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Determination of the Propagation Constants of Dielectric-Waveguide Eigenmodes by Methods of Potential Theory

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Abstract—The problem of determining the propagation constants of dielectric-waveguide eigenmodes is considered. A set of singular integral equations is obtained by methods of potential theory. Conditions that are sufficient for the nontrivial solutions of this set to correspond to nontrivial solutions of the problem considered are presented. A method for calculating approximate values of the propagation constants is proposed.

This paper is devoted to determining the propagation constants of eigenmodes of dielectric waveguides by methods of potential theory. The ray method, the method of normal waves, asymptotic methods [1, 2], the finite-difference method [3], the method of integral equations [4–6], the method of partial domains, variational methods, etc. [7–9] have been applied to the analysis of dielectric waveguides. Integral equations in [4–6] were constructed on the basis of Green's identity. In recent years, the representation of fields in the form of single-layer potentials has been used successfully in solving spectral problems of wave scattering by open screens and the problems of diffraction of electromagnetic waves [10–12]. Such an approach allows one to economize considerably CPU time.

In this paper, we obtain a homogeneous set of real singular integral equations by representing the desired functions in the form of single-layer potentials. We present sufficient conditions for the nontrivial solutions of the set of integral equations to correspond to nontrivial solutions of the original problem. We propose a method for calculating approximate values of propagation constants. As test examples, we solved the problems for waveguides of elliptical, rectangular, and triangular cross-sections, as well as for dielectric strip waveguides.

1. The problem of determining the propagation constants of eigenmodes of a cylindrical dielectric waveguide with a constant refractive index n_1, surrounded by a medium with a constant refractive index n_2, is reduced (see, e.g., [7]) to the determination of the values of the parameter β from the interval $G = (k_0n_2, k_0n_1)$ such that there exist nontrivial exponentially decreasing solutions of the set

$$\Delta u + \chi^2 u = 0,\ \Delta v + \chi^2 v = 0,\ (x, y) \in S,\ \Delta u - \rho^2 u = 0,\ \Delta v - \rho^2 v = 0,\ (x, y) \notin \bar{S}, \tag{1}$$

that satisfy the boundary conditions

$$\begin{gathered} u^+ - u^- = 0,\ v^+ - v^- = 0, \\ \frac{1}{\chi^2}\left(\beta\frac{\partial v}{\partial \tau} + \varepsilon_1\omega\frac{\partial u^+}{\partial \nu}\right) + \frac{1}{\rho^2}\left(\beta\frac{\partial v}{\partial \tau} + \varepsilon_2\omega\frac{\partial u^-}{\partial \nu}\right) = 0, \\ \frac{1}{\chi^2}\left(\beta\frac{\partial u}{\partial \tau} - \mu_0\omega\frac{\partial v^+}{\partial \nu}\right) + \frac{1}{\rho^2}\left(\beta\frac{\partial u}{\partial \tau} - \mu_0\omega\frac{\partial v^-}{\partial \nu}\right) = 0,\ (x, y) \in C. \end{gathered} \tag{2}$$

Here, S is a bounded domain with the boundary C; $\chi^2 = k_0^2n_1^2 - \beta^2$; $\rho^2 = \beta^2 - k_0^2n_2^2$; $k_0^2 = \omega^2\varepsilon_0\mu_0$; ε_0 and μ_0 are the permittivity and the permeability of vacuum, respectively; ω is the frequency of electromagnetic oscillations; $\varepsilon_j = \varepsilon_0 n_j^2$; $\partial u/\partial\nu$ is the derivative normal to the contour C; $\partial u/\partial\tau$ is the derivative tangential to the contour C; and f^+ (f^-) is the limit value of the function f from the interior (exterior) of the contour C.

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Using the representation of the functions u and v in the form of single-layer potentials [13], we obtain a nonlinear spectral problem for the following set of real integral equations:

$$T_1\varphi_1 - T_2\varphi_2 = 0,\ T_1\psi_1 - T_2\psi_2 = 0,$$
$$\frac{1}{\chi^2}\left[\beta K_1\psi_1 + \varepsilon_1\omega\left(\frac{1}{2}\varphi_1 + P_1\varphi_1\right)\right] + \frac{1}{\rho^2}\left[\beta K_2\psi_2 + \varepsilon_2\omega\left(-\frac{1}{2}\varphi_2 + P_2\varphi_2\right)\right] = 0, \tag{3}$$
$$\frac{1}{\chi^2}\left[\beta K_1\varphi_1 - \mu_0\omega\left(\frac{1}{2}\psi_1 + P_1\psi_1\right)\right] + \frac{1}{\rho^2}\left[\beta K_2\varphi_2 - \mu_0\omega\left(-\frac{1}{2}\psi_2 + P_2\psi_2\right)\right] = 0,\ M \in C.$$

Here,[1]

$$(T_j\varphi)(M) = \int_C \Phi_j(M, M_0)\varphi(M_0)dC_{M_0}, \quad (P_j\varphi)(M) = \int_C \frac{\partial}{\partial\nu_M}\Phi_j(M, M_0)\varphi(M_0)dC_{M_0},$$
$$(K_j\varphi)(M) = \int_C \frac{\partial}{\partial\tau_M}\Phi_j(M, M_0)\varphi(M_0)dC_{M_0},\ M \in C,\ j = 1, 2,$$
$$\Phi_1(M, M_0) = -\frac{1}{4}N_0(\chi r_{MM_0}),\ \Phi_2(M, M_0) = \frac{1}{2\pi}K_0(\rho r_{MM_0}),$$
$$M = (x, y),\ M_0 = (x_0, y_0),\ r_{MM_0} = \sqrt{(x-x_0)^2 + (y-y_0)^2}.$$

The operators T_j have logarithmic singularities, the kernels of the operators P_j are continuous, and K_j are singular integral operators with a Cauchy kernel.

Let us analyze the relation between the solutions to problems (1), (2), and (3). The proof of the following lemma is similar to the proof of Theorem 2 in [15]:

Lemma 1. *If, for a certain* $\beta = \beta_0 \in G$, *the potential* u *given by the relation*

$$u(M) = \int_C \Phi_2(M, M_0)\varphi(M_0)dC_{M_0},\ M \in \mathbb{R}^2\backslash\bar{S}$$

vanishes in $\mathbb{R}^2\backslash\bar{S}$, *then its density* $\varphi = 0$ *on* C.

Lemma 2. *If, for a certain* $\beta = \beta_0 \in G$, *the potential* u *given by the relation*

$$u(M) = \int_C \Phi_1(M, M_0)\varphi(M_0)dC_{M_0},\ M \in S$$

vanishes in S *and the problem*

$$T_1\varphi_1 = 0,\ M \in C \tag{4}$$

has only a trivial solution, then the density $\varphi = 0$ *on* C.

The validity of this lemma follows from the continuity of the single-layer potential with the kernel Φ_1. Lemmas 1 and 2 imply the following theorem.

Theorem 1. *If, for a certain* $\beta = \beta_0 \in G$, *problem* (4) *has only a trivial solution and system* (3) *has a nontrivial solution, then the latter corresponds to a nontrivial solution of problem* (1), (2).

2. Following [10, 12], in order to solve the problem numerically, we replace (3) by a set of linear algebraic equations by approximating the integral operators by the Galerkin method. As basis functions, we use the trigonometric functions. Integrating by parts, we preliminarily reduce the integral operators with the

[1] We use the notation from [14] for special functions.

Cauchy kernel to integro-differential operators with a logarithmic singularity. The logarithmic singularities of the kernels are separated analytically. Approximate values of β are determined from the conditions

$$\det(A(\beta)) = 0, \tag{5a}$$

$$\det(D(\beta)) \neq 0, \tag{5b}$$

where A is the matrix of the set of equations constructed and D is the matrix that arises upon discretizing the operator T_1.

A similar numerical method for solving a nonlinear spectral problem for a scalar integral operator function with a logarithmic singularity was substantiated in [16, p. 168]. This method is based essentially on the results of [17].

3. In order to assess the efficiency of the method described, we solved problem (1), (2) for waveguides of elliptic, rectangular, and triangular cross-sections, as well as for dielectric strip waveguides.

The dispersion characteristics of the fundamental modes of an elliptic cross-section waveguide with the ratio of semiaxes equal to 1.31 were constructed by the method proposed. It was found that even for the number of basis functions $N = 2$ they coincided exactly, up to a graphical representation, with the dispersion characteristics [18] obtained by the method of separation of variables. A further increase in N did not improve the computational accuracy.

The solution of problem (1), (2) for a waveguide of rectangular cross-section was based on the approximation of the contour by the curve [4]

$$r(t) = \left[\left(\frac{\cos t}{a}\right)^{2M} + \left(\frac{\sin t}{b}\right)^{2M}\right]^{-1/2M}, \quad t \in [0, 2\pi].$$

As $M \longrightarrow \infty$, this curve tends to a rectangle with sides $2a$ and $2b$.

As in [19], we obtained the dispersion characteristics: the dependence of $h = \beta/k_0$ on $p = 2b/\lambda$, $\lambda = 2\pi/\omega$, for fixed values of ε_1, ε_2, and a/b. The results of computations for $a/b = 1.5$, $\varepsilon_1 = 2.08$, and $\varepsilon_2 = 1$ are shown in Fig. 1 by a solid curve for the fundamental modes and by a dashed curve for higher-order modes. The circles indicate the experimental data of [19]. The results demonstrated in Fig. 1 were obtained for $N = 3$. The method exhibits a stable internal convergence. For instance, the modulus of the difference between the values of h obtained for $N = K$ and $N = K + 1$ did not exceed $\Delta \approx 10^{-2}$ for $K = 2$, $\Delta \approx 10^{-4}$ for $K = 3$, and $\Delta \approx 10^{-6}$ for $K = 4$. All computations were carried out for $M = 20$. A further increase in M did not influence the accuracy of the computations.

We also considered problem (1), (2) for a waveguide with a cross-section in the form of an equilateral triangle. The contour was approximated by a curvilinear triangle [12]. The results of calculations were compared with those obtained by the pointwise matching method in [20]. The accuracy of the calculated prop-

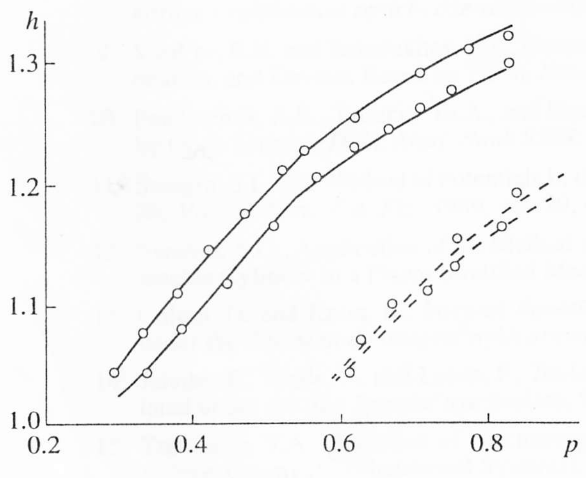

Fig. 1.

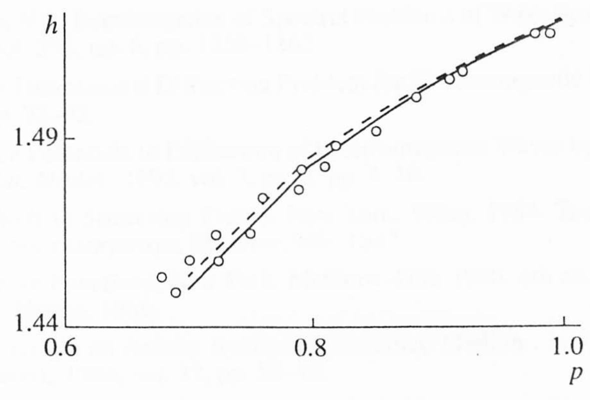

Fig. 2.

agation constants of the fundamental modes of a triangular cross-section waveguide and the internal convergence of the method were of the same order as in the previous case.

We also solved problem (1), (2) for dielectric strip waveguides [19], i.e., waveguides of rectangular cross-section that are situated on a dielectric substrate or embedded into the letter. Similarly to [21], the solution was based on the approximate separation of the fundamental modes into electric and magnetic modes. In this case, set (3) is decomposed into two independent sets similar to those described in [12]. As the kernel Φ_2, we used the Green's function of the Helmholtz equation with piecewise-constant coefficients [22]. The results of calculations for a waveguide situated on a substrate (for $\varepsilon_1 = 2.52$, $\varepsilon_2 = 1$, $\varepsilon_3 = 2.085$, and $a/b = 1.5$) are represented in Fig. 2 by a solid curve for the fundamental E-polarized waves and by a dashed curve for H-polarized waves. Here, ε_1, ε_2, and ε_3 are the dielectric permittivities of the cylinder, the surrounding medium, and the substrate, respectively. The circles represent the experimental data of [19]. The results shown in Fig. 2 were obtained for $N = 2$. The accuracy of the method depends on the parameters N and M, just as in the case of a homogeneous surrounding medium.

Note that, in order to verify the fulfillment of condition (5b), we determined all roots of the equation $\det(D(\beta)) = 0$ in each case and compared these roots with the solutions of equation (5a).

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