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Exact solution of a boundary-value problem for a rectangular checkerboard field

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A class of two-phase composite materials with a biperiodic structure is investigated by the methods of complex analysis. Two interface conditions – continuity of normal component of a desired vector $\boldsymbol{w}$ and tangential component of $\widehat{\rho}\boldsymbol{w}$ at the contact boundary as well as the double-periodicity condition – are involved in rigorous form. The exact analytic solution of the corresponding generalized Riemann boundary-value problem is obtained. The explicit values of the effective parameters, namely effective resistivity and dissipation of energy of an elementary cell and resistivities along the symmetry axes are calculated in closed analytic form. The coincidence of our formulae with the well-known effective resistivity (conductivity) formula of Keller (1964), Dykhne (1970) and Mendelson (1975) and the dissipation formula of Dykhne (1970) is shown in the case of square checkerboard field. The Keller (1963) identity is generalized for the heterogeneous structure studied.

1. Introduction

The distribution of physical fields (currents, hydraulic heads, concentrations, temperature, deformations, etc.) in composite materials are investigated intensively in many mathematically equivalent areas of mechanics of continua (Bear 1972; Berdichevski 1983; Buchholz 1957; Carslaw & Jaeger 1959; Crank 1975; Emets 1986; Honein *et al.* 1992; Milton *et al.* 1981; Missenard 1965). The problem is often reduced to definition of a piecewise-continuous vector-field $\boldsymbol{w} = (w_x, w_y, w_z) = \boldsymbol{w}_p$ by the conditions:

$$\operatorname{div} \boldsymbol{w}_p = 0, \quad \operatorname{curl}(\hat{\rho}\boldsymbol{w}_p) = 0 \tag{1.1}$$

in all the uniform components $\boldsymbol{\Omega}_p$ of a considered medium;

$$(\boldsymbol{w}_p)_n = (\boldsymbol{w}_q)_n, \quad (\hat{\rho}_p\boldsymbol{w}_p)_\tau = (\hat{\rho}_q\boldsymbol{w}_q)_\tau \tag{1.2}$$

along the boundary dividing unlike phases $\boldsymbol{\Omega}_p$ and $\boldsymbol{\Omega}_q$. In (1.1), (1.2) $\boldsymbol{w}_n$ and $(\hat{\rho}\boldsymbol{w})_\tau$ are the normal and tangential components of the corresponding limit boundary values of vectors $\boldsymbol{w}$ and $\hat{\rho}\boldsymbol{w}$ ($\hat{\rho} \equiv \hat{\rho}_p \equiv$ const. is a parameter, characterizing phase $\boldsymbol{\Omega}_p$ property). In what follows we use terminology from electrodynamics. By virtue of analogy mentioned all the results are easily reformulated in terms of other applications for which equations (1.1), (1.2) are valid.

Few explicit analytical solutions were found that involve boundary conditions in rigorous form for two- and three-dimensional cases when ratio $\hat{\rho}_p/\hat{\rho}_q$ is arbitrary (Berdichevski 1985; Carslaw & Jaeger 1959; Gheorghiţă 1966).

Of special interest in applications is the case of doubly and multiply periodic structures when inhomogeneities repeat themselves in space. This case involves additional periodicity condition (PC) along the boundary of an elementary cell and, to our knowledge, the only rigorous explicit solution was obtained by Berdichevski (1985) for the special case of square checkerboard field.

Since the classical works by Rayleigh (1892) and Maxwell (1904) either one of the boundary conditions (1.2) or PC were weakened. Alternatively assumptions on small/large value of ratio $\hat{\rho}_p/\hat{\rho}_q$ or large distance between neighbour inclusions were made.

Sometimes a detailed description of the spatially periodic fields is not necessary and only effective integral parameters of the medium are of interest. For this purpose many asymptotic and variational methods were developed (Bauer 1993; Dykhne 1970; Golden & Papanicolaou 1985; Hashin & Shtrinkman 1962; Helsing 1991; Keller 1963, 1964; Lurie & Cherkaev 1983; Mendelson 1975; Nicorovici *et al.* 1993; Schulgasser 1992; Talbot & Willis 1994; Torquato 1991).

A review of the numerous approximate and numerical methods is not within the scope of this paper and we reference some works containing a detailed bibliography (Bakhvalov & Panasenko 1984; Berdichevski 1983; Clark & Milton 1995; Oleinik *et al.* 1994; Sanchez-Palencia 1980).

The object of the present work is to investigate the two-component planar composites with isotropic rectangular compounds arranged in chess order. In the paper the technique of complex analysis is employed. After a mathematically strict statement (§ 2) we obtain general solution and proof the uniqueness theorem for the problem (1.1), (1.2) under natural auxiliary condition fixing exterior field (§§ 3 and 4). Section 5 deals with special cases when phase characteristics and geometrical parameters tend to their limit values. In § 6 effective parameters of the medium are analytically derived on the base of explicit solutions obtained. Comparisons with known formulae following from asymptotic and variational-topological methods are made.

2. The statement of the problem

We consider a double periodic (with main periods $l \pm \mathrm{i}h$, $l > 0$, $h > 0$) rectangular double component parqueted cover of the plane z. Namely, $\boldsymbol{\Omega}_p$ is an union of all rectangles congruent to a rectangle $\Omega_p = \{z : 0 < \operatorname{Re} z < l, 0 < (-1)^{p-1} \operatorname{Im} z < h\}$, $p = 1, 2$, with respect to a group of linear transformations generated by the translations $z + l \pm \mathrm{i}\,h$. Let T be the set of all corner points of the contour $\partial\boldsymbol{\Omega}_1$ and L, H are subsets of all horizontal and vertical intervals of the set $\partial\boldsymbol{\Omega}_1 \setminus T$, respectively (figure 1). It is required to find a double periodic vector field $\boldsymbol{w}(z) = (w_x, w_y) = \boldsymbol{w}_p(z)$, $z \in \boldsymbol{\Omega}_p$, $\boldsymbol{w}_p(z + l \pm \mathrm{i}\,h) \equiv \boldsymbol{w}_p(z)$, $p = 1, 2$, satisfying the conditions (1.1), (1.2) with the piecewise-constant, generally speaking, complex coefficient being $\hat{\rho}(z) \equiv \hat{\rho}_p = \rho_p(1 - \mathrm{i}\beta_p) = \text{const.}$, $z \in \boldsymbol{\Omega}_p$. For every particular physical model the parameters $\rho_p \geqslant 0, \beta_p \in \boldsymbol{R}$ have their special meaning, e.g. ρ_p is resistivity and β_p is the Hall parameter (sometimes the term refractive index is used (Milton *et al.* 1981)) of the uniform component $\boldsymbol{\Omega}_p$ in electrodynamics; $k_p = 1/\rho_p$ is conductivity of the phase $\boldsymbol{\Omega}_p$ in hydrology and thermodynamics, the magnetic permeability in magnetodynamics, etc. The mean values of the normal part of a desired vector $\boldsymbol{w}(z)$ along the couple of adjoint sides of the rectangle Ω_1 are also given.

The problem (1.1), (1.2) with above-mentioned additional condition is equivalent (Emets & Obnosov 1989) to a homogeneous generalized Riemann boundary-value

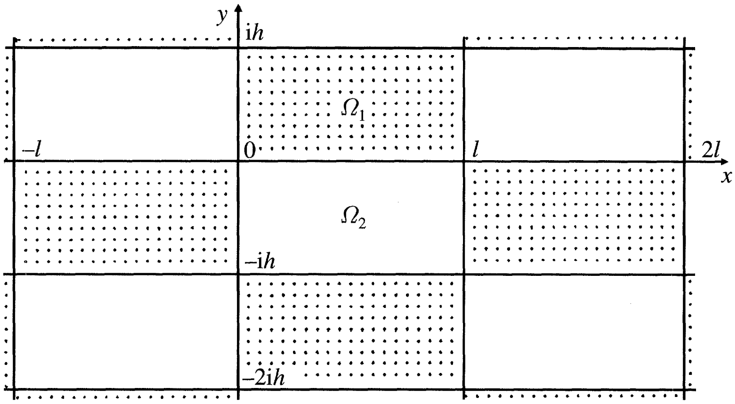

Figure 1. The rectangular checkerboard field.

problem (problem of $\boldsymbol{R}$-linear conjugation):

$$w_1(t) = Aw_2(t) - B[t'(s)]^{-2}\overline{w_2(t)}, \quad t \in L \cup H, \tag{2.1}$$

$$\frac{1}{h}\int_0^h \operatorname{Re} w_1(\mathrm{i}y)\,\mathrm{d}y = a, \quad \frac{1}{l}\int_0^l \operatorname{Im} w_1(x)\,\mathrm{d}x = b. \tag{2.2}$$

Here a, b are given real parameters; the coefficients A, B may be considered generally as arbitrary complex numbers, satisfying the inequality $|A| \geqslant |B|$. In terms of the electrodynamical model these coefficients are of the form

$$A = [\rho_1 + \rho_2 - \mathrm{i}(\rho_1\beta_1 - \rho_2\beta_2)]/2\rho_1, \quad B = 1 - \overline{A}. \tag{2.3}$$

The derivative $t'(s)$ of the function of a point of the contour $L \cup H$ with respect to its natural parameter is ± 1 or $\pm \mathrm{i}$ when $t \in L$ or H, respectively; $w(z) = w_x(x, y) - \mathrm{i}w_y(x, y) = w_p(z) \in \mathcal{H}(\boldsymbol{\Omega}_p) \cap C(\overline{\boldsymbol{\Omega}}_p \setminus T)$ is a piecewise-holomorphic function. At the points of the set T the function $w(z)$ can have utmost, if any, integrable singularities.

Beginning investigation of the problem (2.1), (2.2), we shall consider the following case.

3. Solution of problem (2.1), (2.2) for the case of real coefficients A, B

First of all let us prove the following assertion.

Lemma 3.1. *Let the coefficients A, B be real, and $A^2 - B^2 \geqslant 0$. Then every even ($w(-z) \equiv w(z)$), or odd ($w(-z) \equiv -w(z)$), solution of problem* (2.1) *satisfies the identity* †

$$w_1^2(z) + \delta^2[\overline{w}_2(z)]^2 \equiv C, \quad z \in \boldsymbol{\Omega}_1, \tag{3.1}$$

where

$$\delta = A\Delta', \quad \Delta' = \sqrt{1 - \Delta^2} \geqslant 0, \quad \Delta = B/A, \tag{3.2}$$

C is a real constant, if solution w is even, and $C = 0$ when w is odd.

† Everywhere below $\overline{f}(z)$ designates $\overline{f(\overline{z})}$.

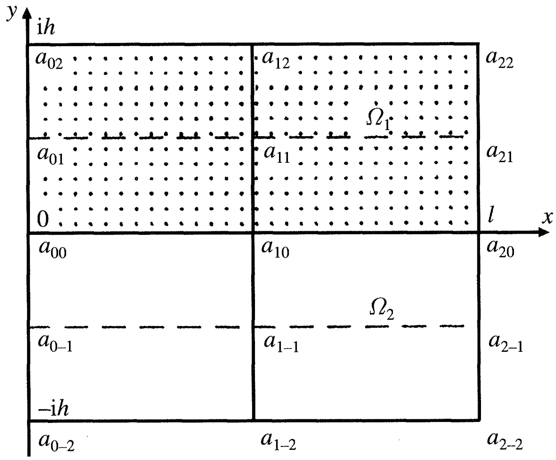

Figure 2. The elementary cell (fundamental period rectangle).

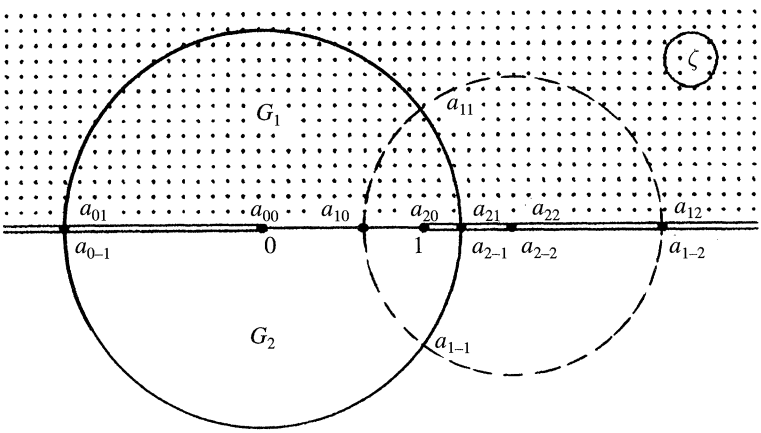

Figure 3. The points a_{pq} here correspond to the nodals in figure 2.

Proof. Squaring both sides of the equality (2.1) and subtracting from the result the square of the equality which is complex conjugate with (2.1), we get

$$\operatorname{Im} w_1^2(t) = \delta^2 \operatorname{Im} w_2^2(t), \quad t \in L \cup H. \tag{3.3}$$

The function $\psi(z) = w_1^2(z) + \delta^2 \overline{w}_2^2(z), z \in \boldsymbol{\Omega}_1$, being double periodic and even together with $w^2(z)$, satisfies the condition $\operatorname{Im}\psi(t) = 0$, $t \in L \cup H$, due to the equality (3.3). Hence the results of analytic continuations of $\psi(z)$ into $\boldsymbol{\Omega}_2$ through L and H coincide ($\overline{\psi}(z) \equiv \overline{\psi}(-z)$). It is evident that the continued function is a real constant as a double periodic, even function with, if any, only simple poles at the points of the set T.

At last, if $w(z)$ is odd, then $w_1(\frac{1}{2}l+\frac{1}{2}\mathrm{i}h) = -w_1(-\frac{1}{2}l-\frac{1}{2}\mathrm{i}h)$, but due to periodicity $w_1(\frac{1}{2}l+\frac{1}{2}\mathrm{i}h) = w_1(-\frac{1}{2}l-\frac{1}{2}\mathrm{i}h)$. Thus $w_1(\frac{1}{2}l+\frac{1}{2}\mathrm{i}h) = 0$, and analogously $w_2(\frac{1}{2}l-\frac{1}{2}\mathrm{i}h) = 0$, i.e. $\psi(\frac{1}{2}l + \frac{1}{2}\mathrm{i}h) = 0$ and therefore $C = 0$ in (3.1). ■

Lemma 3.2. *The problem* (2.1) *with real coefficients* A, B *has only a trivial odd solution and just two linearly independent even solutions satisfying the relation* (3.1) *with* $C = 0$, *if* $0 < |B| < |A|$.

Proof. Let us consider the conformal mapping of the domain $\Omega = \Omega_1 \cup (0, l) \cup \Omega_2$ (figure 2) by the function

$$\zeta = \zeta(z) \equiv \overline{\zeta}(z) = \mathrm{sn}^2(Kz/l|m), \tag{3.4}$$

where the parameter $m = m(h/l)$ may be determined by the exact formula 8.197 from Gradshteyn & Ryzik (1980) or approximately by the tables 17.3 from Abramowitz & Stegun (1970); $K = K(m)$ is the corresponding complete elliptic integral of first genus; $\mathrm{sn}(\,\cdot\,)$ is the Jacobian elliptic sinus. The function (3.4) maps the domain Ω onto the entire plane with a cut along the part of the real axis: $\{x : x < 0, x > 1\}$ (figure 3), where $-a_{01} = a_{21} = 1/\sqrt{m}$, $a_{10} = 1/(1+\sqrt{m_1})$, $a_{22} = 1/m$, $a_{12} = 1/(1-\sqrt{m_1})$ and the semicircles within the domains G_1, G_2 are images of the corresponding middle-lines of the rectangles Ω_1, Ω_2 ($\zeta(a_{02}) = \zeta(a_{0-2}) = \infty$).

Let

$$W(\zeta) = w[z(\zeta)], \tag{3.5}$$

where $z(\zeta)$ is the inversion of the function (3.4). From the character of the conformal mapping and from the presupposed properties of the function $w(z)$ it follows that $W(\zeta) = W_p(\zeta) \in \mathcal{H}(G_p) \cap C(\overline{G}_p \setminus \{0, 1, 1/m, \infty\})$, $p = 1, 2$, and

$$|W(\zeta)| = o(|\zeta - d|^{1/2}) \tag{3.6}$$

in the vicinity of any of the nodal points $d \in \{0, 1, 1/m, \infty\}$. Besides, it may be shown that, due to the periodicity and evenness or oddness $w(z)$, the function (3.5) will satisfy the corresponding conditions:

$$W\left(\frac{1/m - \zeta}{1 - \zeta}\right) = \pm W(\zeta), \tag{3.7}$$

respectively. By means of identities (3.1) with $C = 0$, (3.4), and the uniqueness theorem for holomorphic functions we obtain

$$W_2(\zeta) \equiv \pm \mathrm{i}\overline{W}_1(\zeta)/\delta, \quad \zeta \in G_2. \tag{3.8}$$

From (2.1), (3.2), (3.5) it follows that

$$\mp \mathrm{i}\Delta' W_1(\xi) = \overline{W}_1(\xi) - (-1)^j \Delta W_1(\xi), \quad \xi \in L_j, \quad j = 1, 2,$$

according to two possibilities (3.8). Here $L_1 = (0, 1) \cap (1/m, \infty)$ and $L_2 = \overline{\boldsymbol{R}} \setminus \overline{L_1}$. The last boundary conditions lead to the corresponding Hilbert problems

$$\mathrm{Im}\{\exp[\mp \mathrm{i}\tfrac{1}{4}\pi(1 - (-1)^j 2\lambda)] W_1(\xi)\} = 0, \quad \xi \in L_j, \quad j = 1, 2, \tag{3.9}$$

where, in view of the inequality $-1 \leqslant B/A \leqslant 1$, it is written

$$\Delta = B/A = \sin \pi\lambda \quad (|\lambda| \leqslant 1/2). \tag{3.10}$$

Let us consider the single-valued branch of the function

$$\chi(\zeta) = \left(\frac{1 - \zeta}{\zeta(1 - m\zeta)}\right)^{\lambda}, \tag{3.11}$$

fixed in $\boldsymbol{C} \setminus L_2$ by the condition $\chi(\xi) > 0$, $\xi \in L_1$. Obviously, the function (3.11) satisfies the first condition (3.7) and it meets the condition (3.6) together with $\chi^{-1}(\zeta)$ in view of (3.10), if $|\Delta| < 1$. The cases $|\Delta| = 1$ and $\Delta = 0$ will be studied later.

It is easy to see, that unique to within a real factor solutions of the problems (3.9) have the following form:

$$W_{11}(\zeta) = \delta e^{i\gamma}\chi(\zeta), \qquad W_{12}(\zeta) = \delta e^{-i\gamma}\chi^{-1}(\zeta),$$

respectively. Here

$$e^{i\gamma} = \exp[i\tfrac{1}{4}\pi(1+2\lambda)] = \sqrt{\frac{1-\Delta}{2}} + i\sqrt{\frac{1+\Delta}{2}}. \tag{3.12}$$

The above-obtained solutions of both problems (3.9) meet first condition (3.7), and there are no other holomorphic solutions of these problems. So the problem (2.1) does not have non-trivial odd solutions.

Taking into account that the chosen branch of the analytic function (3.11) satisfies the identity

$$\overline{\chi}(\zeta) \equiv \chi(\zeta) \tag{3.13}$$

and by means of relations (3.4), (3.5), (3.8), (3.10)–(3.12) a desired couple of linearly independent solutions of the problem (2.1) can be determined by formulae:

$$\left.\begin{aligned} w_{11}(z) &= \Delta' e^{i\gamma}\chi(z), & w_{12}(z) &= \Delta' e^{-i\gamma}\chi^{-1}(z), \\ w_{21}(z) &= iA^{-1}e^{-i\gamma}\chi(z), & w_{22}(z) &= -iA^{-1}e^{i\gamma}\chi^{-1}(z). \end{aligned}\right\} \tag{3.14}$$

Here

$$\chi(z) = \left(\frac{1-\zeta}{\zeta(1-m\zeta)}\right)^{\lambda} = \left(\frac{1+\mathrm{cn}(2Kz/l|m)}{1-\mathrm{cn}(2Kz/l|m)}\right)^{\lambda} \tag{3.15}$$

is a double periodic branch fixed in the plane $\boldsymbol{C}\backslash\overline{H}$ by the condition that $\chi(l/2) = 1$.

Thus the lemma is proved. ■

Corollary 3.3. *Any solution of the problem (2.1) with real coefficients A, B is an even function, if $0 < |B/A| < 1$.*

Obviously $w(z)$ is a solution of the problem (2.1), together with $w(-z)$ and with an odd function $w(z) - w(-z)$. The last one vanishes identically by virtue of lemma 3.2, which proves the assumption stated.

Corollary 3.4. *For every solution of the problem* (2.1) *with real coefficients A, B the identity* (3.1) *is true if $0 < |B/A| < 1$.*

Using lemmas 3.1 and 3.2 and the preceding corollaries, one can easily prove

Theorem 3.5. *The general solution of the problem* (2.1) *with real coefficients A, B $(0 < |B/A| < 1)$ is representable in the form,*

$$w_p(z) = c_1 w_{p1}(z) + c_2 w_{p2}(z), \qquad z \in \boldsymbol{\Omega}_p, \quad p = 1, 2, \tag{3.16}$$

where the functions $w_{pq}(z)$, $p, q = 1, 2$ are determined by formulae (3.2), (3.10), (3.12), (3.14), and c_1, c_2 are arbitrary real constants.

Proof. Let $w(z)$ be a certain fixed solution of problem (2.1) and w_{11}, w_{21} are defined by the corresponding relations (3.14). Then

$$w(z; c_1) = w_p(z) - c_1 w_{p1}(z), \qquad z \in \boldsymbol{\Omega}_p, \quad p = 1, 2, \tag{3.17}$$

will be a one-parameter family of solutions of the same problem for any real value c_1. Let $w(z)$ meet the condition (3.1) with right-hand side C_0. It follows from lemma 3.2 that the theorem is valid if $C_0 = 0$. Hence, it is supposed $C_0 \neq 0$. Every solution from the family (3.17) satisfies the identity (3.1) for the corresponding value of constant $C = C(c_1)$, i.e.

$$\begin{aligned} C(c_1) &\equiv (w_1(z) - c_1 w_{11}(z))^2 + \delta^2(\overline{w}_2(z) - c_1\overline{w}_{21}(z))^2 \\ &= [w_1^2(z) + \delta^2\overline{w}_2^2(z)] + c_1^2[w_{11}^2(z) + \delta^2\overline{w}_{21}^2(z)] \\ &\qquad -2c_1[w_1(z)w_{11}(z) + \delta^2\overline{w}_2(z)\overline{w}_{21}(z)]. \end{aligned}$$

From the last relation it follows immediately that

$$w_1(z)w_{11}(z) + \delta^2\overline{w}_2(z)\overline{w}_{21}(z) \equiv [C_0 - C(c_1)]/(2c_1).$$

The left-hand side of the identity obtained does not depend on c_1. Consequently, its right-hand side does not depend on c_1 either. Denoting this side as d and choosing $c_1 = C_0/(2d)$ one can get $C(c_1) = 0$. Therefore, the appropriate solution (3.17) will satisfy the homogeneous condition (3.1). By virtue of lemma 3.2 and the assumption made ($C_0 \neq 0$) the chosen solution differs from the second solutions (3.14) by certain real multiplier only. This proves theorem 3.5. ∎

Now we can state the main result of this section.

Theorem 3.6. *The problem* (2.1), (2.2) *for* $|A| > |B| > 0$ $(A, B \in \boldsymbol{R})$ *is unconditionally and uniquely solvable. Its solution is determined by the formulae:*

$$\left.\begin{aligned} w_1(z) &= \sqrt{\tfrac{1}{2}(1-\Delta)}\{a_1 e^{i\gamma}\chi(z) + b_1 e^{-i\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_1; \\ w_2(z) &= \frac{iA^{-1}}{\sqrt{2(1+\Delta)}}\{a_1 e^{-i\gamma}\chi(z) - b_1 e^{i\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_2, \end{aligned}\right\} \tag{3.18}$$

where $\chi(z)$ *can be found via the formulae (3.15), (3.10), and*

$$a_1 = a\sigma_1 + b\sigma, \quad b_1 = a\sigma_1 - b\sigma. \tag{3.19}$$

Here $\sigma = \sigma(m)$, $\sigma_1 = \sigma(1-m)$ ($m_1 = 1 - m$ *is the complementary parameter),*

$$\sigma(s) = F(\tfrac{1}{2}, \tfrac{1}{2}; 1; s)/F(\tfrac{1}{2} + \lambda, \tfrac{1}{2} - \lambda; 1; s). \tag{3.20}$$

Proof. By virtue of theorem 3.5 it is sufficient to prove that the arbitrary constants c_1, c_2 in the general solution (3.17) are determined by the conditions (2.2) uniquely. To this purpose we prove the validity of the next two equalities:

$$\frac{1}{h}\int_0^h \chi^{\pm 1}(iy)\,dy = \frac{\exp(\mp i\pi\lambda)}{\Delta'\sigma_1}, \qquad \frac{1}{l}\int_0^l \chi^{\pm 1}(x)\,dx = \frac{1}{\Delta'\sigma}. \tag{3.21}$$

Indeed, by the substitution $\zeta = \mathrm{sn}^2(Kz/l|m)$, taking into account the representation,

$$dz = \frac{l\,d\zeta}{2K\sqrt{\zeta(1-\zeta)(1-m\zeta)}} \tag{3.22}$$

(Abramowitz & Stegun 1970, formula 16.16.1), equalities (3.12)–(3.14), and figures 2 and 3 the first integral (3.21) can be transformed into the following:

$$\frac{\exp(\mp i\pi\lambda)}{2K'}\int_{-\infty}^0 |\xi|^{\mp\lambda-1/2}(1-\xi)^{\pm\lambda-1/2}(1-m\,\xi)^{\mp\lambda-1/2}\,d\xi.$$

The replacement $\xi = \tau/(\tau - 1)$ reduces the last integral to the next hypergeometric one (Abramowitz & Stegun 1970, formula 15.3.1)

$$\frac{\exp(\mp \mathrm{i}\pi\lambda)}{2K'} \int_0^1 \tau^{\mp\lambda-1/2}(1-\tau)^{\pm\lambda-1/2}(1-m_1\tau)^{\mp\lambda-1/2}\mathrm{d}\tau = \frac{\pi \exp(\mp \mathrm{i}\pi\lambda)}{2K' \cos \pi\lambda} F(1/2+\lambda, 1/2-\lambda; 1; m_1).$$

By these means, from the definitions (3.19), (3.12), (3.10), (3.2), and the relation

$$K' = K'(m) = (h/l)K(m) = K(m_1) = \tfrac{1}{2}\pi F(\tfrac{1}{2}, \tfrac{1}{2}; 1; m_1) \tag{3.23}$$

(Abramowitz & Stegun 1970, formula 17.3.9) one can get the first equality (3.21). The second integral (3.21) is calculated in a similar way.

On the basis of the relations (3.12), (3.14)–(3.16), (3.21) the conditions (2.2) yield

$$a\sigma_1 = (c_1 + c_2)\sin\gamma, \qquad b\sigma = (c_1 - c_2)\sin\gamma.$$

Now, taking into account the inequality $\sin\gamma = \sqrt{(1+\Delta)/2} \neq 0$, which is the consequence of the theorem restrictions, we obtain

$$c_1 = \frac{a\sigma_1 + b\sigma}{\sqrt{2(1+\Delta)}}, \qquad c_2 = \frac{a\sigma_1 - b\sigma}{\sqrt{2(1+\Delta)}}.$$

Thus theorem 3.6 is proved. ■

Note that the last theorem makes it possible to define more precisely the value of the constant C in the right-hand side of the identity (3.1). Namely, according to the relations (3.13), (3.18)–(3.20) it can be found:

$$C = 2(1-\Delta)(a\sigma_1^2 - b\sigma^2).$$

4. The case of complex coefficients A, B

As well as in author's papers (1991, 1992) the general case of arbitrary complex coefficients ($|A| > |B| > 0$) by means of the substitution,

$$w_1(z) = \sqrt{AB}W_1(z), \qquad w_2(z) = \sqrt{\bar{A}B}W_2(z),$$

is reduced to the problem (2.1) with real coefficients $|A|, |B|$ with respect to the function $W(z) = W_p(z)$, $z \in \boldsymbol{\Omega}_p$.

The general solution of the last problem can be written out via the formulae (3.10), (3.12)–(3.15), but A and B must be replaced there by $|A|$ and $|B|$ respectively. Hence, the general solution of the problem (2.1) is determined by formulae,

$$\begin{cases} w_1(z) = \Delta'\sqrt{A\,B}\{c_1\mathrm{e}^{\mathrm{i}\gamma}\chi(z) + c_2\mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(z)\}, & z \in \boldsymbol{\Omega}_1; \\ w_2(z) = \mathrm{i}\sqrt{B/A}\{c_1\mathrm{e}^{-\mathrm{i}\gamma}\chi(z) - c_2\mathrm{e}^{\mathrm{i}\gamma}\chi^{-1}(z)\}, & z \in \boldsymbol{\Omega}_2, \end{cases} \tag{4.1}$$

where $0 < \Delta = |B|/|A| = \sin\pi\lambda < 1$.

If $\sqrt{AB} = r\mathrm{e}^{\mathrm{i}\alpha}$, $r > 0$, $-\pi < \alpha \leqslant \pi$, then the conditions (2.2) by the help of equalities (3.20), (3.21) lead to the linear system with respect to the constants c_1, c_2. The solution of the system obtained in such a manner, will have the form,

$$c_1 = \frac{a\sigma_1 \sin(\gamma - \alpha) + b\sigma\sin(\gamma + \alpha)}{1 + \Delta\cos 2\alpha}, \quad c_2 = \frac{a\sigma_1 \sin(\gamma + \alpha) - b\sigma\sin(\gamma - \alpha)}{1 + \Delta\cos 2\alpha}.$$

The following statement is proved.

Theorem 4.1. *The problem* (2.1), (2.2) *is unconditionally and uniquely solvable when* $A, B \in \boldsymbol{C}$ *and* $|A| > |B| > 0$. *Its solution is representable in the form,*

$$\begin{cases} w_1(z) = \dfrac{e^{i\alpha}\Delta'}{1+\Delta\cos 2\alpha}\{a_1 e^{i\gamma}\chi(z) + b_1 e^{-i\gamma}\chi^{-1}(z)\}, & z \in \boldsymbol{\Omega}_1; \\ w_2(z) = \dfrac{ie^{i\alpha}A^{-1}}{1+\Delta\cos 2\alpha}\{a_1 e^{-i\gamma}\chi(z) - b_1 e^{i\gamma}\chi^{-1}(z)\}, & z \in \boldsymbol{\Omega}_2, \end{cases} \tag{4.2}$$

where $-\pi < \alpha = [\arg A + \arg B]/2 \leqslant \pi$, $0 < \Delta = |B/A| < 1$, *and*

$$a_1 = a\sigma_1 \sin(\gamma - \alpha) + b\sigma \sin(\gamma + \alpha), \quad b_1 = a\sigma_1 \sin(\gamma + \alpha) - b\sigma \sin(\gamma - \alpha), \tag{4.3}$$

and $\chi(z)$, λ, γ *are determined by* (3.10), (3.12), (3.15), *respectively.*

Note that by using (3.10), (3.15), (4.3) among the solutions (4.2) two subgroups are derived for which

$$\frac{a}{b} = \pm\frac{\sigma\sin(\gamma \mp \alpha)}{\sigma_1 \sin(\gamma \pm \alpha)},$$

correspondingly. The solutions of the first group have integrable singularities at the points of the subset

$$T_1 = \{t_{kn} : t_{kn} = kl + i(2n-k)h, \quad k, n \in \boldsymbol{Z}\}$$

and zeros of the power λ at the points of the subset $T_2 = T \setminus T_1$ if $\rho_1 > \rho_2$, and on the contrary when $\rho_1 < \rho_2$. For the solutions of the second subgroup the opposite situation is true.

5. Degenerated, specific and limit situations

(*a*) *The degenerated cases* $0 \leqslant |B| \leqslant |A|$

It is easy to show that the unique solution of the problem (2.1), (2.2), when $0 = |B| < |A|$ ($\Delta = 0$), will be a piecewise-constant function

$$w_1(z) \equiv a + ib, \quad z \in \boldsymbol{\Omega}_1, \qquad w_2(z) \equiv A^{-1}(a+ib), \quad z \in \boldsymbol{\Omega}_2.$$

One can obtain the same result using the formula (4.2), if B tends to zero in such a manner that the limit of $\arg B$ exists. Therefore, theorem 4.1 is valid for the considered case. In particular, if the representation (2.3) takes place, then the equality $0 = |B|$ is possible iff $\rho_1 = \rho_2$, $\beta_1 = \beta_2$. Hence, $A = 1$, i.e. $w_1(z) \equiv w_2(z)$.

If $0 < |B| = |A|$ ($\Delta = 1$), then the corresponding problem (2.1) has a trivial solution only. In view of the remark made at the very beginning of the §4 it is sufficient to prove the assertion stated for the real case $A = B > 0$, only. Obviously, the problem (2.1) is reduced under such condition to the next mixed boundary value problem:

$$\operatorname{Re} W_1(\xi) = 0, \quad \xi \in L_1, \qquad \operatorname{Im} W_1(\xi) = 0, \quad \xi \in L_2,$$

with respect to the function (3.5). The last problem has (Gakhov 1966, pp. 454–456) only a trivial solution under the restriction (3.6). If $W_1(\zeta) \equiv 0$ and hence $w_1(z) \equiv 0$, then on the basis of the homogeneous condition (2.1) one can obtain the same mixed problem with respect to the function $iW_2(\zeta)$. Thus, $w_2(z) \equiv 0$, as was to be shown.

Remark. A non-trivial solution of the problem (2.1) for the case $A = B > 0$ can be obtained, by admitting non-integrable singularities at the points of the set T. By the formula (3.16) and in accordance with the relations (3.10), (3.12), (3.14), (3.15) one can get the general solution of the corresponding problem (2.1). Namely, by tending Δ to 1 we get:

$$w_1(z) \equiv 0, \quad z \in \boldsymbol{\Omega}_1, \qquad w_2(z) = c_1\chi(z) + c_2\chi^{-1}(z), \quad z \in \boldsymbol{\Omega}_2,$$

where c_1, c_2 are arbitrary real constants. In accordance with what was said above about fixing the branch of the function (3.15) the corresponding limit is

$$\chi(z) = \left(\frac{1+\mathrm{cn}(2Kz/l|m)}{1-\mathrm{cn}(2Kz/l|m)}\right)^{1/2} = (-1)^j \frac{\mathrm{cn}(Kz/l|m)}{\mathrm{sn}(Kz/l|m)\mathrm{dn}(Kz/l|m)},$$

when $jl \leqslant \mathrm{Re}\, z \leqslant (j+1)l, \quad j \in \boldsymbol{Z}$. Just the same solution could be derived by solving the problem (1.1), (1.2) with $\rho_1 = \infty$, or $\rho_2 = 0$ (i.e. the phase $\boldsymbol{\Omega}_1$ is a non-conductor, or the phase $\boldsymbol{\Omega}_2$ is superconductor, respectively). The last general solution can be rewritten when $z \in \Omega_2$ in the following equivalent form (Abramowitz & Stegun 1970, formulae 16.18.1 and 16.20.1–3):

$$w_2(z) = \frac{c_1}{\mathrm{sn}(2Kz/l|m)} + \mathrm{i}\frac{c_2}{\mathrm{sn}(\mathrm{i}2Kz/l\,|\,m_1)}.$$

This solution has simple poles at all the corner points of the rectangle Ω_2. Hence, the conditions (2.2) are incorrect under this situation and they have to be replaced. One of the simplest non-contradicting conditions (ensuring the uniqueness of the desired solution) will be the following:

$$w_2(\tfrac{1}{2}l - \mathrm{i}\tfrac{1}{2}h) = a + \mathrm{i}b.$$

Due to this equality one can get

$$c_1 = a/\sqrt{m}, \qquad c_2 = b/\sqrt{m_1}.$$

At last, the homogeneous problem (2.1), (2.2) ($a = b = 0$) is incorrect and the inhomogeneous one is unsolvable if $A = B = 0$.

(b) *The specific case of square checkerboard field*

If $l = h$, then $m = m_1 = \frac{1}{2}$ and in accordance with (3.20) and formula 15.1.24 from Abramowitz & Stegun (1970) we evaluate

$$\sigma = \sigma_1 = \frac{F(\frac{1}{2},\frac{1}{2};1;\frac{1}{2})}{F(\frac{1}{2}+\lambda,\frac{1}{2}-\lambda;1;\frac{1}{2})} = \frac{\Gamma^2(\frac{1}{4})/\Delta'}{\Gamma(\frac{1}{4}-\frac{1}{2}\lambda)\Gamma(\frac{1}{4}+\frac{1}{2}\lambda)}.$$

By using the expression obtained and the relations (3.18), (3.19) the solution of the problem (2.1), (2.2) with real coefficients (2.3) ($\beta_1 = \beta_2 = 0$) can be written out in the following way:

$$\left.\begin{aligned} w_1(z) &= \Lambda^{-1}\sqrt{\frac{2}{1+\Delta}}\{a_1 \mathrm{e}^{\mathrm{i}\gamma}\chi(z) + b_1\mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_1; \\ w_2(z) &= \mathrm{i}\Lambda^{-1}\sqrt{\frac{2}{1-\Delta}}\{a_1 \mathrm{e}^{-\mathrm{i}\gamma}\chi(z) - b_1\mathrm{e}^{\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_2, \end{aligned}\right\} \tag{5.1}$$

where $a_1 = a + b$, $b_1 = a - b$, $\Lambda = \Gamma(\frac{1}{4} - \frac{1}{2}\lambda)\Gamma(\frac{1}{4}+\frac{1}{2}\lambda)\Gamma^{-2}(\frac{1}{4})$.

This result agrees with that obtained by Berdichevski (1985) and Emets & Obnosov (1989).

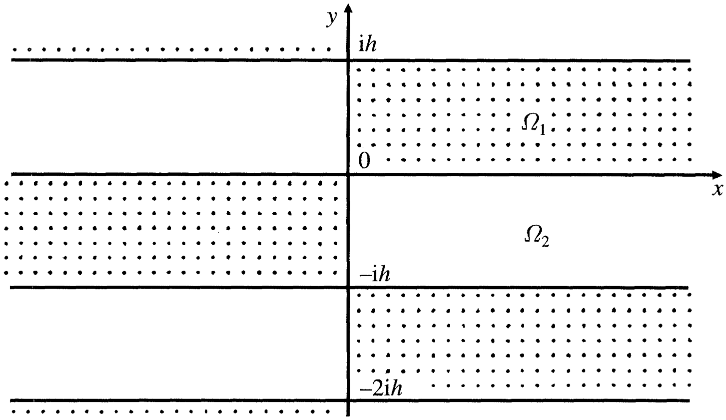

Figure 4. The displaced flakes.

(*c*) *Displaced periodic flakes*

This structure can be obtained as a limiting case of the earlier considered when the value l tends to infinity for a fixed value of h. Hence, $\boldsymbol{\Omega}_p$ $(p = 1, 2)$ is the union of all half-strips congruent (with respect to the translations $z + \mathrm{i}2nh$, $n \in \boldsymbol{Z}$) to the couple $\Omega_p = \{z : \operatorname{Re} z > 0, 0 < (-1)^{p-1} \operatorname{Im} z < h\}$ and the half-strip symmetric with Ω_p about origin (figure 4). In this case the second additional condition (2.2) is a limit when l tends to infinity.

The solution of the stated problem can be derived by the help of the corresponding limit process on the basis of the relations (4.2), (4.3), (3.20), (3.15), (3.12). Omitting intermediate calculations we write out the final result:

$$\left.\begin{aligned} w_1(z) &= \frac{\mathrm{e}^{\mathrm{i}\alpha}}{1 + \varDelta \cos 2\alpha}\{a_1 \mathrm{e}^{\mathrm{i}\gamma}\chi(z) + b_1 \mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_1; \\ w_2(z) &= \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\alpha}(\varDelta\varDelta')^{-1}}{1 + \varDelta \cos 2\alpha}\{a_1 \mathrm{e}^{-\mathrm{i}\gamma}\chi(z) - b_1 \mathrm{e}^{\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_2. \end{aligned}\right\}$$

Here

$$a_1 = \varDelta' a \sin(\gamma - \alpha) + b \sin(\gamma + \alpha), \quad b_1 = \varDelta' a \sin(\gamma + \alpha) - b \sin(\gamma - \alpha),$$

and

$$\chi(z) = (\coth(\pi z / 2h))^{2\lambda}$$

is the $2h\mathrm{i}$-periodic branch fixed in the right half-plane by the condition $\chi(x) > 0$ for $x > 0$ and continued into the left half-plane in an even way.

(*d*) *Two-components four-lobes regular fan*

The structure considered below can be obtained from the above by tending both values l and h to infinity. Thus, here $\boldsymbol{\Omega}_1(\boldsymbol{\Omega}_2)$ is the union of the first, $\boldsymbol{C}^+_+ = \boldsymbol{\Omega}_1$, and the third, $\boldsymbol{C}^-_-$ (the second, $\boldsymbol{C}^+_-$, and the fourth, $\boldsymbol{C}^-_+ = \boldsymbol{\Omega}_2$) quadrants of the plane z (figure 5).

Using the scheme of §§ 2 and 3 one can easily get the general solution of the corresponding problem (2.1) in the form (4.1). Only in this case is $\chi(z)$ a single-valued branch of the function $\chi(z) = z^{-2\lambda}$, fixed in the right half-plane by the condition $\chi(x) > 0$ for $x > 0$ and continued in the left half-plane in an even way. To

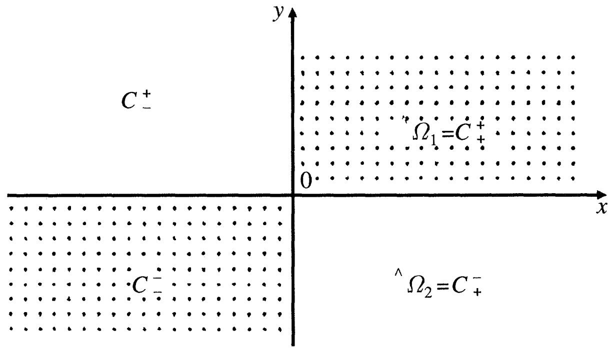

Figure 5. The four-lobes regular fan.

get this result it is necessary to prove the next assertion preceding lemma 3.1: *any solution of the problem* (2.1) *for the four-lobes fan is even.*

Indeed, the function

$$W_2(z) = \begin{cases} w_2(z), & z \in \boldsymbol{C}^-_+ \cup \boldsymbol{C}^+_-; \\ w_1(z)/A - \Delta \overline{w}_2(-z), & z \in \boldsymbol{C}^+_+ \cup \boldsymbol{C}^-_-, \end{cases}$$

is holomorphic in the upper and lower half-planes. In accordance with (2.1) on the real axis the limit values $W_2^{\pm}(x)$ satisfy the condition

$$AW_2^+(x) + B\overline{W_2^+(-x)} = AW_2^-(x) - B\overline{W_2^-(x)}, \quad x > 0;$$
$$AW_2^-(x) + B\overline{W_2^-(-x)} = AW_2^+(x) - B\overline{W_2^+(x)}, \quad x < 0.$$

It follows from these relations that

$$W_2^+(x) + W_2^+(-x) = W_2^-(x) + W_2^-(-x), \quad x \in \boldsymbol{R} \setminus \{0\}.$$

Hence, the odd function $W(z) = W_2(z) - W_2(-z)$, being a holomorphic and continuous in $\boldsymbol{C} \setminus \{0\}$ with, if any, integrable singularities at the points $0, \infty$, vanishes identically.

The following question now arises naturally. What kind of additional conditions must one choose instead of the conditions (2.2) which are incorrect in this case? For instance, setting

$$w_1(\mathrm{e}^{\mathrm{i}\pi/4}) = \Delta' \sqrt{AB}(a + b),$$

it is easy to find

$$c_1 = (a + b)/\sqrt{2}, \qquad c_2 = (a - b)/\sqrt{2}.$$

Other kinds of restriction providing uniqueness of the desired solution will not be discussed here.

Let us consider the solution (4.1) at the point $z - \frac{1}{2}l$. By tending l to infinity we obtain

$$\lim_{l \to \infty} \chi(z - \tfrac{1}{2}l) = 1.$$

Hence the solution of the problem (2.1) for the corresponding flaky system is a piecewise-constant function (Bear 1972).

6. Calculation of effective parameters values

The coefficients A, B below will be taken in real form (2.3):

$$A = \frac{\rho_1 + \rho_2}{2\rho_1}, \qquad B = \frac{\rho_1 - \rho_2}{2\rho_1}. \tag{6.1}$$

This is of special interest in certain applications. We calculate the following functionals.

(*a*) An effective resistivity along the axes of symmetry:

$$\rho^x_{\text{eff}} = \frac{1}{2l}\int_{-l}^{l} \operatorname{Re}[\rho(x+\tfrac{1}{2}\mathrm{i}h)\boldsymbol{w}(x+\tfrac{1}{2}\mathrm{i}h)]\,\mathrm{d}x \Big/ \frac{1}{h}\int_0^h \operatorname{Re}\boldsymbol{w}(\mathrm{i}y)\,\mathrm{d}y, \tag{6.2}$$

$$\rho^y_{\text{eff}} = \frac{1}{2h}\int_{-h}^{h} \operatorname{Im}[\rho(\tfrac{1}{2}l+\mathrm{i}y)\boldsymbol{w}(\tfrac{1}{2}l+\mathrm{i}y)]\,\mathrm{d}y \Big/ \frac{1}{l}\int_0^l \operatorname{Im}\boldsymbol{w}(x)\,\mathrm{d}x. \tag{6.3}$$

(*b*) An effective resistivity of the elementary cell: $\Omega = \Omega_1 \cup (0,l) \cup \Omega_2$

$$\rho_{\text{eff}} = \langle \rho(z)\boldsymbol{w}(z)\rangle / \langle \boldsymbol{w}(z)\rangle, \tag{6.4}$$

where

$$\langle \boldsymbol{f}(z)\rangle = \frac{1}{2lh}\int_{\Omega} \boldsymbol{f}(z)\,\mathrm{d}s.$$

(*c*) A functional of the whole dissipation:

$$D = \langle \rho(z)|\boldsymbol{w}(z)|^2\rangle. \tag{6.5}$$

For the convenience let us bring together all the formulae needed for the case:

$$\left.\begin{aligned} w_1(z) &= \sqrt{\frac{1-\Delta}{2}}\{a_1\mathrm{e}^{\mathrm{i}\gamma}\chi(z) + b_1\mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_1; \\ w_2(z) &= \mathrm{i}\sqrt{\frac{1+\Delta}{2}}\{a_1\mathrm{e}^{-\mathrm{i}\gamma}\chi(z) - b_1\mathrm{e}^{\mathrm{i}\gamma}\chi^{-1}(z)\}, \quad z \in \boldsymbol{\Omega}_2. \end{aligned}\right\} \tag{6.6}$$

Here $a_1 = a\sigma_1 + b\sigma$, $b_1 = a\sigma_1 - b\sigma$,

$$\left.\begin{aligned} \Delta = (\rho_1-\rho_2)/(\rho_1+\rho_2), \qquad & \mathrm{e}^{\mathrm{i}\pi\lambda} = \Delta' + \mathrm{i}\Delta, \\ \Delta' = \sqrt{1-\Delta^2} \geqslant 0, \qquad & \mathrm{e}^{\mathrm{i}\gamma} = (\sqrt{1-\Delta} + \sqrt{1+\Delta})/\sqrt{2}; \end{aligned}\right\} \tag{6.7}$$

$$-1 \leqslant \Delta \leqslant 1, \qquad -\tfrac{1}{2} \leqslant \lambda \leqslant \tfrac{1}{2}, \qquad 0 \leqslant 2\gamma = \pi(\lambda + \tfrac{1}{2}) \leqslant 1; \tag{6.8}$$

$$\sigma = \sigma(\lambda, m), \quad \sigma_1 = \sigma(\lambda, m_1), \quad \sigma(\lambda, m) = \frac{F(\frac{1}{2}, \frac{1}{2}; 1; m)}{F(\frac{1}{2}+\lambda, \frac{1}{2}-\lambda; 1; m)}. \tag{6.9}$$

and

$$\chi(z) = \left(\frac{1-\zeta}{\zeta(1-m\zeta)}\right)^{\lambda} = \left(\frac{1+\operatorname{cn}(2Kz/l|m)}{1-\operatorname{cn}(2Kz/l|m)}\right)^{\lambda} \tag{6.10}$$

is an even double-periodic branch fixed in the plane $\boldsymbol{C}\backslash\overline{H}$ by the condition $\chi(\frac{1}{2}l) = 1$. As a function of variable ζ the branch (6.10) is fixed in $\boldsymbol{C}\setminus L_2$ (recall that $L_2 = (-\infty, 0) \cup (1, 1/m)$) by the condition $\chi(\xi) > 0$, for $\forall\xi \in (0,1)$. The branch chosen satisfies the identities:

$$\overline{\chi}(\pm z) \equiv \chi(z), \qquad \overline{\chi}(l-z) \equiv \overline{\chi}(\mathrm{i}h + z) \equiv \chi^{-1}(z), \tag{6.11}$$

$$\overline{\chi}(\zeta) \equiv \chi(\zeta), \qquad \overline{\chi}\left(\frac{1}{m\zeta}\right) \equiv \overline{\chi}\left(\frac{1-\zeta}{1-m\zeta}\right) \equiv \chi^{-1}(\zeta), \tag{6.12}$$

$$\operatorname{Im}\chi^{\pm}(\xi) = \operatorname{Im}\chi(\xi) = 0, \xi \in L_1, \quad \operatorname{Im}[\mathrm{e}^{\pm\mathrm{i}\pi\lambda}\chi^{\pm}(\xi)] = 0, \xi \in L_2. \tag{6.13}$$

(*a*) Beginning the calculation of the functional (6.2) note that

$$\frac{1}{h}\int_0^h \operatorname{Re}\boldsymbol{w}_1(\mathrm{i}y)\,\mathrm{d}y = \frac{1}{h}\int_0^h \operatorname{Re} w_1(\mathrm{i}y)\,\mathrm{d}y = a,$$

due to the first condition (2.2). From relations (6.6)–(6.11) and figure 2 the following can be derived:

$$\begin{aligned} I &= \frac{1}{2l}\int_{-l}^{l} \operatorname{Re}[\rho(x+\mathrm{i}\tfrac{1}{2}h)\boldsymbol{w}(x+\mathrm{i}\tfrac{1}{2}h)]\,\mathrm{d}x \\ &= \frac{1}{2l}\int_{a_{01}}^{a_{21}} \operatorname{Re}[\rho_1 w_1(t) + \rho_2\overline{w}_2(-t)]\,\mathrm{d}t \\ &= \frac{\sqrt{\rho_1\rho_2}}{2l}\left\{\int_{a_{01}}^{a_{21}} \operatorname{Re}[\sqrt{\tfrac{1}{2}(1+\Delta)}(a_1\mathrm{e}^{\mathrm{i}\gamma}\chi(t) + b_1\mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(t))\right. \\ &\qquad \left. -\mathrm{i}\sqrt{\tfrac{1}{2}(1-\Delta)}(a_1\mathrm{e}^{\mathrm{i}\gamma}\chi(t) - b_1\mathrm{e}^{-\mathrm{i}\gamma}\chi^{-1}(t))]\,\mathrm{d}t\right\} \\ &= \frac{\sqrt{\rho_1\rho_2}}{2l}\operatorname{Re}\int_{a_{01}}^{a_{21}} [(a_1\mathrm{e}^{\mathrm{i}\lambda}\chi(t) + b_1\mathrm{e}^{-\mathrm{i}\lambda}\chi^{-1}(t)]\,\mathrm{d}t. \end{aligned}$$

Consider the integral

$$I_x(\lambda) = \operatorname{Re}\left[\mathrm{e}^{\mathrm{i}\lambda}\int_{a_{01}}^{a_{21}} \chi(t)\,\mathrm{d}t\right].$$

In accordance with the Cauchy integral theorem and taking into account that the function $\mathrm{e}^{\mathrm{i}\lambda}\chi(t)$ is real when $t \in [a_{00}, a_{01}] \cup [a_{20}, a_{21}]$ it is possible to get

$$\begin{aligned} I_x(\lambda) &= \operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\lambda}\left[\int_{a_{00}}^{a_{20}} \chi(x)\,\mathrm{d}x + \int_{a_{20}}^{a_{21}} \chi(t)\,\mathrm{d}t - \int_{a_{00}}^{a_{01}} \chi(t)\,\mathrm{d}t\right]\right\} \\ &= \Delta'\int_0^l \chi(x)\,\mathrm{d}x \end{aligned}$$

and due to the second relation (3.21)

$$I_x(\lambda) = I_x(-\lambda) = l/\sigma.$$

Thus

$$I = a\sqrt{\rho_1\rho_2}\sigma_1/\sigma.$$

At last

$$\rho^x_{\text{eff}} = \sqrt{\rho_1\rho_2}\sigma_1/\sigma. \tag{6.14}$$

In the same manner it may be shown that

$$\rho^y_{\text{eff}} = \sqrt{\rho_1\rho_2}\sigma/\sigma_1. \tag{6.15}$$

The last result can be obtained from the previous one by the replacement m on m_1 (l on h), i.e.

$$\rho^x_{\text{eff}}(m) = \rho^y_{\text{eff}}(m_1), \qquad (\rho^x_{\text{eff}}(l,h) = \rho^y_{\text{eff}}(h,l)).$$

(*b*) It will be shown below that for functional (6.4) the representation,

$$\rho_{\mathrm{eff}} = \sqrt{\rho_1\rho_2}\frac{a\sigma_1/\sigma - \mathrm{i}b\sigma/\sigma_1}{a - \mathrm{i}b} \tag{6.16}$$

is valid. To prove the equality (6.16) it is necessary to calculate the integral

$$\Lambda(\lambda) = \frac{1}{lh}\int_{\Omega_1}\chi(z)\,\mathrm{d}s = \frac{\mathrm{i}}{2lh}\int_{\Omega_1}\chi(z)\,\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

By means of the substitution $\zeta = \mathrm{sn}^2(Kz/l|m)$, taking into account the relations $l/h = K/K'$, (3.22), (6.12), figures 2 and 3 and Stokes's theorem, the last integral transforms to the following one

$$\Lambda(\lambda) = \frac{\mathrm{i}}{8KK'}\int_{\boldsymbol{R}}\frac{1}{\overline{X(\tau)}}\int_0^{\tau}\frac{\chi(\xi)\,\mathrm{d}\xi}{X(\xi)}\,\mathrm{d}\tau.$$

Here $X(\tau)$ is the boundary value on the real axis from the upper half-plane of the single-valued branch of the function

$$X(\zeta) = \sqrt{\zeta(1-\zeta)(1-m\zeta)}, \tag{6.17}$$

fixed in the plane $\boldsymbol{C}\setminus L_2$ by the condition $X(x) > 0,\quad x \in (0,1)$. It may be shown that the chosen branch (6.17) is characterized by the following properties:

$$\overline{X}(\zeta) \equiv X(\zeta),\quad \overline{X}\left(\frac{1}{m\zeta}\right) \equiv -\frac{X(\zeta)}{m\zeta^2},\quad \overline{X}\left(\frac{1-\zeta}{1-m\zeta}\right) \equiv \frac{m_1X(\zeta)}{(1-m\zeta)^2}; \tag{6.18}$$

$$\mathrm{Im}\,X(\tau) = 0, \tau\in L_1,\quad \mathrm{Re}\,X^{\pm}(\tau) = 0, \tau \in L_2. \tag{6.19}$$

By means of the equalities (6.19) and the Cauchy integral theorem the last integral can be transformed to the following form:

$$\begin{aligned}\Lambda(\lambda) &= \frac{\mathrm{i}}{8KK'}\int_{\boldsymbol{R}}\frac{1}{X(\tau)}\int_0^{\tau}\frac{\chi(\xi)\,\mathrm{d}\xi}{X(\xi)}\,\mathrm{d}\tau - \frac{\mathrm{i}}{4KK'}\int_{L_2}\frac{1}{X(\tau)}\int_0^{\tau}\frac{\chi(\xi)\,\mathrm{d}\xi}{X(\xi)}\,\mathrm{d}\tau\\ &= -\frac{\mathrm{i}}{4KK'}\int_{L_2}\frac{\omega_\lambda(\tau)}{X(\tau)}\,\mathrm{d}\tau.\end{aligned}$$

Here

$$\omega_\lambda(\zeta) = \int_0^{\zeta}\frac{\chi(\tau)}{X(\tau)}\,\mathrm{d}\tau = \int_0^{\zeta}\tau^{-\lambda-1/2}(1-\tau)^{\lambda-1/2}(1-m\tau)^{-\lambda-1/2}\,\mathrm{d}\tau. \tag{6.20}$$

The Schwarz–Christoffel integral (6.20) maps conformally the half-plane $\boldsymbol{C}^+$ onto the parallelogram P^+ with vertexes at the points $\{0, R, R + R_1\mathrm{e}^{\mathrm{i}\pi(1/2-\lambda)}, R_1\mathrm{e}^{\mathrm{i}\pi(1/2-\lambda)}\}$, where $R = R(\lambda, m) = \omega_\lambda(1)$, $R_1 = |\omega_\lambda(\infty)| = R(\lambda, m_1)$. It is not difficult to find

$$R = \frac{\pi}{\Delta'}F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\lambda; 1; m) = R(\pm\lambda, m). \tag{6.21}$$

Analytic continuation of the function (6.20) into the half-plane $\boldsymbol{C}^-$ through the interval $(0,1)$ maps $\boldsymbol{C}^-$ onto the parallelogram P^- which is symmetric with P^+ about the real axis. The following identities are valid for the continued function (6.20):

$$\overline{\omega}_\lambda\left(\frac{1}{m\zeta}\right) \equiv R_1\mathrm{e}^{-\mathrm{i}\pi(1/2-\lambda)} + \omega_{-\lambda}(\zeta),\quad \overline{\omega}_\lambda\left(\frac{1-\zeta}{1-m\zeta}\right) \equiv R - \omega_{-\lambda}(\zeta) \tag{6.22}$$

and $\overline{\omega}_\lambda(\zeta) \equiv \omega_\lambda(\zeta)$. Let us consider now an auxiliary integral

$$I = \frac{\mathrm{i}}{4KK'} \int_\Gamma \frac{\omega_\lambda(\tau)}{X(\tau)}\,\mathrm{d}\tau,$$

where Γ is a two-sided cut along $\boldsymbol{R} \setminus (0,1)$. It is obvious that $I = 0$ on the basis of the Cauchy integral theorem. On the other hand, taking into account (6.21) and the relationship between limit values of the functions $\omega^{\pm}(\tau)$ and $X^{\pm}(\tau)$ on Γ, it can be shown that

$$I = (1 - \mathrm{e}^{\mathrm{i}2\pi\lambda})\Lambda(\lambda) - \frac{\mathrm{i}R(1+\mathrm{e}^{\mathrm{i}2\pi\lambda})}{4KK'} \int_1^{1/m} \frac{\mathrm{d}\tau}{X^+(\tau)} + \frac{\pi\Delta' R_1}{2KK'} \int_{1/m}^{\infty} \frac{\mathrm{d}\tau}{X(\tau)} = 0.$$

After some algebra we obtain

$$\Lambda(\lambda) = \frac{\mathrm{i}}{\Delta}\left(\frac{\mathrm{e}^{-\mathrm{i}\pi\lambda}}{\sigma_1} - \frac{1}{\sigma}\right). \tag{6.23}$$

Now by means of the relations (6.6)–(6.10), (6.23) we get

$$\begin{aligned}\langle \boldsymbol{w}(z)\rangle &= \frac{1}{2lh}\int_{\Omega_1} (\overline{w_1(z)} + \overline{w}_2(z))\,\mathrm{d}s \\ &= \tfrac{1}{2}\{\surd(\tfrac{1}{2}(1-\Delta))(a_1\mathrm{e}^{-\mathrm{i}\gamma}\overline{\Lambda} + b_1\mathrm{e}^{\mathrm{i}\gamma}\Lambda) - \mathrm{i}\surd(\tfrac{1}{2}(1+\Delta))(a_1\mathrm{e}^{\mathrm{i}\gamma}\Lambda - b_1\mathrm{e}^{-\mathrm{i}\gamma}\overline{\Lambda})\} \\ &= a\sigma_1\,\mathrm{Re}\,\Lambda - \mathrm{i}b\sigma\,\mathrm{Re}(\mathrm{e}^{\mathrm{i}\pi\lambda}\Lambda) = a - \mathrm{i}b.\end{aligned}$$

Similarly

$$\begin{aligned}\langle \rho(z)\boldsymbol{w}(z)\rangle &= \frac{1}{2lh}\int_{\Omega_1} (\rho_1\overline{w_1(z)} + \rho_2\overline{w}_2(z))\,\mathrm{d}s \\ &= \sqrt{\rho_1\rho_2}\left(a\frac{\sigma_1}{\sigma} - \mathrm{i}b\frac{\sigma}{\sigma_1}\right).\end{aligned}$$

Thus the validity of the formula (6.16) is established.

From the formulae (6.14)–(6.16) we conclude the following.

First, note the following useful relations:

$$\rho_{\mathrm{eff}} = \rho_{\mathrm{eff}}(\lambda, a, b; m) = \rho_{\mathrm{eff}}(-\lambda, a, b; m) = \rho_{\mathrm{eff}}(\lambda, b, -a; m_1).$$

Being evident from the physical point of view these are an indirect proof of the validity of formula (6.16).

Second, from (6.16) we deduce

$$\rho_{\mathrm{eff}} = \sqrt{\rho_1\rho_2},$$

iff $m = m_1 = \frac{1}{2}$ $(l = h)$. The last relation coincides with the computations by Helsing (1991) (figure 1 and table 1, $\beta = \pi/4$), and with the analytic Keller–Dykhne–Mendelson formula which was stated for arbitrary two-phase composite with statistically equivalent phases. Thus, our result shows that phases of the rectangular checkerboard field are equivalent in this sense just for the square case.

Third, from (6.14)–(6.16) follows:

$$\begin{aligned}\rho_{\mathrm{eff}}(\lambda, a, 0; m) &= \rho_{\mathrm{eff}}(-\lambda, \infty, b; m) = \rho^x_{\mathrm{eff}}, \\ \rho_{\mathrm{eff}}(\lambda, 0, b, m) &= \rho_{\mathrm{eff}}(-\lambda, a, \infty, m) = \rho^y_{\mathrm{eff}}, \\ \rho^x_{\mathrm{eff}}(m) &= \rho^y_{\mathrm{eff}}(m_1), \quad \rho^x_{\mathrm{eff}}\rho^y_{\mathrm{eff}} = \rho_1\rho_2.\end{aligned}$$

The last of these relations generalizes in some sense the Keller formula (1963).

(*c*) To calculate the dissipation functional (6.5) we consider the integral

$$D_p = \frac{\rho_p}{lh}\int_{\Omega_p} |\boldsymbol{w}_p|^2\,\mathrm{d}s, \qquad p = 1,2.$$

Due to (6.6) and the identity $|w|^2 = w\overline{w}$ dissipation of energy in the cell Ω_1 is

$$D_1 = \frac{\rho_1(1-\Delta)}{2lh}\left\{\int_{\Omega_1}(a_1^2|\chi|^2 + b_1^2|\chi|^{-2})\,\mathrm{d}s + 2a_1b_1\,\mathrm{Re}[\mathrm{e}^{-2\mathrm{i}\gamma}\int_{\Omega_1}\overline{\chi}\chi^{-1}\,\mathrm{d}s]\right\}. \tag{6.24}$$

Analogously, using Stokes's theorem and relations (6.11)–(6.13), (6.18)–(6.22) we obtain

$$I(\lambda,m) = \frac{1}{lh}\int_{\Omega_1}|\chi|^2\,\mathrm{d}s = -\frac{\mathrm{i}\Delta'\mathrm{e}^{\mathrm{i}\pi\lambda}}{4KK'}\int_{L_2}\frac{\chi(\tau)\omega_\lambda(\tau)}{X(\tau)}\,\mathrm{d}\tau.$$

Then on the basis of the equality,

$$\int_\Gamma \frac{\chi(\tau)\omega_\lambda(\tau)}{X(\tau)}\,\mathrm{d}\tau = 0,$$

we deduce

$$I(\lambda,m) = I(-\lambda,m) = 1/(\Delta'\sigma\sigma_1).$$

Unfortunately, we could not calculate in such an exact form the integral

$$J(\lambda,m) = \frac{1}{lh}\,\mathrm{Re}\left[\mathrm{e}^{-2\mathrm{i}\gamma}\int_{\Omega_1}\overline{\chi}\chi^{-1}\,\mathrm{d}s\right].$$

Nevertheless, it is not difficult to show that

$$J(\lambda,m) = \frac{\Delta'}{8KK'}\{j(\lambda,m) - j(-\lambda,m) + j(-\lambda,m_1) - j(\lambda,m_1)\}, \tag{6.25}$$

where

$$\begin{aligned} j(\lambda,m) &= \int_0^1 \frac{\chi(\tau)\omega_{-\lambda}(\tau)}{X(\tau)}\,\mathrm{d}\tau \\ &= \int_0^1 \frac{\tau^{-\lambda}(1-\tau)^{\lambda}(1-m\tau)^{-\lambda}}{\sqrt{\tau(1-\tau)(1-m\tau)}}\int_0^\tau \frac{\xi^\lambda(1-\xi)^{-\lambda}(1-m\xi)^\lambda}{\sqrt{\xi(1-\xi)(1-m\xi)}}\,\mathrm{d}\xi\mathrm{d}\tau. \end{aligned}$$

From representation (6.24) it follows immediately that

$$J(\lambda,m) = -J(-\lambda,m) = -J(\lambda,m_1). \tag{6.26}$$

Now we derive

$$D_1 = \sqrt{\rho_1\rho_2}\{a^2\sigma_1/\sigma + b^2\sigma/\sigma_1 + \Delta'(a^2\sigma_1^2 - b^2\sigma^2)J(\lambda,m)\}. \tag{6.27}$$

Taking into account that $D_2(\rho_1,\rho_2;m) = D_1(\rho_2,\rho_1;m)$ and in accordance with relations (6.7), (6.9), (6.25) we obtain

$$D_2 = \sqrt{\rho_1\rho_2}\{a^2\sigma_1/\sigma + b^2\sigma/\sigma_1 - \Delta'(a^2\sigma_1^2 - b^2\sigma^2)J(\lambda,m)\}. \tag{6.28}$$

From (6.26), (6.27) it follows that the functional (6.5) of whole dissipation within an elementary cell Ω is

$$D = \tfrac{1}{2}(D_1 + D_2) = \sqrt{\rho_1\rho_2}(a^2\sigma_1/\sigma + b^2\sigma/\sigma_1). \tag{6.29}$$

In particular, for the square checkerboard field ($l = h$, $m = m_1 = \frac{1}{2}$) $J(\lambda, \frac{1}{2}) = 0$ on the basis of (6.25). Therefore

$$D = D_1 = D_2 = \sqrt{\rho_1 \rho_2}(a^2 + b^2). \qquad (6.30)$$

This expression meets with the corresponding Dykhne's assertion (1970).

For an isotropic medium ($\rho_1 = \rho_2 = \rho$, $\lambda = 0$) formulae (6.9), (6.25)–(6.28) give

$$D = D_1 = D_2 = \rho(a^2 + b^2).$$

For other cases ($l \neq h$, $\rho_1 \neq \rho_2$) $D_1 \neq D_2$, except of one particular case when the exterior field is directed in such a manner that $a/b = \pm\sigma/\sigma_1$.

The expressions for effective integral characteristics on the case of complex coefficients A, B are also obtained but they will be presented in our next paper.

7. Conclusion

In this paper we investigated a classical boundary-value problem, including proof of existence and uniqueness of solution, for a biperiodic two-phase material. Many real composites exhibit more complex structure and nonlinearity of physical processes involved that call for numerical methods. However, the explicit solutions derived can serve as check procedures both for various approximations and routine finite-difference–finite-element approaches. So far we evaluated specific functionals and characteristics (dissipation, effective resistivities), the determination of which is often sufficient for engineering purposes. However, our solutions allow for detailed description of the field such as tracking of individual particles, time evolution of fronts, calculation of isochrones, etc. (Bear 1972). This is important in problems of contaminant transport in subsurface, civil and chemical engineering, electrodynamics when 'fine' flow, current, transport structure is required. For example, to characterize fingering induced by porous inclusions (Kacimov & Obnosov 1994), forces and torques arising as a result of exposure of dielectric particles to outer fields (Emets *et al.* 1993), etc., one needs detailed field distributions. For arbitrary resistivities ratios the known asymptotic, variational, lumped-parameter methods and empiric formulae can fail to do this. I plan to broaden the class of composites allowing for explicit solutions and compare them with approximate, including numerical (FDM and FEM), results. My final goal is optimization of field characteristics by proper choice of geometry and resistivity distribution.

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References

Abramowitz, M. & Stegun, I. A. 1970 *Handbook of mathematical functions.* New York: Dover.

Bakhvalov, M. C. & Panasenko, G. P. 1984 *Homogenization of processes in periodic media.* Moscow: Nauka. (In Russian.)

Bauer, T. H. 1993 A general analytical approach toward the thermal conductivity of porous media. *Int. Jl Heat Mass Transfer* **36**, 4181–4191.

Bear, J. 1972 *Dynamics of fluids in porous media*, pp. 154. New York: Elsevier.

Berdichevski, V. L. 1983 *Variational principles of continuum mechanics.* Moscow: Nauka. (In Russian.)

Berdichevski, V. L. 1985 The thermal conductivity of chess structures. *Vestn. Mosk. Univ. Mat. Mech.* **40**(4), 56–63. (In Russian.)

Buchholz, H. 1957 *Elektrische und Magnetische Potentialfelder.* Berlin: Springer.

Carslaw, H. S. & Jaeger, J. C. 1959 *Conduction of heat in solids.* London: Oxford University Press.

Clark K. E. & Milton, G. W. 1995 Optimal bounds correlating electric, magnetic and thermal properties of two-phase, two-dimensional composites. *Proc. R. Soc. Lond.* A **448**, 161–190.

Crank, J. 1975 *The mathematics of diffusion.* Oxford: Clearendon Press.

Dykhne, A. M. 1970 Conductivity of a two-dimensional two-phase system. *Zh. Eksp. Teor. Fiz.* **59**, 110–113. (Soviet. Phys. JETP **32** (1971), 63–65.)

Emets, Yu. P. 1986 *Electrical characteristics of composite materials with regular structure.* Kiev: Naukova dumka. (In Russian.)

Emets, Yu. P. & Obnosov, Yu. V. 1989 Exact solution of the problem of current generation in a doubly periodic heterogeneous system. *Dokl. Akad. Nauk SSSR* **309**, 319–322. (*Soviet. Phys. Dokl.* **34** (1989), 972–974.)

Emets, Yu. P. & Obnosov, Yu. V. 1990 Compact analog of a heterogeneous system with a checkerboard field structure. *Zh. Tekh. Fiz.* **60**, 59–66. (*Soviet. Phys. Tech. Phys.* **35** (1990), 907–913.)

Emets, Yu. P., Obnosov, Yu. V. & Onofriichuk, Ju. P. 1993 Interaction between touching circular dielectric cylinders in a uniform electric field. *Zh. Tekh. Fiz.* **63**(12), 12–24. (*Tech. Phys.* **38** (1993), 1043–1047.)

Gakhov, F. D. 1966 *Boundary value problems.* New York: Addison-Wesley.

Gheorghiţă, Şt. I. 1966 *Metode matematice in hidrogasodinamica subterană.* Bucureşti: ed. Acad. RSR. (In Romanian.)

Golden, K. & Papanicolaou, G. 1985 Bounds for effective parameters of multicomponent media by analytic continuation. *J. Statist. Phys.* **40**, 655–667.

Gradshteyn, I. S. & Ryzik, I. M. 1980 *Table of integrals, series and products.* New York: Academic Press.

Hashin, Z. & Shtrikman, S. 1962 A variational approach to the theory of effective magnetic permeability of multyphase materials. *J. Appl. Phys.* **33**, 3125–3131.

Helsing, J. 1991 Transport properties of two-dimensional tilings with corners. *Phys. Rev.* B **44**, 11677–11682.

Honein, N., Honein, T. & Herrmann, G. 1992 On two circular inclusions in harmonic problems. *Q. Appl. Math.* **50**, 479–499.

Kacimov, A. R. & Obnosov, Yu. V. 1994 Minimization of ground water contamination by lining of a porous waste repository. *Proc. Indian Nat. Sci. Acad.* A **60**, 783–792.

Keller, J. B. 1963 Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders. *J. Appl. Phys.* **34**, 991–993.

Keller, J. B. 1964 A theorem on the conductivity of a composite medium. *J. Math. Phys.* **5**, 548–549.

Lurie, K. A. & Cherkaev, A. V. 1985 Optimization of properties of multicomponent isotropic composites. *J. Opt. Theory Appl.* **46**, 571–580.

Maxwell, J. C. 1904 *A treaties on electricity and magnetism*, 3rd edn, vol. 1. Oxford University Press.

Mendelson, K. S. 1975 A theorem on the effective conductivity of a two-dimensional heterogeneous medium. *J. Appl. Phys.* **46**, 4740–4741.

Milton, G. W., McPhedran, R. C. & McKenzie, D. R. 1981 Transport properties of intersecting cylinders. *Appl. Phys.* **25**, 23–30.

Missenard, A. 1965 *Conductivité thermique des solides, liquides, gaz et de leurs mélanges.* Paris: Editions Eyrolles. (In French.)

Muskhelishvili, N. I. 1972 *Singular integral equation.* Groningen: Wolters-Nordorf.

Nicorovici, N. A., McPhedran, R. C. & Milton, G. W. 1993 Transport properties of a three-phase composite material: the square array of coated cylinders. *Proc. R. Soc. Lond.* A **442**, 599–620.

Obnosov, Yu. V. 1991 Solution of a Markushevich problem in the class of double-periodic functions with orthogonal periods. *Dokl. Akad. Nauk SSSR* **319**, 1125–1127. (*Soviet. Phys. Dokl.* **36** (1991), 573–575.)

Obnosov, Yu. V. 1992 Solution of a problem of ***R***-linear conjugation for a triangular regular checkerboard field. *Dokl. Akad. Nauk SSSR* **327**, 326–330. (*Soviet. Phys. Dokl.* **37** (1992), 546–549.)

Oleinik, O. A., Kozlov, S. M. & Zhikov, V. V. 1994 *Homogenization of differential operators and integral functionals*. Berlin: Springer.

Lord Rayleigh 1892 On the influence of obstacles arranged in rectangular order upon the properties of medium. *Phil. Mag.* **34**, 481–502.

Sanchez-Palencia, E. 1980 *Non-homogeneous media and vibration theory.* New York: Springer-Verlag.

Schulgasser, K. 1992 A reciprocal theorem in two-dimensional heat transfer and its implications. *Int. Comm. Heat Mass Transfer* **19**, 639–649.

Talbot, D. R. S. & Willis, Y. R. 1994 Upper and lower bounds for the overall properties of a nonlinear composite dielectric. I, II. Random microgeometry. *Proc. R. Soc. Lond.* A **447**, 365–396.

Torquato, S. 1991 Random heterogeneous media: microstructure and improved bounds on effective properties. *Appl. Mech. Rev.* **44**, 37–76.

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