F_3(z_{13}^*) \\ &- \frac{\delta_{13}}{1+\Delta_2} V_3^+(z_{13}^*) - \frac{\delta_{12}}{1+\Delta_2} V_3^+(z_{12}^*) + \left( \frac{(1-\Delta_3)}{1+\Delta_2} \overline{a_4} + \frac{\overline{a_3}}{1+\Delta_2} - \Delta_1 \overline{a_2} \right)/z. \end{aligned}$$

Finally, substitution of the last three representations and (7)–(9) into the fourth equation (6) leads to the following functional equation about $V_3^+(z)$

$$V_3^+(z) = -(K_{12} + K_{13} + K_{23}) V_3^+(z) + F_0(z), \tag{10}$$

where the operator $K_{ij}$ is defined as $K_{ij} V(z) = \delta_{ij} V(z_{ij}^*)$, and

$$\begin{aligned} F_0(z) &= (1+\Delta_1)(1+\Delta_2)F_1(z) + (1+\Delta_2)F_2(z) - \delta_{13} F_3(z_{13}^*) - \delta_{23} F_3(z_{23}^*) \\ &- \left[ (1+\Delta_2)\delta_1 \overline{F_2(z_1^*)} + \delta_1 \overline{F_3(z_1^*)} + \delta_2 \overline{F_3(z_2^*)} + (1-\Delta_3) \left( \delta_1 \overline{F_4(z_1^*)} + \delta_2 \overline{F_4(z_2^*)} \right) \right] z^{-2} \\ &- \left[ \Delta_1 (1+\Delta_2) \overline{a_2} + (\Delta_1 + \Delta_2) \left( \overline{a_3} + (1-\Delta_3) \overline{a_4} \right) \right]/z = G_1(z) - G_2(z)/z^2 - c_0/z. \end{aligned} \tag{11}$$

The function (11) is holomorphic in the circle $S_2^+$, as $G_2(z)/z^2$ has a simple pole at $z=0$ and

$$c_0 = \Delta_1(1+\Delta_2)\overline{a_2} + (\Delta_1 + \Delta_2) \left( \overline{a_3} + (1-\Delta_3)\overline{a_4} \right) = -\mathrm{res}_0 [G_2(z)/z^2].$$

Exactlier, according to the assumptions of this section $F(z)$ is holomorphic at the interface, hence $F_0(z)$ is holomorphic into the closed circle $\overline{S_2^+}$.

Let the equation (10) is solvable and $V_3^+(z)$ is its solution holomorphic in the circle $S_2^+$, then each function $K_{ij} V_3^+(z)$ is holomorphic in the circle of radius $r_2(r_i/r_j)^2 > r_2$ if $i<j$. It means that the right-hand side of the equality (10) is holomorphic into the closed circle $\overline{S_2^+}$. Hence the same is true for a required solution $V_3^+(z)$.

So, all terms of the equation (10) are holomorphic in the closed circle $\overline{S_2^+}$ and they can be represented there as a converging absolutely and uniformly Taylor series:

$$V_3^+(z) = \sum_{l=0}^{\infty} c_l z^l, \quad K_{ij} V_3^+(z) = \sum_{l=0}^{\infty} \delta_{ij} (r_j/r_i)^{2l} c_l z^l, \quad F_0(z) = \sum_{l=0}^{\infty} \frac{F_0^{(l)}(0)}{l!} z^l.$$

We find all unknown coefficients $c_l$ by equating coefficients at the same powers $z$ on the left- and right-hand sides of the equation (10). Thus, we get

$$V_3^+(z) = \sum_{l=0}^{\infty} \frac{F_0^{(l)}(0)/l! z^l}{1 + \delta_{12}(r_2/r_1)^{2l} + \delta_{13}(r_3/r_1)^{2l} + \delta_{23}(r_3/r_2)^{2l}}. \tag{12}$$

It is clear that the denominator of $c_l$ tends to one when $l$ tends to infinity as $|\delta_{ij}| < 1$ and $r_j/r_i < 1$ if $i < j$. Hence the series (12) converges absolutely and uniformly in $\overline{S_2^+}$.

Now we can find consequentially the required solution of the problem (3) from the system (6) and in accordance with the definitions (4), (5).

$$v_4(z) = F_4(z) + (1+\Delta_3)(F_3(z) + V_3^+(z)) - \delta_3 z^{-2}\overline{F_4(z_3^*)} - C_3(1+\Delta_3)/z,$$

$$v_3(z) = F_3(z) + V_3^+(z) + (1-\Delta_3)F_4(z) + \frac{\delta_3}{z^2}[\overline{F_3(z_3^*)} + \overline{V_3^+(z_3^*)}] - \frac{C_3 + \Delta_3\overline{C_3}}{z},$$

$$\begin{aligned} v_2(z) &= (1+\Delta_1)F_1(z) + F_2(z) + A_2F_3(z) + A_2(1-\Delta_3)F_4(z) - A_2\delta_{13}F_3(z_{13}^*) \\ &+ \frac{A_2}{z^2}\left(\delta_3\overline{F_3(z_3^*)} - \delta_1\overline{F_3(z_1^*)} - (1+\Delta_2)\delta_1\overline{F_2(z_1^*)} - (1-\Delta_3)\delta_1\overline{F_4(z_1^*)}\right) \\ &+ \frac{A_2}{z^2}\left(\delta_3\overline{V_3^+(z_3^*)} + \delta_2\overline{V_3^+(z_2^*)}\right) - A_2\left(\delta_{13}V_3^+(z_{13}^*) + \delta_{12}V_3^+(z_{12}^*)\right) \\ &+ \left(A_2((\Delta_1 - \Delta_3)\overline{C_3} + (\Delta_1\Delta_3 - 1)C_3) + \Delta_1\overline{C_2} - C_2 - (1+\Delta_1)C_1\right)/z, \end{aligned}$$

$$\begin{aligned} v_1(z) &= F_1(z) + (1-\Delta_1)\left[F_2(z) + A_2F_3(z) + A_2(1-\Delta_3)F_4(z)\right] \\ &+ \frac{A_2(1-\Delta_1)}{z^2}\left(\delta_3\left[\overline{F_3(z_3^*)} + \overline{V_3^+(z_3^*)}\right] + \delta_2\overline{V_3^+(z_2^*)}\right) + \delta_1 z^{-2}\overline{F_1(z_1^*)} \\ &- \left(A_2(1-\Delta_1)(C_3 + \Delta_3\overline{C_3}) + (1-\Delta_1)C_2 + \Delta_1\overline{C_1} + C_1\right)/z, \end{aligned}$$

where the parameters $C_k$ are defined in (7) through (9).

In conclusion of this section we consider the most important, in view of possible applications, case when $F(z)$ has only simple poles with real residues. It means that we are looking for a complex potential generated by finite set of sinks and sources. Sufficiently to consider the case when each of four summands $F(z)$ has no more than one, may be none, pole, i.e.

$$F_k(z) = c_k/(z - z_k), \quad z_k \in S_k, \quad c_k \in \mathbb{R}, \quad k = \overline{1,4}.$$

If, in particular, there is no sink, no source at infinity then $a_1 + a_2 + a_3 + a_4 = 0$, where $a_k = -c_k = \mathrm{res}_\infty F_k(z)$.

Omitting laborious algebra based on (11), (12), (7), (8), (9) and last four presentations for $v_k(z)$, we summarise the final results

$$\begin{aligned} F_0(z) &= \frac{c_1(1+\Delta_1)(1+\Delta_2)}{z - z_1} + \frac{c_2(1+\Delta_2)}{z - z_2} - \frac{c_3\Delta_1\Delta_3}{z - z_3r_1^2/r_3^2} - \frac{c_3\Delta_2\Delta_3}{z - z_3r_2^2/r_3^2} \\ &+ \frac{c_2\Delta_1(1+\Delta_2)}{z - r_1^2/\overline{z}_2} + \frac{c_3\Delta_1}{z - r_1^2/\overline{z}_3} + \frac{c_3\Delta_2}{z - r_2^2/\overline{z}_3} + \frac{c_4\Delta_1(1-\Delta_3)}{z - r_1^2/\overline{z}_4} + \frac{c_4\Delta_2(1-\Delta_3)}{z - r_2^2/\overline{z}_4}, \end{aligned}$$

$$\begin{aligned} v_1(z) &= \frac{c_1}{z - z_1} + \frac{(1-\Delta_1)c_2}{z - z_2} + \frac{A_2(1-\Delta_1)c_3}{z - z_3} + \frac{A_2(1-\Delta_3)(1-\Delta_1)c_4}{z - z_4} \\ &+ \frac{A_2(1-\Delta_1)}{z^2}\left[\delta_3\overline{V_3^+(z_3^*)} + \delta_2\overline{V_3^+(z_2^*)}\right] + \frac{A_2(1-\Delta_1)\Delta_3c_3}{z - r_3^2/\overline{z_3}} + \frac{\Delta_1c_1}{z - r_1^2/\overline{z_1}} \\ &+ \left(\Delta_1(c_1 + c_2) + (1 - A_2(1-\Delta_1)(1-\Delta_3))(c_3 + c_4)\right)/z, \end{aligned}$$

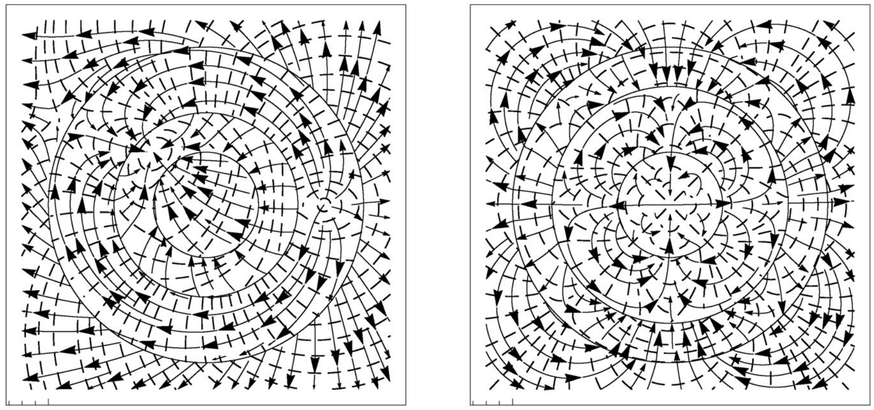

**Fig. 1.** $\rho_1 = 1$, $\rho_2 = 0.2$, $\rho_3 = 5$, $\rho_4 = 0.5$ at the left; $\rho_1 = 1$, $\rho_2 = 20$, $\rho_3 = 0.1$, $\rho_4 = 10$ at the right.

$$
\begin{aligned}
v_2(z) &= \frac{(1+\Delta_1)c_1}{z-z_1} + \frac{c_2}{z-z_2} + \frac{A_2 c_3}{z-z_3} + \frac{A_2(1-\Delta_3)c_4}{z-z_4} - \frac{A_2\Delta_1\Delta_3 c_3}{z-(r_1/r_3)^2 z_3} \\
&\quad - A_2\left(\frac{\Delta_3 c_3}{z-r_3^2/\overline{z_3}} - \frac{\Delta_1 c_3}{z-r_1^2/\overline{z_3}} - \frac{(1+\Delta_2)\Delta_1 c_2}{z-r_1^2/\overline{z_2}} - \frac{(1-\Delta_3)\Delta_1 c_4}{z-r_1^2/\overline{z_4}}\right) \qquad (13) \\
&+ \frac{A_2}{z^2}\left(\delta_3\overline{V_3^+(z_3^*)} + \delta_2\overline{V_3^+(z_2^*)}\right) - A_2\left(\delta_{13}V_3^+(z_{13}^*) + \delta_{12}V_3^+(z_{12}^*)\right) + A_2(\Delta_2+\Delta_3)(c_3+c_4)/z, \\
v_3(z) &= \frac{c_3}{z-z_3} + \frac{(1-\Delta_3)c_4}{z-z_4} - \frac{\Delta_3 c_3}{z-r_3^2/\overline{z_3}} + V_3^+(z) + \frac{\delta_3}{z^2}\overline{V_3^+(z_3^*)} + \Delta_3(c_3+c_4)/z, \\
v_4(z) &= \frac{c_4}{z-z_4} + \frac{(1+\Delta_3)c_3}{z-z_3} + \frac{\Delta_3 c_4}{z-r_3^2/\overline{z_4}} + (1+\Delta_3)V_3^+(z).
\end{aligned}
$$

Here $V_3^+(z)$ is given by equation (12) with $F_0(z)$ defined in (13).

**Example 1.** Let $r_1 = 6$, $r_2 = 4$, $r_3 = 2$. In Fig. 1 the streamlines and equipotential lines (dashed) are plotted for two different complex potentials $f_1(z) = 2\ln(z-5) - 2\ln(z+2-2\mathrm{i})$ (left panel) and $f_2(z) = -0,5z^{-2}$ (right panel).

## 4. SOLUTION OF THE PROBLEM (3) WITH SINGULARITIES AT THE INTERFACE

Let all poles of $F(z)$ are at the interface components $l_k$, $k = 1, 2, 3$. For the sake of simplicity we consider the case of no more than one singular point at each component $l_k$ i.e.,

$$F(z) = \sum_{j=1}^{3}\sum_{k=1}^{n_j} \frac{b_k^j}{(z-\tau_j)^k} = F_{01}(z) + F_{02}(z) + F_{03}(z),$$

We use here the same representation (4) for a required solution with principal parts $F_k(z)$ defined as follows:

$$F_1(z) = \sum_{k=1}^{n_1} \frac{b_{1k}^1}{(z-\tau_1)^k}, \quad F_4(z) = \sum_{k=1}^{n_3} \frac{b_{2k}^3}{(z-\tau_3)^k},$$

$$F_j(z) = \sum_{k=1}^{n_{j-1}} \frac{b_{2k}^{j-1}}{(z-\tau_{j-1})^k} + \sum_{k=1}^{n_j} \frac{b_{1k}^j}{(z-\tau_j)^k} = F_{j1}(z) + F_{j2}(z), \quad j = 2, 3. \qquad (14)$$

In contrast with above considered case of internal singularities, here we have to define not only unknown holomorphic in $S_k$ and continuous in $\overline{S_k}$ functions $V_k(z)$, but also all coefficients of rational functions (14).

In accordance with conservation law should be $F_1(z) + F_{21}(z) = 2F_{01}(z)$, $F_{22}(z) + F_{31}(z) = 2F_{02}(z)$, $F_{32}(z) + F_4(z) = 2F_{03}(z)$, wherefrom we get the following set of relations

$$b^j_{1k} + b^j_{2k} = 2b^j_k, \quad k = \overline{1, n_j}, \quad j = 1, 2, 3. \tag{15}$$

For to get additional relations connecting unknown coefficients we, in analogy with (3), introduce here three functions

$$\Phi_1 = \begin{cases} -V_1(z) + A_1[F_{22}(z) + V_2^-(z)] + B_1 r_1^2 z^{-2}\overline{V_2^+(z_1^*)}, & z \in S_1^-, \\ F_1(z) - A_1 V_2^+(z) - A_1 F_{21}(z) - B_1 r_1^2 z^{-2}[\overline{F_2(z_1^*)} + \overline{V_2^-(z_1^*)}], & z \in S_1^+, \end{cases}$$

$$\Phi_2 = \begin{cases} -V_2^-(z) + A_2[F_{32}(z) + V_3^-(z)] + B_2 r_2^2 z^{-2}\overline{V_3^+(z_2^*)}, & z \in S_2^-, \\ F_2(z) + V_2^+(z) - A_2 F_{31}(z) - A_2 V_3^+(z) - B_2 r_2^2 z^{-2}[\overline{F_3(z_2^*)} + \overline{V_3^-(z_2^*)}], & z \in S_2^+, \end{cases}$$

$$\Phi_3 = \begin{cases} -V_3^-(z) + B_3 r_3^2 z^{-2}\overline{V_4(z_3^*)}, & z \in S_3^-, \\ F_3(z) + V_3^+(z) - A_3 F_4(z) - A_3 V_4(z) - B_3 r_3^2 z^{-2}\overline{F_4(z_3^*)}, & z \in S_3^+. \end{cases}$$

It is clear, that each function $\Phi_k(z)$ is holomorphic in the domains $S_k^-$, $S_k^+ \setminus \{0\}$ and continuous across $l_k$ everywhere for exception possibly the point $\tau_k$. Hence, $\Phi_k(z)$ is holomorphic in $\mathbb{C} \setminus \{0, \tau_k\}$ due to the theorem of analytical continuation via continuity. But, evidently, limit value $\Phi_k^-(t)$ is continuous everywhere on $l_k$ including $\tau_k$, hence the same should be true for $\Phi_k^+(t)$. The last demand holds if functions

$$\begin{aligned} \Psi_1(z) &= F_1(z) - A_1 F_{21}(z) - B_1 r_1^2 z^{-2}\overline{F_{21}(z_1^*)}, \\ \Psi_2(z) &= F_{22}(z) - A_2 F_{31}(z) - B_2 r_2^2 z^{-2}\overline{F_{31}(z_2^*)}, \\ \Psi_3(z) &= F_{32}(z) - A_3 F_4(z) - B_3 r_3^2 z^{-2}\overline{F_4(z_3^*)} \end{aligned} \tag{16}$$

are holomorphic at the points $\tau_1$, $\tau_2$, and $\tau_3$ respectively. Let us consider the last summands of functions (16). Omitting for the sake of simplicity almost all indexes, we derive

$$r^2 z^{-2}\overline{F(r^2/\overline{z})} = r^2 \sum_{j=1}^{n} \frac{\overline{b}_j z^{j-2}}{(r^2 - z\overline{\tau}_0)^j} = r^2 \sum_{j=1}^{n} \frac{(-1)^j \overline{b}_j (z - \tau_0 + \tau_0)^{j-2}}{\overline{\tau}_0^j (z - \tau_0)^j}$$

$$= \frac{\overline{b}_1}{z} - \frac{\overline{b}_1}{z - \tau_0} + r^2 \sum_{j=2}^{n} \sum_{i=0}^{j-2} \frac{(-1)^j \overline{b}_j \tau_0^i C^i_{j-2}}{\overline{\tau}_0^j (z - \tau_0)^{i+2}} = \frac{\overline{b}_1}{z} - \frac{\overline{b}_1}{z - \tau_0} - \sum_{j=2}^{n} \frac{\tau_0^{j-1}}{(z - \tau_0)^j} \sum_{i=j}^{n} \frac{\overline{b}_i C^{j-2}_{i-2}}{(-\overline{\tau}_0)^{i-1}}.$$

Equating to zero all coefficients of functions (16) at all powers of $z - \tau_k$ we get

$$\Psi_k(z) = -B_k b^k_{21}/z, \quad k = 1, 2, 3, \tag{17}$$

and

$$b^l_{1j} - A_l b^l_{2j} + B_l \tau_l^{j-1} \sum_{i=j}^{n_l} C^{j-2}_{i-2} \overline{b^l_{2i}} (-\overline{\tau}_l)^{1-i} = 0, \quad j = \overline{1, n_l}, \quad l = 1, 2, 3. \tag{18}$$

Relations (15), (18) give the system for determination of all unknown coefficients $b^l_{1j}$, $b^l_{2j}$ through the given coefficients $b^l_j$, $l = 1, 2, 3$. After simple algebra we get the following recursion formula for determination of $b^l_{2j}$:

$$\begin{aligned} b^l_{2j} = {} & \frac{(2 + \Delta_l) b^l_j}{2} + \frac{\Delta_l \tau_l^{j-1} (-\overline{\tau}_l)^{1-j} \overline{b^l_j}}{2} + \frac{\Delta_l + B_l}{4} \tau_l^{j-1} \sum_{i=j+1}^{n_l} C^{j-2}_{i-2} \frac{\overline{b^l_{2i}}}{(-\overline{\tau}_l)^{i-1}} \\ & + \frac{(-1)^{1-j} \Delta_l B_l}{4} \tau_l^{j-1} \sum_{i=j+1}^{n_l} C^{j-2}_{i-2} \frac{b^l_{2i}}{(-\tau_l)^{i-1}}. \end{aligned} \tag{19}$$

If $j = n_l$ then from (19) we find $b^l_{2n_l}$

$$b^l_{2n_l} = \left[(2+\Delta_l)b^l_{n_l} + \Delta_l \tau_l^{n_l-1}(-\overline{\tau}_l)^{1-n_l}\overline{b^l_{n_l}}\right]/2.$$

Then, using (19), we will sequentially find $b^l_{2n_l-1}, b^l_{2n_l-2}, \cdots, b^l_{21}$, $l = 1,2,3$. From the first equation (15) we get $b^l_{1j} = 2b^l_j - b^l_{2j}$, $l = 1,2,3$, $j = \overline{1, n_l}$.

Thus, each function $\Phi_k$ ($k = \overline{1,3}$), as well as in previous section, is holomorphic in the domains $S_k^+/\{0\}$ and $S_k^-$ and due to the corresponding boundary condition (3) continuous everywhere, including the point $z = \tau_k$, across the line $l_k$. At origin these functions have simple poles and they vanish at infinity. By the generalized Liouville theorem $\Phi_k(z) = C_k/z$, where $C_1$, $C_2$, $C_3$ are constants to be determined. From (14), (16), (17) follows

$$\begin{cases} -V_1(z) + A_1[F_{22}(z) + V_2^-(z)] + B_1 r_1^2 z^{-2}\overline{V_2^+(z_1^*)} = C_1/z, \quad z \in S_1^-, \\ -A_1 V_2^+(z) - B_1 r_1^2 z^{-2}[\overline{F_{22}(z_1^*)} + \overline{V_2^-(z_1^*)}] = (C_1 + B_1\overline{b^1_{21}})/z, \quad z \in S_1^+, \\ -V_2^-(z) + A_2[F_{32}(z) + V_3^-(z)] + B_2 r_2^2 z^{-2}\overline{V_3^+(z_2^*)} = C_2/z, \quad z \in S_2^-, \\ F_{21}(z) + V_2^+(z) - A_2 V_3^+(z) - B_2 r_2^2 z^{-2}[\overline{F_{32}(z_2^*)} + \overline{V_3^-(z_2^*)}] \\ = (C_2 + B_2\overline{b^2_{21}})/z, \quad z \in S_2^+, \\ -V_3^-(z) + B_3 r_3^2 z^{-2}\overline{V_4(z_3^*)} = C_3/z, \quad z \in S_3^-, \\ F_{31}(z) + V_3^+(z) - A_3 V_4(z) = (C_3 + B_3\overline{b^3_{21}})/z, \quad z \in S_3^+. \end{cases} \tag{20}$$

Similar to the case of inner multipoles we find here

$$\begin{aligned} C_3 &= -B_3\overline{b^3_{21}}, \quad C_2 = -B_2(\overline{b^2_{21}} + \overline{b^3_{11}} - \overline{C}_3), \\ C_1 &= -B_1(\overline{b^1_{21}} + \overline{b^2_{11}} + A_2\overline{b^3_{11}} - \overline{C}_2 - A_2\overline{C_3}). \end{aligned} \tag{21}$$

Solution of the system (20) leads again to the functional equation (10) about $V_3^+(z)$ with $F_0(z) = G_1(z) - z^{-2}G_2(z) + c_0/z$, where

$$\begin{aligned} G_1(z) &= (1+\Delta_2)F_{21}(z) - \delta_{13}F_{31}(z_{13}^*) - \delta_{23}F_{31}(z_{23}^*), \\ G_2(z) &= (1+\Delta_2)\delta_1\overline{F_{22}(z_1^*)} + \delta_1\overline{F_{32}(z_1^*)} + \delta_2\overline{F_{32}(z_2^*)}, \\ c_0 &= (\Delta_1+\Delta_2)\overline{C_3} - \Delta_2\overline{b^2_{21}} + (1+\Delta_2)\left(\Delta_1\overline{C_2} - C_2 - (1+\Delta_1)C_1 - \Delta_1\overline{b^1_{21}}\right). \end{aligned} \tag{22}$$

The function $F_0(z)$ with components (22), parameters (21), and coefficients of the functions (14) defined in (19), (15) is holomorphic into the closed circle $\overline{S_2^+}$. As well as earlier, we get $V_3^+(z)$ as the absolutely and uniformly convergent series (12). Then from the system (20) one can find all other components of functions $V_k(z)$. Finally, we get the required solution of the stated problem in accordance with (4), (5):

$$\begin{aligned} v_4(z) &= F_4(z) + (1+\Delta_3)(F_{31}(z) + V_3^+(z)), \\ v_3(z) &= F_{31}(z) + F_{32}(z) + V_3^+(z) + \delta_3 z^{-2}[\overline{F_{31}(z_3^*)} + \overline{V_3^+(z_3^*)}] - C_3/z, \\ v_2(z) &= F_{21}(z) + F_{22}(z) - \delta_1 z^{-2}\overline{F_{22}(z_1^*)} + A_2 F_{32}(z) \\ &+ A_2 z^{-2}\left(\delta_3[\overline{F_{31}(z_3^*)} + \overline{V_3^+(z_3^*)}] + \delta_2\overline{V_3^+(z_2^*)} - \delta_1\overline{F_{32}(z_1^*)}\right) \\ &- A_2\left(\delta_{13}[F_{31}(z_{13}^*) + V_3^+(z_{13}^*)] - \delta_{12}V_3^+(z_{12}^*)\right) \\ &+ \left(A_2(\Delta_1\overline{C_3} - C_3) + \Delta_1\overline{C_2} - C_2 - (1+\Delta_1)(C_1 + B_1\overline{b^2_{21}})\right)/z, \\ v_1(z) &= F_1(z) + (1-\Delta_1)(F_{22}(z) + A_2F_{32}(z)) \\ &+ A_2(1-\Delta_1)z^{-2}\left(\delta_2\overline{V_3^+(z_2^*)} + \delta_3[\overline{F_{31}(z_3^*)} + \overline{V_3^+(z_3^*)}]\right) \end{aligned}$$

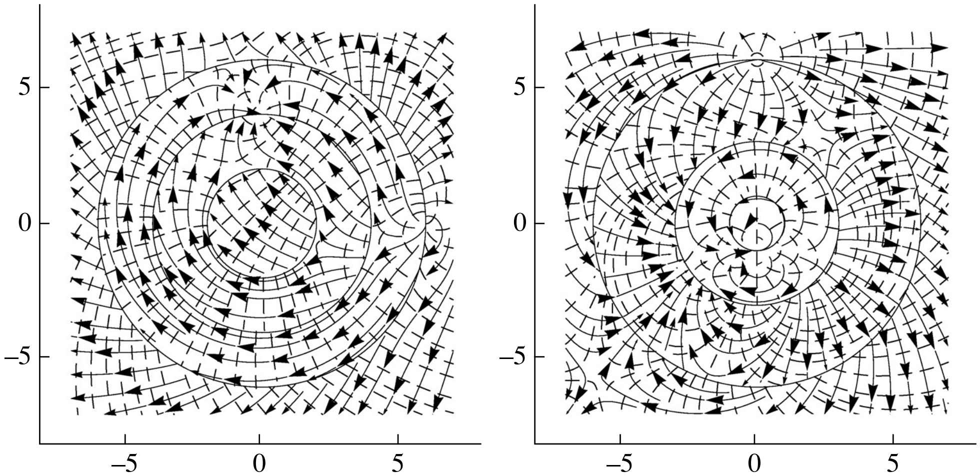


**Fig. 2.** $\rho_1 = 1$, $\rho_2 = 0.1$, $\rho_3 = 10$, $\rho_4 = 1000$ at the left; $\rho_1 = 1$, $\rho_2 = 5$, $\rho_3 = 0.1$, $\rho_4 = 10$ at the right.

$$-\left((1-\Delta_1)(A_2C_3+C_2)+\Delta_1(\overline{C_1}+B_1b_{21}^1)+C_1\right)/z.$$

The case of arbitrary number of multipoles at the interface can be easily gotten as a corresponding sum of the above derived solutions.

**Example 2**. Examples of the corresponding flow nets for complex potentials $f_1(z) = (2+\mathrm{i})\ln(z-6) - \ln(z-4\mathrm{i})$ ($r_1 = 6$, $r_2 = 4$, $r_3 = 2$) and $f_2(z) = 0.1\ln(z-6\mathrm{i}) - 2/(z+\mathrm{i})$, ($r_1 = 6$, $r_2 = 3$, $r_3 = 1$) are presented in Fig. 2.

## 5. CONCLUSION

As a continuation of investigation of two-phase [12] and three-phase ([14], p. 92) concentric circular structures we have given a constructive explicit solution of the corresponding four-phase problem. It was shown that the same basic idea as in the above cited papers is also working here. Namely, we have considered a given boundary condition as a law of analytical continuation. It has allowed to reduce the initial boundary value problem to an equivalent functional equation. Solvability of the last equation was established by the method of undefined coefficients.

We hope that the present structure should be of value for several reasons: first, it provides a non-trivial solution allowing to give an exact picture of the flow nets, that may be useful for solution of a corresponding heterogeneous media problems. Second, it increases the number of not many examples of exactly solvable problems of ℝ-linear conjugation. Finally, the ideas used here one can apply to solve a general $n$-phase concentric circular problem.

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