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AFFILIATED SUBSPACES AND THE STRUCTURE OF VON NEUMANN ALGEBRAS

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ABSTRACT. The interplay between order-theoretic properties of structures of subspaces affiliated with a von Neumann algebra M and inner structure of the algebra M is studied. The following characterization of finiteness is given: von Neumann algebra M is finite if and only if in each representation space of M on has that closed affiliated subspaces are given precisely by strongly closed left ideals in M. Moreover, it is shown that if the modular operator of a faithful normal state φ is bounded, then all important classes of affiliated subspaces in the GNS representation space of φ coincide. Orthogonally closed affiliated subspaces are characterized in terms of the supports of normal functionals. It is proved that complete affiliated subspaces correspond to left ideals generated by finite sums of orthogonal atomic projections.

KEYWORDS: *Subspaces affiliated with a von Neuman algebra, states and weights on von Neumann algebras, modular theory.*

MSC (2000): 46L10, 46L30, 46K15, 46C05.

INTRODUCTION AND PRELIMINARIES

The set $L(H)$ of all closed subspaces of a Hilbert space H ordered by set inclusion and endowed with orthogonal complement constitutes one of the most important examples of a complete orthomodular lattice. From the point of view of the lattice theory the lattice $L(H)$ has many specific properties. It is irreducible, atomic, and usually not modular. In order to obtain more general subspace lattices interesting from both mathematical and physical point of view one can select only distinguished subspaces of H. A natural way how to do it is to replace the lattice $L(H)$ by its sublattice consisting of those closed subspaces of H for which the corresponding projections belong to a given von Neumann algebra M acting on H. (The lattice $L(H)$ can be then recovered as a special case by taking M to be the algebra of all bounded operators on H.) In other words, one can choose only subspaces which are invariant under operators from the commutant M' of M. This "selection procedure" leads to lattices which may be atomless, modular, and