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NUMERICAL METHODS

Implicit Euler Scheme for an Abstract Evolution Inequality

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Abstract—For a triple $\{V, H, V^*\}$ of Hilbert spaces, we consider an evolution inclusion of the form $u'(t) + A(t)u(t) + \partial\phi(t, u(t)) \ni f(t)$, $u(0) = u_0$, $t \in (0, T]$, where $A(t)$ and $\phi(t, \cdot)$, $t \in [0, T]$, are a family of nonlinear operators from V to V^* and a family of convex lower semicontinuous functionals with common effective domain $D(\phi) \subset V$. We indicate conditions on the data under which there exists a unique solution of the problem in the space $H^1(0, T; V) \cap W^1_\infty(0, T; H)$ and the implicit Euler method has first-order accuracy in the energy norm.

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The aim of the present paper is to state conditions guaranteeing the first-order accuracy of the implicit Euler method in the energy norm for the following problem: find a function $u(t) \in D(\phi)$ with $u(0) = u_0$ such that the inequality

$$\langle u'(t) + A(t)u(t) - f(t), v - u(t)\rangle + \phi(t, v) - \phi(t, u(t)) \geq 0 \tag{1}$$

holds for each $v \in D(\phi)$ and for almost all $t \in (0, T]$, which is equivalent to the relation $u'(t) + A(t)u(t) + \partial\phi(t, u(t)) \ni f(t)$, where $\partial\phi : \ \mathcal{V} \to \mathcal{V}^*$ is the subdifferential of ϕ. This problem was earlier considered in [1, 2] for the case in which the functional ϕ does not explicitly depend on t. (The operators $A(t)$ were assumed in [1] to be linear, and the case of $A(t) = 0$ was considered in [2].) We generalize the results of these papers by using the studies in [1]. The energy norm is defined by the formula

$$\|v\|_E = \|v\|_{L_\infty(0,T;H)} + \|v\|_{L_2(0,T,V)}.$$

1. NOTATION AND ASSUMPTIONS

Let V and H be separable Hilbert spaces with dense continuous embeddings $V \subset H = H^* \subset V^*$, and let $\langle\cdot\,,\cdot\rangle$ be the duality pairing between V^* and V. For a given Banach space X, we define the spaces $L_p(0, T; X)$ and $W^k_p(0, T; X)$, $p \in [1, \infty]$, $k \geq 0$, and the norms in them in a standard way (e.g., see [3, Chap. 4]). Set

$$\mathcal{V} = L_2(0, T; V), \qquad \mathcal{V}^* = L_2(0, T; V^*), \qquad H^1(0, T; X) = W^1_2(0, T; X),$$
$$\mathcal{D}(\phi) = \{v \in \mathcal{V} : \ \phi(t, v(t)) \in L_1(0, T)\}.$$

We impose the following constraints on data of problem (1):

(A_1) $A(t)0 = 0$; the estimates

$$\langle A(t)u - A(t)v, u - v\rangle \geq \alpha\|u - v\|^2_V, \qquad \alpha = \text{const} > 0,$$
$$\|A(t)u - A(t)v\|_{V^*} \leq m_0(t)\|u - v\|_V, \qquad \|A'(t)v\|_{V^*} \leq m_1(t)\|v\|_V,$$
$$M = \|m_0\|_{L_2(0,T)} + \|m_1\|_{L_2(0,T)} < \infty$$

$[A'(t) = dA(t)/dt]$ hold for arbitrary $u, v \in V$ and for almost all $t \in [0, T]$.

(A_2) The functional $v \to \phi(t,v)$ is proper convex and lower semicontinuous on V for each $t \in [0,T]$, and its effective domain

$$D(\phi) = \{v \in V : \ \phi(t,v) < \infty\}$$

is independent of t; $0 \in D(\phi)$.

(A_3) If $\chi : \ [0,T] \to V^*$ is a subgradient of ϕ at zero, i.e., if $\phi(t,v) - \phi(t,0) \geq \langle \chi(t), v\rangle$ for all $v \in D(\phi)$, then $\chi \in H^1(0,T;V^*)$.

(A_4) $\int_0^T |\phi_t(t,u(t)) - \phi_t(t,v(t))|\, dt \leq \varrho(\|u\|_{\mathcal{V}}, \|v\|_{\mathcal{V}})\|u-v\|_{\mathcal{V}}$ for all $u, v \in \mathcal{D}(\phi)$, where the function ϱ is continuous function and nondecreasing with respect to each argument and $\phi_t(t,u) = d\phi(t,u)/dt$, $u \in D(\phi)$.

(A_5) $f \in H^1(0,T;V^*)$, $u_0 \in D(\phi)$, and $C_0 = \|u_0\|_V + \inf_{v \in M(u_0,f)} \|v\|_H < \infty$, where the set

$$M(u_0,f) = \{w \in H : \ \langle w + A(0)u_0 - f(0), v - u_0\rangle + \phi(0,v) - \phi(0,u_0) \geq 0 \ \forall\, v \in D(\phi)\}$$

is nonempty.

Note that condition (A_1) implies the continuity of the function $t \to A(t)$ on $[0,T]$ and the pseudomonotonicity and coercivity of $A(t)$ for each $t \in [0,T]$ (e.g., see [4, p. 190]), and condition (A_3) permits one to assume without loss of generality that

$$\phi(t,v) \geq \phi(t,0) = 0 \qquad \forall\, v \in D(\phi).$$

In what follows, we assume that this condition is satisfied. Indeed, otherwise the problem can be reduced to problem (1) with data $\bar{f}(t) = f(t) - \chi(t)$ and $\bar{\phi}(t,v) = \phi(t,v) - \phi(t,0) - \langle \chi(t), v\rangle$ and with the same solution u; moreover, one can readily see that conditions (A_2), (A_3), and (A_5), as well as condition (A_4), remain valid for the new data, because $\|\chi'(t)\|_{V^*} \in L_2(0,T)$ and

$$\begin{aligned} |\bar{\phi}_t(t,u) - \bar{\phi}_t(t,v)| &= |\phi_t(t,u) - \phi_t(t,v) + \langle \chi'(t), u - v\rangle| \\ &\leq |\phi_t(t,u) - \phi_t(t,v)| + \|\chi'(t)\|_{V^*}\|u - v\|_V. \end{aligned}$$

The condition $C_0 < \infty$ in (A_5) is the matching condition for the data and is necessary for the problem to be solvable in the space $E^1 = H^1(0,T;V) \cap W^1_\infty(0,T;H)$. Indeed, if $u \in E^1$, then $u \in C([0,T];V)$, $u' \in C([0,T];H)$, and one can consider inequality (1) for $t = 0$ by continuity. Then we obtain $u'(0) \in M(u_0,f)$ and $C_0 < \infty$.

2. IMPLICIT EULER SCHEME

Let us fix the grid increment $\tau = T/N$ and the corresponding partition of the interval $[-\tau, T]$ into the elements $I_n = [t_{n-1}, t_n)$, $n = 0, 1, \ldots, N$, where $t_j = j\tau$, $j = -1, 0, \ldots, N$. Set $y^n \approx u(t_n)$, $A^n = A(t_n)$, $f^n = f(t_n)$, $\phi^n(\cdot) = \phi(t_n, \cdot)$, $y^{-1} = u_0$, $A(t_{-1}) = A(0)$, $f(t_{-1}) = f(0)$, and $\phi(t_{-1}, \cdot) = \phi(0, \cdot)$.

Let us define an implicit scheme as follows: find $y^n \in D(\phi)$ such that the inequalities

$$\langle (y^n - y^{n-1})/\tau + A^n y^n - f^n, v - y^n\rangle + \phi^n(v) - \phi^n(y^n) \geq 0 \qquad \forall\, v \in D(\phi) \tag{2}$$

hold for $n = 0, 1, \ldots, N$. Note that, unlike the traditional statement of the implicit Euler scheme, for $n = 0$, we solve the following problem to find y^0:

$$\langle (y^0 - u_0)/\tau + A(0)y^0 - f(0), v - y^0\rangle + \phi(0,v) - \phi(0,y^0) \geq 0 \qquad \forall\, v \in D(\phi). \tag{3}$$

This approximation proves useful in connection with condition (A_5).

From inequality (2), we successively find y^n (starting from y^0). For $y^n \in D(\varphi) = D(\phi)$, we obtain the inequality

$$\langle By - G, v - y\rangle + \varphi(v) - \varphi(y) \geq 0 \qquad \forall\, v \in D(\varphi),$$

where $B = 1/\tau + A^n$ is a pseudomonotone coercive operator from V into V^*,

$$G = y^{n-1}/\tau + f^n \in V^*,$$

and $\varphi(y) = \phi(t_n, y)$ is a proper convex lower semicontinuous functional on V. It is well known that there exists a uniquely determined solution of this inequality (e.g., see [4, Th. 8.5, p. 265]). Therefore, the sequence $\{y^n\}_{n=0}^N$ is well defined and lies in $D(\phi)$.

3. A PRIORI ESTIMATES

The piecewise constant and piecewise linear extensions of a grid function g^n, $n = -1, 0, \ldots, N$, will be denoted by $g_\tau(t)$ and $\hat{g}_\tau(t)$, $t \in [-\tau, T)$, respectively; by $\check{u}$ we denote the shift $\check{u}(t) = u(t-\tau)$ of a function u. In addition, let $\ell(t)$ stand for the τ-periodic function equal to $(t - t_{n-1})/\tau$ on the interval $[t_{n-1}, t_n)$. Then, for all $t \in (-\tau, T)$, we have[1]

$$\hat{y}_\tau(t) = (1 - \ell(t))\check{y}_\tau(t) + \ell(t) y_\tau(t), \quad \hat{y}_\tau(t) - y_\tau(t) = (1 - \ell(t))(\check{y}_\tau(t) - y_\tau(t)),$$
$$\hat{y}'_\tau(t) = (y_\tau(t) - \check{y}_\tau(t))/\tau.$$

Lemma 1. *One has the estimates*

$$\|(A - A_\tau)v\|_{\mathcal{V}^*} + \tau\|\hat{A}'_\tau v\|_{\mathcal{V}^*} \le 2M\tau\|v\|_{L_\infty(0,T;V)}$$

and

$$\|f - f_\tau\|_{\mathcal{V}^*} + \tau\|\hat{f}'_\tau\|_{\mathcal{V}^*} \le 2F\tau,$$

where $F = \|f\|_{H^1(0,T;V^*)}$.

Proof. We have

$$\int_0^T \langle (A - A_\tau)v, w\rangle\, dt = \sum_{n=1}^N \int_{I_n} \langle (A(t) - A(t_n))v(t), w(t)\rangle\, dt = \sum_{n=1}^N \int_{I_n} \left(\int_{t_n}^t \langle A'(s)v(t), w(t)\rangle\, ds \right) dt$$
$$\le \sum_{n=1}^N \int_{I_n} \left(\int_{I_n} m_1(s)\, ds \right) \|v(t)\|_V \|w(t)\|_V\, dt \le M\tau\|v\|_{L_\infty(0,T;V)}\|w\|_{\mathcal{V}}.$$

Hence it follows that $\|(A - A_\tau)v\|_{\mathcal{V}^*} \le M\tau\|v\|_{L_\infty(0,T;V)}$. The estimate $\|\hat{A}'_\tau v\|_{\mathcal{V}^*} \le M\|v\|_{L_\infty(0,T;V)}$, as well as similar estimates for f, can be proved in a completely similar way.

Lemma 2. *Let* y^0 *be a solution of problem* (3). *Then* $\|y^0 - u_0\|_H^2 + \alpha\tau\|y^0 - u_0\|_V^2 \le C_0^2\tau^2$.

Proof. Let w be some element of the set $M(u_0, f)$ [see condition (A_5)]. We take $v = y^0$ in the inequality defining w and $v = u_0$ in (3). By adding the resulting inequalities, we obtain

$$\langle (y^0 - u_0)/\tau - w + A(0)y^0 - A(0)u_0, y^0 - u_0\rangle \le 0.$$

Let us use the strong monotonicity of A. Then

$$\|y^0 - u_0\|_H^2 + \alpha\tau\|y^0 - u_0\|_V^2 \le \tau\|w\|_H\|y^0 - u_0\|_H.$$

Hence it follows that $\|y^0 - u_0\|_H^2 + \alpha\tau\|y^0 - u_0\|_V^2 \le \tau^2\|w\|_H^2$. By minimizing this estimate with respect to w, we obtain the desired assertion.

[1] At the points of discontinuity, $\hat{y}'_\tau$ is assumed to be left continuous; $\hat{f}_\tau$ and $\hat{A}_\tau$ are defined in a similar way.

Lemma 3. *Let y be a solution of the scheme* (2). *Then*[2]

$$\|y_\tau\|_E \le C, \qquad \|\check{y}_\tau\|_E \le C, \qquad \|\hat{y}_\tau\|_E \le C, \qquad C = c(T,\alpha)(C_0 + F).$$

Proof. Let us multiply inequality (2) by τ and set $v = 0$. By virtue of conditions (A_1) and (A_3), we obtain the inequality $\langle y^n - y^{n-1} + \tau A^n y^n, y^n\rangle \le \tau\langle f^n, y^n\rangle$. We use the strong monotonicity A^n $(A^n 0 = 0)$, the inequality

$$2\langle u - v, u\rangle = \|u\|_H^2 + \|u - v\|_H^2 - \|v\|_H^2 \ge \|u\|_H^2 - \|v\|_H^2 \qquad \forall\, u, v \in V, \tag{4}$$

and the ε-inequality $2ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$, $\varepsilon > 0$, $a, b \in R$. As a result, we have

$$\|y^n\|_H^2 - \|y^{n-1}\|_H^2 + 2\alpha\tau\|y^n\|_V^2 \le 2\tau\|f^n\|_{V^*}\|y^n\|_V \le \tau\varepsilon^{-1}\|f^n\|_{V^*}^2 + \tau\varepsilon\|y^n\|_V^2.$$

By setting $\varepsilon = \alpha$ and by summing the resulting inequalities with respect to n from 1 to $m \le N$, we obtain the estimate

$$\|y^m\|_H^2 + \alpha\tau\sum_{n=1}^{m}\|y^n\|_V^2 \le \|y^0\|_H^2 + \alpha^{-1}\tau\sum_{n=1}^{N}\|f^n\|_{V^*}^2 \le C^2, \quad C = c(T,\alpha)(C_0 + F),$$

since Lemmas 1 and 2 imply that $\|f_\tau\|_{\mathcal{V}^*} \le 3F$ and $\|y^0\|_H \le \|u_0\|_H + C_0\tau \le (1+T)C_0$.

Consequently, we have the estimates

$$\|y_\tau\|_{L_\infty(0,T;H)} = \max_{1\le m\le N}\|y^m\|_H \le C, \qquad \|y_\tau\|_{L_2(0,T;V)} \le C/\alpha.$$

These two estimates imply the first assertion of the lemma. Since

$$\|\check{y}_\tau\|_{L_\infty(0,T;H)} = \max_{0\le m\le N-1}\|y^m\|_H, \qquad \|\check{y}_\tau\|_{\mathcal{V}}^2 \le \tau\|y^0\|_V^2 + \|y_\tau\|_{\mathcal{V}}^2,$$

it is clear that $\|\check{y}_\tau\|_E \le C$. The final estimate in the lemma follows from the inequality $\|\hat{y}_\tau\|_E \le \|\check{y}_\tau\|_E + \|y_\tau\|_E$.

Lemma 4. *Let y be a solution of the scheme* (2). *Then $\|\hat{y}'_\tau\|_E \le C$, $C = C(T,\alpha,C_0,F,M,\varrho)$.*

Proof. Let us introduce the notation $y_t^n = (y^{n+1} - y^n)/\tau$. Set $v = y^{n+1}$ in inequality (2) preliminarily divided by τ and $v = y^n$ in the same inequality at the next time step; by adding the resulting inequalities, we arrive at the relation

$$\langle y_t^n - y_t^{n-1}, y_t^n\rangle + \langle A^n y^{n+1} - A^n y^n, y_t^n\rangle \le \tau\langle f_t^n - A_t^n y^{n+1}, y_t^n\rangle + \tau^{-1}\Phi^n,$$

where $\Phi^n = (\phi(t^n, y^{n+1}) - \phi(t^{n+1}, y^{n+1})) - (\phi(t^n, y^n) - \phi(t^{n+1}, y^n))$. We again use the strong monotonicity of A^n and inequality (4) and obtain the relation

$$\|y_t^n\|_H^2 - \|y_t^{n-1}\|_H^2 + 2\alpha\tau\|y_t^n\|_V^2 \le 2\tau(\|f_t^n\|_{V^*} + \|A_t^n y^{n+1}\|_{V^*})\|y_t^n\|_V + 2\Phi^n/\tau.$$

By applying the ε-inequality with $\varepsilon = \alpha$ to the right-hand side and by summing the resulting inequalities with respect to n from 0 to $m < N$, we obtain

$$\|y_t^m\|_H^2 + \alpha\tau\sum_{n=0}^{m}\|y_t^n\|_V^2 \le \|y_t^{-1}\|_H^2 + \alpha^{-1}\tau\sum_{n=0}^{N-1}(\|f_t^n\|_{V^*}^2 + \|A_t^n y^{n+1}\|_{V^*}^2) + 2\tau^{-1}\sum_{n=0}^{N-1}\Phi^n = S_1 + S_2 + S_3.$$

Just as above, it follows from this estimate that

$$\max_{0\le m\le N-1}\|y_t^m\|^2 + \tau\sum_{n=0}^{N-1}\|y_t^n\|_V^2 \le cS^2, \qquad c = c(\alpha), \qquad S^2 = S_1 + S_2 + S_3.$$

[2] Here and throughout the following, various constants like c and C, possibly with subscripts, are independent of τ.

This implies an estimate for the quantity required in the statement of the lemma,

$$\|\hat{y}'_\tau\|_E \le cS. \tag{5}$$

Let us estimate the terms occurring in S^2. By Lemma 2, we have

$$S_1 = \tau^{-2}\|y^0 - u_0\|_H^2 \le cC_0^2. \tag{6}$$

To estimate S_2, we use Lemma 1. Obviously,

$$S_2 = \alpha^{-1}\int_0^T \left(\|\hat{f}'_\tau\|_{V^*}^2 + \|\hat{A}'_\tau y_\tau\|_{V^*}^2\right)dt = \alpha^{-1}\left(\|\hat{f}'_\tau\|_{\mathcal{V}^*}^2 + \|\hat{A}'_\tau y_\tau\|_{\mathcal{V}^*}^2\right) \le \alpha^{-1}\left(F^2 + M^2\|y_\tau\|_{L_\infty(0,T;V)}^2\right).$$

Since $\|y_\tau\|_{L_\infty(0,T;V)}^2 \le \|y^0\|_V^2 + 2\|\hat{y}'_\tau\|_{L_2(0,T;V)}\|y_\tau\|_{L_2(0,T;V)}$, it follows from Lemmas 2 and 3 that

$$\|y_\tau\|_{L_\infty(0,T;V)}^2 \le C(1 + \|\hat{y}'_\tau\|_E), \qquad C = C(T,\alpha,C_0,F,M). \tag{7}$$

Consequently,

$$S_2 \le C(1 + \|\hat{y}'_\tau\|_E). \tag{8}$$

Finally, let us estimate S_3,

$$S_3 = -2\tau^{-1}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} (\phi_t(t,y^{n+1}) - \phi_t(t,y^n))\,dt = -2\tau^{-1}\int_0^T (\phi_t(t,y_\tau(t)) - \phi_t(t,\check{y}_\tau(t)))\,dt$$
$$\le 2\tau^{-1}\varrho(\|y_\tau\|_{\mathcal{V}}, \|\check{y}_\tau\|_{\mathcal{V}})\|y_\tau - \check{y}_\tau\|_{\mathcal{V}} = 2\varrho(\|y_\tau\|_{\mathcal{V}}, \|\check{y}_\tau\|_{\mathcal{V}})\|\hat{y}'_\tau\|_{\mathcal{V}}.$$

Since the norms $\|y_\tau\|_{\mathcal{V}}$ and $\|\check{y}_\tau\|_{\mathcal{V}}$ are bounded uniformly with respect to τ by Lemma 3, we have

$$S_3 \le C\|\hat{y}'_\tau\|_E, \qquad C = C(T,\alpha,C_0,F,\varrho).$$

By summing this estimate with (6) and (8), we obtain the estimate $S^2 \le C(1 + \|\hat{y}'_\tau\|_E)$. By using it in (5), we obtain the assertion of the lemma.

Lemma 5. *The inequality*

$$\langle \hat{y}'_\tau + A(t)\hat{y}_\tau - f, v - \hat{y}_\tau\rangle + \phi(t,v) - \phi(t,\hat{y}_\tau) \ge -R_\tau(t,v) \tag{9}$$

holds for all $t \in [0,T]$ *and* $v \in D(\phi)$; *moreover, if* $v \in \mathcal{D}(\phi)$, *then there exists a constant* C *depending on* $\|v\|_{\mathcal{V}}$ *such that*

$$\int_0^T R_\tau(t,v(t))\,dt \le C(\tau^2 + \tau\|\hat{y}_\tau - v\|_{\mathcal{V}}).$$

Proof. Let $\sigma = \sigma(t)$, $t \in [-\tau,T)$, be the piecewise constant extension of the grid function t_n, $\sigma(t) = t_n$, $t \in [t_{n-1},t_n)$, $n = 0,1,\dots,N$. By writing out the grid inequality (2) in the index-free form, for all $t \in (-\tau,T)$, we have

$$\langle \hat{y}'_\tau + A_\tau(t)y_\tau - f_\tau, v - y_\tau\rangle + \phi(\sigma(t),v) - \phi(\sigma(t),y_\tau) \ge 0 \qquad \forall\, v \in D(\phi). \tag{10}$$

Since $\hat{y}_\tau$, $\check{y}_\tau$, and y_τ are related by the identities

$$\hat{y}_\tau(t) = (1 - \ell(t))\check{y}_\tau(t) + \ell(t)y_\tau(t), \qquad \hat{y}_\tau(t) - y_\tau(t) = (1 - \ell(t))(\check{y}_\tau(t) - y_\tau(t)),$$

it follows from the convexity of ϕ and inequality (10) that ($L = \langle A\hat{y}_\tau - A_\tau y_\tau, \hat{y}_\tau - v\rangle$)

$$\begin{aligned}
&\langle \hat{y}'_\tau + A\hat{y}_\tau - f_\tau, \hat{y}_\tau - v\rangle + \phi(\sigma, \hat{y}_\tau) - \phi(\sigma, v)\\
&\quad = \langle \hat{y}'_\tau + A_\tau y_\tau - f_\tau, \hat{y}_\tau - v\rangle + \phi(\sigma, (1-\ell)\check{y}_\tau + \ell y_\tau) - \phi(\sigma, v) + L\\
&\quad \le \langle \hat{y}'_\tau + A_\tau y_\tau - f_\tau, y_\tau - v\rangle + \langle \hat{y}'_\tau + A_\tau y_\tau - f_\tau, \hat{y}_\tau - y_\tau\rangle\\
&\qquad + (1-\ell)\phi(\sigma, \check{y}_\tau) + \ell\phi(\sigma, y_\tau) - \phi(\sigma, v) + L\\
&\quad \le (1-\ell)[\langle \hat{y}'_\tau + A_\tau y_\tau - f_\tau, \check{y}_\tau - y_\tau\rangle + \phi(\sigma, \check{y}_\tau) - \phi(\sigma, y_\tau)] + L.
\end{aligned} \tag{11}$$

By multiplying relation (10) for $t-\tau$ by $(1-\ell)$ and by setting $v = y$, we obtain the inequality

$$(1-\ell)[\langle \hat{y}'_\tau(t-\tau) + \check{A}_\tau \check{y}_\tau - \check{f}_\tau, y_\tau - \check{y}_\tau\rangle + \phi(\check{\sigma}, y_\tau) - \phi(\check{\sigma}, \check{y}_\tau)] \ge 0.$$

By adding this quantity to the right-hand side in (11) and by making simple transformations, we obtain the desired inequality (9) with

$$R_\tau(t, v) = r_1 + r_2 + r_3 + r_4 + r_5 + L,$$

where

$$\begin{aligned}
&r_1 = (1-\ell)\langle \hat{y}'_\tau - \hat{y}'_\tau(t-\tau), \check{y}_\tau - y_\tau\rangle,\\
&r_2 = (1-\ell)\langle \check{f}_\tau - f_\tau + A_\tau y_\tau - \check{A}_\tau \check{y}_\tau, \check{y}_\tau - y_\tau\rangle,\\
&r_3 = (1-\ell)(\phi(\check{\sigma}, y_\tau) - \phi(\sigma, y_\tau) - \phi(\check{\sigma}, \check{y}_\tau) + \phi(\sigma, \check{y}_\tau)),\\
&r_4 = \phi(\sigma(t), \hat{y}_\tau) - \phi(t, \hat{y}_\tau) - \phi(\sigma(t), v) + \phi(t, v), \qquad r_5 = \langle f - f_\tau, v - \hat{y}_\tau\rangle.
\end{aligned}$$

Let us estimate the integral of each term in $R_\tau(t, v)$. Note that the identity

$$\int_0^T (1-\ell) g_\tau\, dt = \frac{1}{2}\int_0^T g_\tau\, dt$$

holds for any piecewise constant function g_τ. Since $\hat{y}'_\tau(t) = (y_\tau(t) - \check{y}_\tau(t))/\tau$, it follows from inequality (4) and Lemma 2 that

$$\begin{aligned}
\int_0^T r_1\, dt &= \tau\int_0^T (1-\ell)\langle \hat{y}'_\tau(t-\tau) - \hat{y}'_\tau, \hat{y}'_\tau\rangle\, dt = \frac{\tau}{2}\int_0^T \langle \hat{y}'_\tau(t-\tau) - \hat{y}'_\tau, \hat{y}'_\tau\rangle\, dt\\
&\le \frac{\tau}{2}\int_0^T (\|\hat{y}'_\tau(t-\tau)\|_H^2 - \|\hat{y}'_\tau(t)\|_H^2)\, dt \le \frac{\tau}{2}\int_0^\tau \left\|\frac{y^0 - u_0}{\tau}\right\|_H^2 dt \le \frac{1}{2}C_0^2\tau^2.
\end{aligned}$$

By taking into account the strong monotonicity of $\check{A}_\tau$, the estimate (8) for the quantity S_2, and Lemma 4, we obtain the estimate

$$\begin{aligned}
\int_0^T r_2\, dt &= \tau\int_0^T (1-\ell)\langle f_\tau - \check{f}_\tau + \check{A}_\tau \check{y}_\tau - A_\tau y_\tau, \hat{y}'_\tau\rangle\, dt\\
&= \frac{\tau^2}{2}\int_0^T \langle \hat{f}'_\tau - \hat{A}'_\tau y_\tau, \hat{y}'_\tau\rangle\, dt - \frac{1}{2}\int_0^T \langle \check{A}_\tau y_\tau - \check{A}_\tau \check{y}_\tau, y_\tau - \check{y}_\tau\rangle\, dt\\
&\le \frac{\tau^2}{2}(\|\hat{f}'_\tau\|_{\mathcal{V}^*} + \|\hat{A}'_\tau y_\tau\|_{\mathcal{V}^*})\|\hat{y}'_\tau\|_{\mathcal{V}} \le c\tau^2.
\end{aligned}$$

The following estimates are similar to the estimate of S_3 in the proof of Lemma 4:

$$\int_0^T r_3\,dt = \frac{1}{2}\int_0^T\int_{\sigma(t)}^{\check\sigma(t)} (\phi_t(\xi, y_\tau) - \phi_t(\xi, \check y_\tau))\,d\xi\,dt = \frac{1}{2}\sum_{n=1}^N \int_{t_{n-1}}^{t_n}\int_{t_n}^{t_{n-1}} (\phi_t(\xi, y_\tau) - \phi_t(\xi, \check y_\tau))\,d\xi\,dt$$

$$= \frac{-\tau}{2}\int_0^T (\phi_t(\xi, y_\tau) - \phi_t(\xi, \check y_\tau))\,d\xi \le c\tau\|y_\tau - \check y_\tau\|_{\mathcal V} = c\tau^2\|\hat y'_\tau\|_{\mathcal V} \le c\tau^2,$$

$$\int_0^T r_4\,dt = \int_0^T\int_t^{\sigma(t)} (\phi_t(\xi, \hat y_\tau) - \phi_t(\xi, v))\,d\xi\,dt \le c(v)\tau\|\hat y_\tau - v\|_{\mathcal V},$$

where $c(v)$ is bounded on $\mathcal{D}(\phi)$. From Lemma 1, we have

$$\int_0^T r_5\,dt \le \|f - f_\tau\|_{\mathcal V^*}\|v - \hat y_\tau\|_{\mathcal V} \le c\tau\|\hat y_\tau - v\|_{\mathcal V}.$$

Finally, by taking into account Lemma 1 and inequality (7), we obtain the estimate

$$\int_0^T L\,dt \le \int_0^T \|A\hat y_\tau - A_\tau y_\tau\|_{V^*}\|\hat y_\tau - v\|_V\,dt \le \int_0^T (\|A\hat y_\tau - Ay_\tau\|_{V^*} + \|(A - A_\tau)y_\tau\|_{V^*})\|\hat y_\tau - v\|_{\mathcal V}\,dt$$

$$\le (\tau M\|\hat y'_\tau\|_{\mathcal V^*} + M\tau\|y_\tau\|_{L_\infty(0,T;\mathcal V)})\|\hat y_\tau - v\|_{\mathcal V} \le c\tau\|\hat y_\tau - v\|_{\mathcal V},$$

because $\hat y_\tau - y_\tau = \tau(\ell - 1)\hat y'_\tau$. By summing this estimate with the estimates for the integrals of the remaining terms in $R_\tau(t, v)$, we arrive at the definitive conclusion of the lemma.

4. EXISTENCE OF A SOLUTION. ERROR ESTIMATE

Lemma 6. *Let $\hat y_\tau$ and $\hat y_k$ be the solutions of the scheme* (2) *with time increments τ and $k = T/M$, respectively. Then $\|\hat y_\tau - \hat y_k\|_E \le c(\tau + k)$, where c is a constant independent of τ and k.*

Proof. By Lemma 5, the inequality

$$\langle \hat y'_k + A\hat y_k - f, v - \hat y_k\rangle + \phi(t, v) - \phi(t, \hat y_k) \ge -R_k(t, v) \tag{12}$$

holds for $t \in [0, T]$ and $v \in D(\phi)$; moreover, if $v \in \mathcal{D}(\phi)$, then $\int_0^T R_k(t, v(t))\,dt \le C(k^2 + k\|\hat y_k - v\|_{\mathcal V})$, $C = C(\|v\|_{\mathcal V})$.

By setting $v = \hat y_k$ in inequality (9) and $v = \hat y_\tau$ in (12) and by adding the resulting inequalities, we obtain $\langle \hat y'_k - \hat y'_\tau + A\hat y_k - A\hat y_\tau, \hat y_k - \hat y_\tau\rangle \le R_k(t, \hat y_\tau) + R_\tau(t, \hat y_k)$. Hence we have the inequality

$$\frac{d}{dt}\|\hat y_k - \hat y_\tau\|_H^2 + 2\alpha\|\hat y_k - \hat y_\tau\|_V^2 \le 2(R_k(t, \hat y_\tau) + R_\tau(t, \hat y_k)).$$

We integrate this inequality with respect to $t \in (0, s)$, $s \le T$, set $s = T$ on the right-hand side, use the estimates for the integrals of R_k and R_τ, and take into account the relation

$$\|\hat y_k(0) - \hat y_\tau(0)\|_H \le \|\hat y_\tau(0) - u_0\|_H + \|\hat y_k(0) - u_0\|_H \le c(k + \tau).$$

As a result, we obtain the inequality

$$\|(\hat y_k - \hat y_\tau)(s)\|_H^2 + 2\alpha\int_0^s \|\hat y_k - \hat y_\tau\|_V^2\,dt \le \|\hat y_k(0) - \hat y_\tau(0)\|_H^2 + c(\tau^2 + k^2) + c(\tau + k)\|\hat y_k - \hat y_\tau\|_{\mathcal V}$$

$$\le c(\tau + k)^2 + c(\tau + k)\|\hat y_k - \hat y_\tau\|_{\mathcal V}.$$

This implies the assertion of the lemma.

Theorem 1. *The following assertions hold.*

(a) *There exists a unique solution u of problem* (1). *It satisfies the inclusion $u \in H^1(0,T;V) \cap W^1_\infty(0,T;H)$.*

(b) *Let u and $\bar{u}$ be the solutions of problem* (1) *with input data $\{u_0, f\}$ and $\{\bar{u}_0, \bar{f}\}$, respectively*; *then $\|\bar{u} - u\|_E \le c(\|\bar{u}_0 - u_0\|_H + \|\bar{f} - f\|_{\mathcal{V}^*})$.*

(c) *If $\hat{y}_\tau$ is the piecewise linear extension of the solution of the implicit scheme* (2), *then $\|u - \hat{y}_\tau\|_E \le c\tau$.*

Proof. We start the proof from assertion (b). Set $v = \bar{u}(t)$ and $v = u(t)$ in inequality (1) defining u and $\bar{u}$, respectively. By adding the resulting inequalities, we obtain

$$\langle (\bar{u} - u)', \bar{u} - u\rangle + \langle A\bar{u} - Au, \bar{u} - u\rangle \le \langle \bar{f} - f, \bar{u} - u\rangle \le \|\bar{f} - f\|_{V^*}\|\bar{u} - u\|_V.$$

This implies the estimate

$$\frac{d}{dt}\|\bar{u} - u\|_H^2 + \alpha\|\bar{u} - u\|_V^2 \le \alpha^{-1}\|\bar{f} - f\|_{V^*}^2,$$

whence, in turn, we obtain the stability estimate in assertion (b); note that it guarantees the uniqueness of the solution.

To prove assertion (a), we arbitrarily fix $v \in \mathcal{D}(\phi)$. By integrating inequality (9), we obtain the relation

$$\int_0^T (\langle \hat{y}'_\tau + A\hat{y}_\tau - f, v - \hat{y}_\tau\rangle + \phi(t, v) - \phi(t, \hat{y}_\tau))\, dt + \varepsilon_\tau \ge 0, \qquad \varepsilon_\tau \le c(\tau^2 + \tau\|\hat{y}_\tau - v\|_{\mathcal{V}}). \tag{13}$$

From the a priori estimates, we have

$$\hat{y}_\tau, \hat{y}'_\tau \in E = L_2(0,T;V) \cap L_\infty(0,T;H);$$

moreover, $\|\hat{y}_\tau\|_E + \|\hat{y}'_\tau\|_E \le C$ uniformly with respect to τ. It follows from Lemma 6 that the sequence $\{\hat{y}_\tau,\ \tau = T/N,\ N = 1, 2, \ldots\}$ is a Cauchy sequence in E. Therefore, it strongly converges to some element $u \in E$; in a standard way, one can show that $\hat{y}'_\tau$ weakly converges to u' in $L_2(0,T;V)$, and $\|u\|_E + \|u'\|_E \le C$. By virtue of the lower semicontinuity of the functional $v \to \phi(t, v)$, $t \in [0,T]$, we have

$$\phi(t, u) \le \liminf_{\tau\to 0} \phi(t, \hat{y}_\tau);$$

i.e., $u \in D(\phi)$. Therefore, by taking into account the continuity of the operators $A(t)$, $t \in [0,T]$, and by passing to the limit in (13), we obtain the inequality

$$\int_0^T (\langle u'(t) + A(t)u(t) - f(t), v(t) - u(t)\rangle + \phi(t, v(t)) - \phi(t, u(t)))\, dt \ge 0.$$

This implies inequality (1), because v is arbitrary. Since the estimate

$$\|\hat{y}_\tau(0) - u_0\|_H \le C_0\tau$$

holds by Lemma 2, it follows that $u(0) = u_0$ and u is a solution of problem (1).

To prove assertion (c), it suffices to pass to the limit as $k \to 0$ in the estimate in Lemma 6.

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REFERENCES

1. Savaré, G., Weak Solutions and Maximal Regularity for Abstract Evolution Inequalities, *Adv. Math. Sci. Appl.*, 1996, no. 6, pp. 377–418.
2. Rulla, J., Error Analysis for Implicit Approximations to Solutions to Cauchy Problems, *SIAM J. Numer. Anal.*, 1996, vol. 33, no. 1, pp. 68–87.
3. Gajewski, H., Gröger, K., and Zacharias, K., *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Berlin: Akademie–Verlag, 1974. Translated under the title *Nelineinye operatornye uravneniya i operatornye differentsial'nye uravneniya*, Moscow: Mir, 1978.
4. Lions, J.-L., *Quelques méthodes de résolution des problèmes aux limites nonlinéaires*, Paris: Dunod, 1969. Translated under the title *Nekotorye metody resheniya nelineinykh kraevykh zadach*, Moscow: Mir, 1972.