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Ando's inequality for uniform submajorization [☆]

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We discuss Ando's inequality involving operator monotone functions and demonstrate that a natural strengthening of Ando's result to uniform submajorization holds if and only if the operator monotone function has a very simple rational form.

1. Introduction

Recall that a function $f : [0, \infty) \to \mathbb{R}$ is called operator monotone if $0 \leq A \leq B \in B(\mathcal{H})$ implies that $f(A) \leq f(B)$, where $B(\mathcal{H})$ stands for the $*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. The present paper is motivated by a result due to

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T. Ando [2] (see also [5] and [17, Chapter 43]), which states that for any non-negative operator monotone function f and any (fully symmetric) unitarily invariant norm $\|\cdot\|_E$ on $B(\mathcal{H})$, we have

$$\|f(X)-f(Y)\|_E \le \|f(|X-Y|)\|_E\,,\ X,Y\in B(\mathcal{H})_+.$$

Recall that given two positive operators A and B, we say that A is submajorized by B (in the sense of Hardy, Littlewood and Pólya) if

$$\sum_{k=0}^{n}\mu(k,A)\le\sum_{k=0}^{n}\mu(k,B),\quad n\ge 0.$$

Here, μ is the singular value function. Submajorization is denoted $A \prec\prec B$.

A similar result to that of Ando can be stated in terms of submajorization. If f is operator monotone, then:

$$|f(X)-f(Y)| \prec\prec f(|X-Y|).$$

In this form, it follows that $\|f(X)-f(Y)\|_E \le \|f(|X-Y|)\|_E$ for all fully symmetric norms $\|\cdot\|_E$, although not for all unitarily invariant norms.

It is natural to consider whether this form of Ando's inequality can be strengthened to *uniform* submajorization, a notion introduced in [11] (see also [12]). Given two positive operators A and B, one says that A is uniformly submajorized by B (written $A\lhd B$) if there exists $n\in\mathbb{N}$ such that:

$$\sum_{k=na}^{b}\mu(k,A)\le\sum_{k=a}^{b}\mu(k,B),\ \text{ for all } a,b\ge 0 \text{ such that } na<b.$$

Uniform submajorization is much stronger than submajorization, and $A\lhd B$ implies (in particular, and in contrast to submajorization) that if B is finite rank then A is finite rank (see Lemma 2.2).

It follows from $A\lhd B$ that $\|A\|_E\le\|B\|_E$ for *all* symmetric norms E. This is the reason for the importance of uniform submajorization, and should be compared with the fact that $A\prec\prec B$ implies that $\|A\|_E\le\|B\|_E$ only for those norms E which are fully symmetric.

It is natural to ask if Ando's inequality can be extended in at least some cases to uniform submajorization. Say that f satisfies Ando's inequality for uniform submajorization if for all bounded positive linear operators X and Y we have

$$|f(X)-f(Y)|\lhd f(|X-Y|).$$

It can be proved (see Theorem 4.5 below) that f obeys Ando's inequality for uniform submajorization if f has the form

$$f(t) = \alpha + \beta t + \sum_{j=1}^{d-1} \frac{\gamma_j t}{t + \delta_j} \tag{1.1}$$

for some $\alpha \in \mathbb{R}$, $\beta, \gamma_j \geq 0$ and $\delta_j > 0$, $j = 1, \ldots, n$ and $d \geq 1$. We show that this is the only case. That is, an operator monotone function f satisfies Ando's inequality for uniform submajorization if and only if f has the form (1.1).

Our proofs are based on the following observation:

Theorem 1.1. *Let f be a Borel function on $\mathbb{R}$. If for all bounded positive operators X and Y such that $X - Y$ has finite rank, the difference $f(X) - f(Y)$ also has finite rank, then f is rational.*

Our proof of Theorem 1.1 is ultimately based on the theory of Hankel operators. Using a recent characterization of rational operator monotone functions due to Nagisa [13], we deduce the following:

Theorem 1.2. *An operator monotone function f obeys Ando's inequality for uniform submajorization if and only if f has the form (1.1).*

The same result holds for uniform submajorization in the setting of an infinite factor.

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2. Preliminaries

We will present our results and proofs in the language of semifinite von Neumann algebras and singular value functions. This has the advantage of some additional generality, but also simplifies the notation in some places and clarifies some computations.

2.1. Singular value functions

Let $\mathcal{M}$ be a semifinite von Neumann algebra on a separable Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ. We denote the norm of $\mathcal{M}$ simply by $\|\cdot\|$.

Let $\mathcal{P}(\mathcal{M})$ denote the lattice of all projections in $\mathcal{M}$, 1 be the unit of $\mathcal{M}$. A linear operator $X : \mathfrak{D}(X) \to \mathcal{H}$, where the domain $\mathfrak{D}(X)$ of X is a linear subspace of $\mathcal{H}$, is said to be *affiliated* with $\mathcal{M}$ if $YX \subseteq XY$ for every $Y \in \mathcal{M}'$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ (notation: $X\eta\mathcal{M}$). For any self-adjoint operator A on $\mathcal{H}$, its spectral measure is denoted by E_A. A self-adjoint operator A is affiliated with $\mathcal{M}$ if and only if $E_A(B) \in \mathcal{P}(\mathcal{M})$ for any Borel set $B \subseteq \mathbb{R}$. A closed and densely defined operator $A\eta\mathcal{M}$ is called τ-measurable if $\tau(E_{|A|}(s,\infty)) < \infty$ for sufficiently large s, where $|A| = \sqrt{A^*A}$. We denote the set of all τ-measurable operators by $S(\mathcal{M},\tau)$. For every $A \in S(\mathcal{M},\tau)$, we define its singular value function $\mu(A)$ by setting

$$\mu(t,A) = \inf\{\|A(1-P)\| : P \in \mathcal{P}(\mathcal{M}), \quad \tau(P) \le t\}, \quad t > 0.$$

Equivalently, for positive self-adjoint operators $A \in S(\mathcal{M},\tau)$, we have

$$n_A(s) = \tau(E_A(s,\infty)), \quad \mu(t,A) = \inf\{s : n_A(s) < t\}, \quad t > 0.$$

If $\mathcal{M} = B(\mathcal{H})$ and τ is the standard trace Tr, then it is not difficult to see that $S(\mathcal{M}) = S(\mathcal{M},\tau) = \mathcal{M}$ and

$$\mu(n,A) = \mu(t,A),\ \ t \in [n,n+1),\ n \in \mathbb{N}.$$

The sequence $\{\mu(n,A)\}_{n\ge 0}$ is just the sequence of singular values of the operator $A \in B(\mathcal{H})$. If we consider $\mathcal{M} = L_\infty(X,m)$ for some σ-finite measure space (X,m) as an Abelian von Neumann algebra acting via multiplication on the Hilbert space $L_2(X,m)$, with the trace given by integration with respect to m, then $S(\mathcal{M},\tau)$ consists of all measurable functions on X which are bounded except on a set of finite measure. In this case for $f \in S(\mathcal{M},\tau)$, the generalized singular value function $\mu(f)$ is precisely the classical decreasing rearrangement of the absolute value $|f|$.

We record for our use the inequality:

$$\mu(t+s, A+B) \le \mu(t,A) + \mu(s,B), \quad t,s \ge 0, A,B \in S(\mathcal{M},\tau). \tag{2.1}$$

See [12, Corollary 2.3.16].

For more details on generalized singular value functions, we refer the reader to [9] and [12].

2.2. *Hardy–Littlewood–Pólya submajorization*

If $A,B \in S(\mathcal{M},\tau)$, then we say that B is submajorized by A (in the sense of Hardy–Littlewood–Pólya), denoted by $\mu(B) \prec\prec \mu(A)$ or simply $B \prec\prec A$ if

$$\int_0^t \mu(s,B)ds \le \int_0^t \mu(s,A)ds, \quad t \ge 0.$$

Ando's inequality for submajorization holds in the semifinite setting (see [5]). That is, if f is an operator monotone function on $[0,\infty)$, then for all $0 \le A,B \in S(\mathcal{M},\tau)$,

$$\int_0^t \mu(s, f(A) - f(B))\, ds \le \int_0^t \mu(s, f(|A-B|))\, ds \text{ for all } t \ge 0. \tag{2.2}$$

2.3. Uniform submajorization

As mentioned in the introduction, *uniform submajorization* is a strengthening of submajorization in the sense of Hardy, Littlewood and Pólya. First introduced in [11], the notion was used extensively in [12] (see Section 3.4 there).

As indicated in the introduction, uniform submajorization is defined as follows:

Definition 2.1. Let $A, B \in S(\mathcal{M}, \tau)$. We say that A is uniformly submajorized by B (written $A \lhd B$) if there exists $n \geq 1$ such that

$$\int_{na}^{b} \mu(s, A) ds \leq \int_{a}^{b} \mu(s, B) ds, \quad \text{for all } a, b \geq 0 \text{ such that } na < b.$$

Note that in the case when $\mathcal{M} = B(\mathcal{H})$ and τ is the canonical trace, this reduces to the definition of uniform submajorization given in the introduction. Clearly, if $\mu(A) \leq \mu(B)$, then $A \lhd B$ with $n = 1$.

Recall that the support projection $\mathrm{supp}(A)$ of an element $A \in S(\mathcal{M}, \tau)$ is defined as the maximal projection $p \in \mathcal{P}(\mathcal{M})$ such that $A(1 - p) = 0$.

Lemma 2.2. *If* $0 \leq A, B \in S(\mathcal{M}, \tau)$ *with* $A \lhd B$ *and* $\tau(\mathrm{supp}(B)) < \infty$*, then*

$$\tau(\mathrm{supp}(A)) \leq n \cdot \tau(\mathrm{supp}(B)) < \infty$$

where n *is the integer appearing in the definition of uniform submajorization.*

Proof. If $\lambda = \tau(\mathrm{supp}(B)) < \infty$, then $\mu(t, B) = 0$ for $t > \lambda$. It follows from the definition of uniform submajorization that for N sufficiently large we have:

$$\int_{n\lambda}^{N} \mu(s, A)\, ds \leq \int_{\lambda}^{N} \mu(s, B)\, ds = 0.$$

Since μ is positive and decreasing, it follows that $\mu(s, A) = 0$ for all $s > n\lambda$. Therefore,

$$\tau(\mathrm{supp}(A)) \leq n \cdot \tau(\mathrm{supp}(B)). \quad \square$$

In the case $\mathcal{M} = B(\mathcal{H})$ and τ is the classical trace, we obtain the following:

Corollary 2.3. *If* $0 \leq A, B \in B(\mathcal{H})$ *and* $A \lhd B$ *and* B *has finite rank, then* A *has finite rank.*

3. Proof of Theorem 1.1

We work on the Hilbert space $L_2(\mathbb{R})$. Let H be the Hilbert transform, defined via Fourier multiplication:

$$H\xi(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{sgn}(\omega) e^{i\omega(t-s)} \xi(s)\, ds d\omega$$

when ξ is a Schwartz class function on $\mathbb{R}$ [10, Section 4.1]. Here, $\operatorname{sgn}(\omega) = 1$ if $\omega \geq 0$ and $\operatorname{sgn}(\omega) = -1$ if $\omega < 0$. Parseval's theorem implies that the Hilbert transform uniquely extends to a unitary and self-adjoint operator on $L_2(\mathbb{R})$, so in particular we have $H^2 = 1$, where 1 is the identity operator on $L_2(\mathbb{R})$.

The Hilbert transform H' on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ is defined similarly, as the Fourier multiplication:

$$H'\xi(z) = \sum_{n \in \mathbb{T}} \operatorname{sgn}(n) z^n \widehat{\xi}(n)$$

where $\widehat{\xi}(n)$ is the nth Fourier coefficient of $\xi \in C^{\infty}(\mathbb{T})$. The operators H and H' are unitarily equivalent, with the unitary equivalence being given by the Cayley transform. To be precise, if $\omega(t) = \frac{t-i}{t+i}$ is the Cayley transform, then the operator:

$$U\xi(t) = \frac{1}{\sqrt{\pi}} \frac{\xi(\omega^{-1}(t))}{t+i}$$

is a unitary equivalence between $L_2(\mathbb{T})$ (where $\mathbb{T}$ is equipped with its normalized Haar measure) and $L_2(\mathbb{R})$ such that $HU = UH'$. This follows from the Paley–Wiener theorem [14, Theorem 19.2], which characterizes the images of $2H + 1$ and $2H' + 1$ as being the boundary values of holomorphic functions in the unit disc and the upper half-plane respectively, and the fact that ω is a conformal equivalence between the upper half-plane and the interior of the unit disc.

Given a bounded function $h \in L_{\infty}(\mathbb{R})$, let M_h be the operator of pointwise multiplication by h on $L_2(\mathbb{R})$, $M_h\xi(t) = h(t)\xi(t)$ for $\xi \in L_2(\mathbb{R})$ and $t \in \mathbb{R}$.

The following result concerning commutators of the form $[H, M_h]$ is essential to our approach. We explain how it can be deduced from a characterization of finite rank Hankel matrices, originally due to L. Kronecker (see [15, Section 1.1.3])

Proposition 3.1. *Let h be a bounded function on the real line. Then the commutator*

$$[H, M_h] = HM_h - M_hH : L_2(\mathbb{R}) \to L_2(\mathbb{R})$$

has finite rank if and only if h is a rational function.

Proof. If $h \in L_\infty(\mathbb{R})$, then $U^* M_h U$ is the function of pointwise multiplication by $h \circ \omega^{-1}$ on $\mathbb{T}$.

As discussed above, we have $HU = UH'$, where H' is the Hilbert transform on the unit circle. Therefore:

$$U^*[H, M_h]U = [H', M_{h\circ\omega^{-1}}].$$

Necessary and sufficient criteria for a commutator of a pointwise multiplier and the Hilbert transform on the circle to be finite rank are known (see [15, Section 1.3]). In the Fourier basis, the commutator $[H', M_{h\circ\omega^{-1}}]$ is unitarily equivalent to a direct sum of Hankel matrices. It follows from a theorem of Kronecker [15, Theorem 1.3.1] that the commutator $[H', M_{h\circ\omega^{-1}}]$ has finite rank if and only if $h \circ \omega^{-1}$ is rational. Thus, $[H, M_h]$ has finite rank if and only if h is rational. □

Lemma 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function. If $f \circ h$ is rational for every bounded real-valued rational function h, then f is rational.*

Proof. Let $h_1(t) = \frac{1}{1+t^2}$: this is a bounded rational function. Then by assumption, $g = f \circ h_1$ is rational. If $s \in (0, 1)$, then there exists $t = \sqrt{s^{-1} - 1}$ such that $h_1(s) = t$. We have

$$f(s) = f(h_1(t)) = g(t) = g(\sqrt{s^{-1} - 1}), \quad s \in (0, 1).$$

Set

$$g_1(u) = \frac{g(u) + g(-u)}{2}, \quad g_2(u) = \frac{g(u) - g(-u)}{2u}, \quad u \in \mathbb{R}.$$

Obviously, g_1 and g_2 are even rational functions such that $g(u) = g_1(u) + ug_2(u)$. We have

$$f(s) = g_1(\sqrt{s^{-1} - 1}) + g_2(\sqrt{s^{-1} - 1}) \cdot \sqrt{s^{-1} - 1}, \quad s \in (0, 1).$$

Since g_k are even rational functions, it follows that the functions

$$s \to g_k(\sqrt{s^{-1} - 1}), \quad s \in (0, 1),$$

admit rational extensions R_k. Thus,

$$f(s) = R_1(s) + R_2(s) \cdot \sqrt{s^{-1} - 1}, \quad s \in (0, 1).$$

Similarly, consider a function $h_2(t) = \frac{2}{1+t^2}$. There exist rational functions R_3 and R_4 such that

$$f(s) = R_3(s) + R_4(s) \cdot \sqrt{2s^{-1} - 1}, \quad s \in (0,2).$$

Combining the last 2 equalities, we obtain

$$(R_1 - R_3)(s) + R_2(s) \cdot \sqrt{s^{-1} - 1} = R_4(s) \cdot \sqrt{2s^{-1} - 1}, \quad s \in (0,1).$$

Taking squares, we obtain

$$R_6(s) \cdot \sqrt{s^{-1} - 1} = R_5(s), \quad s \in (0,1),$$

where

$$R_5(s) = R_4^2(s) \cdot (2s^{-1} - 1) - (R_1 - R_3)^2(s) - R_2^2(s) \cdot (s^{-1} - 1),$$
$$R_6(s) = 2(R_1 - R_3)(s)R_2(s), \quad s \in \mathbb{R}.$$

Since both R_5 and R_6 are rational functions and since $s \to \sqrt{s^{-1} - 1}$, $s \in (0,1)$, is not, it follows that $R_5 = R_6 = 0$. Thus, $R_2 \cdot (R_1 - R_3) = 0$.

If $R_1 = R_3$, then

$$R_2(s) \cdot \sqrt{s^{-1} - 1} = R_4(s) \cdot \sqrt{2s^{-1} - 1}, \quad s \in (0,1).$$

Since both R_2 and R_4 are rational functions and since $s \to \sqrt{\frac{s^{-1}-1}{2s^{-1}-1}}$, $s \in (0,1)$, is not, it follows that $R_2 = R_4 = 0$.

Hence, $f(s) = R_1(s)$, $s \in (0,1)$. In other words, f is rational on the interval $(0,1)$. Similarly, one can show it is rational on every interval $(0,\alpha)$, $\alpha > 0$, and an every interval $(-\alpha, 0)$, $\alpha > 0$. Thus, f is rational on $\mathbb{R}_+$ and on $\mathbb{R}_-$.

Let f_+ and f_- be rational functions such that $f = f_+$ on $\mathbb{R}_+$ and $f = f_-$ on $\mathbb{R}_-$. By subtracting f_- from f, we may assume without loss of generality that $f_- = 0$. Consider the function $h_3(t) = \frac{t}{1+t^2}$. By assumption, $f \circ h_3 = (f_+ \circ h_3) \cdot \chi_{(0,\infty)}$ is rational on $\mathbb{R}$. However, every rational function with infinitely many zeroes is 0. Hence, $f_+ \circ h_3 = 0$. Thus, $f_+ = 0$ and, therefore in this case we have $f = 0$. This proves that f is rational. □

The proof of Theorem 1.1 follows very similar lines to the methods of Aleksandrov and Peller [1, Section 9].

Proof of Theorem 1.1. We may assume without loss of generality that $f(0) = 0$. Let h be a bounded real-valued rational function on $\mathbb{R}$, and let

$$X = HM_hH, \quad Y = M_h.$$

We have that:

$$X - Y = HM_hH - M_h = [H, M_h]H.$$

Since h is rational, $[H, M_h]$ has finite rank due to Proposition 3.1, and so $X - Y$ has finite rank. Thus, $f(|X - Y|)$ has finite rank.

Recall that H is unitary. It follows that for any Borel function f we have:

$$f(X) = HM_{f\circ h}H.$$

Thus,

$$f(X) - f(Y) = HM_{f\circ h}H - M_{f\circ h} = [H, M_{f\circ h}]H.$$

By assumption, $f(X) - f(Y)$ has finite rank, and thus $[H, M_{f\circ h}]$ has finite rank. Therefore the "only if" component of Proposition 3.1 implies that $f \circ h$ is rational. Since h is an arbitrary bounded rational function, it follows from Lemma 3.2 that f is itself rational. □

4. Proof of Theorem 1.2

We now explain how Theorem 1.2 follows from Theorem 1.1. Lemma 2.2 implies that if f is an operator monotone function satisfying Ando's inequality for uniform submajorization then f has the property that if $f(|X - Y|)$ has finite rank then $|f(X) - f(Y)|$ has finite rank for all bounded positive linear operators X and Y.

Note that if $X - Y$ has finite rank, then $f(|X - Y|)$ also has finite rank. It now follows from Theorem 1.1 that f is rational. However, a recent result of Nagisa [13] states that a rational operator monotone function necessarily has the form (1.1).

Remark 4.1. If $(\mathcal{M}, \tau)$ is an infinite semifinite factor (that is, the centre of $\mathcal{M}$ is trivial and $\tau(1) = \infty$), then there exists a subalgebra $\mathcal{N} \subseteq \mathcal{M}$ and a unitary equivalence $\mathcal{N} \approx B(L_2(\mathbb{R}))$ (this essentially follows from [18, Theorem 4.22]). Hence, if f obeys Ando's inequality for uniform submajorization in the von Neumann algebra $\mathcal{M}$, then by restricting to the subalgebra $\mathcal{N}$ it follows that f obeys Ando's inequality for uniform submajorization in $B(\mathcal{H})$, and hence by the above argument f has the form (1.1).

To complete the proof, we only need to prove the converse statement: if f has the form (1.1) then f obeys Ando's inequality for uniform submajorization. For the sake of maximal generality and also notational simplicity, we prove this theorem in the language of semifinite von Neumann algebras.

We recall that $\tau(\mathrm{supp}(X + Y)) \leq \tau(\mathrm{supp}(X)) + \tau(\mathrm{supp}(Y))$ for $X, Y \in S(\mathcal{M}, \tau)$, $\tau(\mathrm{supp}(ZX)) \leq \tau(\mathrm{supp}(X))$ if $Z \in S(\mathcal{M}, \tau)$ and $\tau(\mathrm{supp}(XZ)) \leq \tau(\mathrm{supp}(X))$ if $Z = Z^* \in \mathcal{M}$ is invertible.

Lemma 4.2. *If f has the form (1.1) and $0 \leq A, B \in \mathcal{M}$ are self-adjoint operators such that $\tau(\mathrm{supp}(A - B))$ is finite, then:*

$$\tau(\mathrm{supp}(f(A)-f(B))) \leq d \cdot \tau(\mathrm{supp}(A-B)).$$

(Recall that d is the number of summands in (1.1).)

Proof. By definition we have

$$f(A)-f(B)=\beta(A-B)-\sum_{j=1}^{d-1}\gamma_j\delta_j(A+\delta_j)^{-1}(A-B)(B+\delta_j)^{-1}.$$

Therefore,

$$\begin{aligned}\tau(\mathrm{supp}(f(A)-f(B))) &\leq \tau(\mathrm{supp}(A-B))+\sum_{j=1}^{d-1}\tau(\mathrm{supp}(A-B))\\ &= d\cdot\tau(\mathrm{supp}(A-B)). \quad \square\end{aligned}$$

The following assertion is standard (compare it with e.g. Lemma 1.3 in [4], [6]). For convenience of the reader, we provide a direct argument. For technical reasons, we work over an atomless von Neumann algebra. Recall that a semifinite von Neumann algebra $\mathcal{M}$ is called atomless if $\mathcal{M}$ contains no non-zero minimal projections. Any von Neumann algebra (in particular, $B(\mathcal{H})$) may be embedded into an atomless von Neumann algebra in a way which preserves the trace. For example, we may consider the embedding $\mathcal{M} \hookrightarrow \mathcal{M}\otimes L_\infty(0,1)$, where $L_\infty(0,1)$ is considered as an Abelian von Neumann algebra with finite trace given the Lebesgue integral, and $\mathcal{M}$ is mapped into the first tensor factor.

The same argument in [6, Lemma 2.5] yields the following lemma (see Lemma 7.7, Chapter III, in the forthcoming book [7] for a complete proof).

Lemma 4.3. *Suppose that the von Neumann algebra $\mathcal{M}$ is non-atomic and $0 \leq T \in S(\mathcal{M},\tau)$. Let $\lambda=\mu(t,T)$ for some $t\in(0,\infty)$ satisfying $t\leq\tau(1)$. If $\lambda>\mu_\infty:=\mu(\infty;T)$ or $\lambda=0$, then there exists $e\in P(\mathcal{M})$ such that*

$$e^T(\lambda,\infty)\leq e\leq e^T[\lambda,\infty) \quad \text{and} \quad \tau(e)=t.$$

The following lemma is folklore (see e.g. [6,7] for the case of finite von Neumann algebras). We give a full proof below for completeness in the setting of general semifinite von Neumann algebras.

Lemma 4.4. *Let $\mathcal{M}$ be an atomless von Neumann algebra. If $T=T^*\in\mathcal{M}$ and if $t\in(0,\infty)$, then there exists $e\in\mathcal{P}(\mathcal{M})$ such that $\tau(1-e)=t$ and*

$$\mu(s,eTe)=\mu(s+t,T), \quad s>0.$$

Moreover, if $\mu(t;T)>\mu(0;T)$ or $\mu(t;T)=0$, then e can be chosen to be commuting with T.

Proof. Assume that $T \geq 0$. Let $\lambda = \mu(t, T)$.

Step **1**. Note that if a projection e commutes with T, then $\mu(eTe) = \mu(Te) = \mu(|T|e)$. Hence, it suffices to prove the case when T is positive.

If $\mu(t, T) > \mu(\infty, T)$ or $\mu(t, T) = 0$. By Lemma 4.3, there exists a projection $q \in \mathcal{M}$ such that

$$e^T(\lambda, \infty) \leq q \leq e^T[\lambda, \infty) \quad \text{and} \quad \tau(q) = t.$$

This implies that q commutes with T.

In particular, for any Borel set $B \in \mathcal{B}(\mathbb{R})$, we have

$$e^{T(1-q)}(B) = (1-q)\, e^T(B) = e^T(B)\,(1-q),$$

and hence,

$$e^{T(1-q)}(r, \infty) = (1-q)\, e^T(r, \infty) = \begin{cases} e^T(r, \infty) - q & \text{if } r < \lambda \\ 0 & \text{if } r \geq \lambda \end{cases}.$$

It follows that

$$d(r; T(1-q)) = \begin{cases} d(r; T) - \tau(q) & \text{if } r < \lambda \\ 0 & \text{if } r \geq \lambda \end{cases}. \tag{4.1}$$

Observe that (4.1) may also be written as $d(r; T(1-q)) = (d(r; T) - \tau(q))^+$. Indeed, if $r < \lambda$, then $q \leq e^T[\lambda, \infty) \leq e^T(r, \infty)$ and so, $d(r; T) - \tau(q) \geq 0$. If $r \geq \lambda$, then $e^T(r, \infty) \leq e^T(\lambda, \infty) \leq q$ and so, $d(r; T) - \tau(q) \leq 0$. Consequently, if $0 \leq t < \tau(q^\perp)$, then

$$\begin{aligned} \mu(t; T(1-q)) &= \inf\{r \in \mathbb{R} : d(r; T(1-q)) \leq t\} \\ &= \inf\left\{r \in \mathbb{R} : (d(r; T) - \tau(q))^+ \leq t\right\} \\ &= \inf\{r \in \mathbb{R} : d(r; T) - \tau(q) \leq t\} = \mu(t + \tau(q); T). \end{aligned}$$

It suffices to take e as $1 - q$.

Step **2**. Assume that $\mu(t, T) = \mu(\infty, T)$. Recall that $\mu(\infty, T)$ is also given by

$$\mu_\infty := \inf\{s \geq 0 : d(s; |T|) < \infty\}. \tag{4.2}$$

It follows that for any $\varepsilon > 0$,

$$d(\mu_\infty - \varepsilon; |T|) = \infty.$$

Since $\mu(t, T) = \mu_\infty$, it follows that $d(\mu_\infty; |T|) < \infty$. Hence, for any $\varepsilon > 0$, we have

$$\tau(e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty]) = d(\mu_\infty - \varepsilon; |T|) - d(\mu_\infty; |T|) = \infty. \tag{4.3}$$

By $d(|T|) = d(\mu(T))$, we have (see also [7, Lemma 7.7 (ii)])

$$d(\mu_\infty; |T|) \le t.$$

Let $a := t - d(\mu_\infty; |T|) \ge 0$.

(i). If $\tau(e^{|T|}\{\mu_\infty\}) = \infty$, then we may take p as a subprojection of $e^{|T|}\{\mu_\infty\}$ with $\tau(p) = a$. In particular, p commutes with T. Take $e = 1 - (p + e^{|T|}(\mu_\infty, \infty))$. It is clear that $\mu(s+t; T) = \mu(s; e|T|e) = \mu(s; eTe) = \mu_\infty$ for any $s > 0$.

(ii). Now, assume that $\tau(e^{|T|}\{\mu_\infty\}) < \infty$. Then, for any $\varepsilon > 0$, by (4.3), we have

$$\tau(e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty)) = \infty. \tag{4.4}$$

In particular, for a fixed ε, there exists a sufficiently small $\delta > 0$ such that

$$\tau(e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty - \delta)) \ge a.$$

Since $\mathcal{M}$ is atomless, it follows that we can take a subprojection p of $e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty - \delta)$ with $\tau(p) = a$. Define

$$e = 1 - (p + e^{|T|}(\mu_\infty, \infty)) = e^{|T|}[0, \mu_\infty] - p.$$

Note that $p_1 := e^{|T|}[\mu_\infty - \delta, \mu_\infty]$, $p_2 := e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty - \delta) - p$ and $p_3 := e^{|T|}[0, \mu_\infty - \varepsilon]$ are pairwise orthogonal. Moreover, since p_1, p_3 commute with T, it follows that

$$\begin{aligned}
eTe &= (e^{|T|}[0, \mu_\infty] - p)T(e^{|T|}[0, \mu_\infty] - p)\\
&= (p_1 + p_2 + p_3)T(p_1 + p_2 + p_3)\\
&= (p_1 + p_2 + p_3)Tp_1 + (p_1 + p_2 + p_3)Tp_2 + (p_1 + p_2 + p_3)Tp_3\\
&= (p_1 + p_2 + p_3)p_1Tp_1 + (p_1 + p_2 + p_3)Tp_2 + (p_1 + p_2 + p_3)p_3Tp_3\\
&= p_1Tp_1 + (p_1 + p_2 + p_3)Tp_2 + p_3Tp_3\\
&= p_1Tp_1 + p_1Tp_2 + p_2Tp_2 + p_3Tp_2 + p_3Tp_3\\
&= p_1Tp_1 + p_2Tp_2 + p_3Tp_3\\
&= e^{|T|}[\mu_\infty - \delta, \mu_\infty]Te^{|T|}[\mu_\infty - \delta, \mu_\infty]\\
&\quad + (e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty - \delta) - p)T(e^{|T|}(\mu_\infty - \varepsilon, \mu_\infty - \delta) - p)\\
&\quad + e^{|T|}[0, \mu_\infty - \varepsilon]Te^{|T|}[0, \mu_\infty - \varepsilon].
\end{aligned}$$

Note that $\tau(e^{Te^{|T|}[\mu_\infty - \delta, \mu_\infty]}[\mu_\infty - \delta', \mu_\infty]) = \tau(e^{|T|}[\mu_\infty - \delta', \mu_\infty]) \overset{(4.4)}{=} \infty$ for any positive number $\delta' < \delta$. Hence, by the definition of singular value function in terms of distribution function, we obtain that $\mu(Te^{|T|}[\mu_\infty - \delta, \mu_\infty]) = \mu_\infty$ is a constant on $\mathbb{R}_+$. Recall that p_1, p_2, p_3 are pairwise orthogonal. Hence, the supports of

$$p_1Tp_1, \ p_2Tp_2, \ \text{and } p_3Tp_3$$

are pairwise disjoint. Moreover, $\|p_2Tp_2\|_\infty, \|p_3Tp_3\|_\infty \leq \mu_\infty - \delta$. Hence, for any $\theta > 0$, we have

$$d(\mu_\infty - \delta + \theta; eTe) = d(\mu_\infty - \delta + \theta; p_1Tp_1 \oplus p_2Tp_2 \oplus p_3Tp_3) = d(\mu_\infty - \delta + \theta; p_1Tp_1).$$

Since $\mu(p_1Tp_1) \overset{(4.4)}{=} \mu_\infty$, it follows that $\mu(eTe) = \mu_\infty$.

The proof is complete. $\square$

Theorem 4.5. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra, and let f have the form (1.1). Then for all $0 \leq A, B \in \mathcal{M}$ we have:*

$$|f(A) - f(B)| \lhd f(|A - B|)$$

with $n \leq 4d$.

Proof. Without loss of generality $\mathcal{M}$ is atomless. Fix $0 < a, b < \infty$ such that $4da < b$. Using Lemma 4.4 select a projection p such that $\tau(1 - p) = a$ and such that

$$\mu(t, p(A - B)p) = \mu(t + a, A - B), \quad t > 0.$$

Since f is increasing, it follows that

$$\mu(f(|pAp - pBp|)) = f(\mu(|pAp - pBp|)) = f(\mu(p(A - B)p)).$$

Thus,

$$\mu(t, f(|pAp - pBp|)) = \mu(t + a, f(|A - B|)), \quad t > 0. \tag{4.5}$$

Now, we write

$$f(A) - f(B) = (f(pAp) - f(pBp)) + (f(A) - f(pAp)) + (f(pBp) - f(B)).$$

Using (2.1), we have:

$$\begin{aligned}\mu(t, f(A) - f(B)) \leq \mu(t - 4da, f(pAp) - f(pBp)) + \mu(2da, f(A) - f(pAp)) \\ + \mu(2da, f(B) - f(pBp)).\end{aligned}$$

By writing

$$A - pAp = A(1-p) + (1-p)Ap$$

it is clear that $\tau(\mathrm{supp}(A - pAp)) \leq 2a$. Therefore by Lemma 4.2, $\tau(\mathrm{supp}(f(A) - f(pAp))) \leq 2da$. Thus,

$$\mu(2da, f(A) - f(pAp)) = 0 \text{ and, similarly, } \mu(2da, f(B) - f(pBp)) = 0.$$

We conclude that

$$\mu(t, f(A) - f(B)) \leq \mu(t - 4da, f(pAp) - f(pBp)), \quad t > 4da.$$

Hence,

$$\begin{aligned}\int_{4da}^{b} \mu(s, f(A) - f(B))\, ds &\leq \int_{4da}^{b} \mu(s - 4da, f(pAp) - f(pBp)) ds \\ &= \int_{0}^{b-4da} \mu(s, f(pAp) - f(pBp)) ds.\end{aligned}$$

By Ando's inequality in the semifinite setting (2.2), we have

$$f(pAp) - f(pBp) \prec\prec f(|pAp - pBp|).$$

Thus,

$$\begin{aligned}\int_{4da}^{b} \mu(s, f(A) - f(B)) ds &\leq \int_{0}^{b-4da} \mu(s, f(|pAp - pBp|)) ds \\ &\overset{(4.5)}{=} \int_{0}^{b-4da} \mu(s + a, f(|A - B|)) ds \\ &\leq \int_{a}^{b} \mu(s, f(|A - B|)) ds. \quad \square\end{aligned}$$

5. Consequences of Ando's inequality for uniform submajorization

While the class of operator monotone functions satisfying Ando's inequality for uniform submajorization is quite small, it is worthwhile to explore the consequences of the inequality in the cases where it holds.

5.1. Inequalities for symmetric norms

A Banach subspace $(E, \|\cdot\|_E)$ of $S(\mathcal{M}, \tau)$ is said to be a symmetric operator space if $B \in E$, $A \in S(\mathcal{M}, \tau)$ and $\mu(A) \leq \mu(B)$ implies that $A \in E$ and $\|A\|_E \leq \|B\|_E$. A symmetric operator space $(E, \|\cdot\|_E)$ is called *fully symmetric* if $A \in S(\mathcal{M}, \tau)$, $B \in E$ and $A \prec\prec B$ implies that $A \in E$ and $\|A\|_E \leq \|B\|_E$ (for these and related notions see [8, Section 4] and [12, Section 2.5]). Fully symmetric operator spaces are precisely those symmetric operator spaces which are exact interpolation spaces for the couple $(L_1(\mathcal{M}, \tau), \mathcal{M})$. Ando's inequality (2.2) implies that $\|f(A) - f(B)\|_E \leq \|f(|A - B|)\|_E$ for operator monotone f and fully symmetric operator spaces E. However, there are symmetric operator spaces which are not exact interpolation spaces for the couple $(L_1(\mathcal{M}, \tau), \mathcal{M})$ and therefore not fully symmetric [11,16].

As indicated in the introduction, $A \lhd B$ implies that if $B \in E$ then $\|A\|_E \leq \|B\|_E$ for all symmetric operator spaces $(E, \|\cdot\|_E)$ (see [12, Corollary 3.4.3]). Therefore, Theorem 4.5 implies the following strengthening of Ando's inequality for functions of the form (1.1):

Theorem 5.1. *Let f be a function of the form* (1.1)*. Then for all symmetric operator spaces $(E, \|\cdot\|_E)$, we have*

$$\|f(A) - f(B)\|_E \leq \|f(|A - B|)\|_E.$$

5.2. Reverse inequalities

In Theorem 2 of [2], Ando proved that if g is the inverse to an operator monotone function, then for any fully symmetric norm $\|\cdot\|$ and all positive operators A and B we have

$$\|g(|A - B|)\| \leq \|g(A) - g(B)\|.$$

See also [17, Theorem 43.2].

The following assertion is known (see e.g. [17, Equation (43.35)]), but we supply a short proof for convenience.

Proposition 5.2. *Let $f : [0, \infty) \to [0, \infty)$ be operator monotone and invertible. Then f^{-1} is convex.*

Proof. Since f is operator monotone, it follows firstly that f is smooth on $(0, \infty)$ [17, Theorem 5.2] and secondly that f is concave [17, Corollary 14.5].

Let $g = f^{-1}$. Then since $g(f(x)) = x$, we have:

$$g'(f(x))f'(x) = 1, \quad g''(f(x))f'(x)^2 + g'(f(x))f''(x) = 0.$$

Since $f' \geq 0$ and $f'' \leq 0$, it follows that $g''(f(x)) \geq 0$, and hence $g = f^{-1}$ is convex. □

The following proposition shows that increasing convex functions are "almost" monotone under uniform submajorization.

Proposition 5.3. *Let $0 \leq A, B \in \mathcal{M}$ (where $(\mathcal{M}, \tau)$ is an arbitrary semifinite von Neumann algebra), and let $F : [0, \infty) \to [0, \infty)$ be an increasing convex function. If $A \lhd B$, then for any $\varepsilon > 0$ we have $F(\frac{A}{1+\varepsilon}) \lhd F(B)$.*

Proof. Note that $A \lhd B$ in the von Neumann algebra $\mathcal{M}$ if and only if $\mu(A) \lhd \mu(B)$ in the von Neumann algebra $L_\infty(0, \infty)$ with the Lebesgue integral as trace.

A special case of [11, Example, page 100] (or [12, Theorem 3.4.2]), states that if $\mu(A) \lhd \mu(B)$ then for any $\varepsilon > 0$ there exist non-negative $\lambda_1, \dots, \lambda_N$ such that $\sum_{k=1}^N \lambda_k = 1 + \varepsilon$, and elements $Y_1 \dots, Y_N \in L_\infty(0, \infty)$ such that

$$\mu(A) = \sum_{k=1}^N \lambda_k Y_k$$

and

$$\mu(Y_k) \leq \mu(B), \quad k = 1, \dots, N$$

Hence, $\mu(A)(1 + \varepsilon)^{-1}$ is a convex combination of $\{Y_1, \dots, Y_N\} \subset L_\infty(0, \infty)$. Since F is convex (Lemma 5.2), we have

$$F\left(\frac{\mu(A)}{1+\varepsilon}\right) \leq \sum_{k=1}^N \frac{\lambda_k}{1+\varepsilon} F(\mu(Y_k)).$$

Since F is monotone, $F(Y_k) \leq F(\mu(B))$ and therefore

$$F\left(\frac{\mu(A)}{1+\varepsilon}\right) \leq \sum_{k=1}^N \frac{\lambda_k}{1+\varepsilon} F(\mu(B)) = F(\mu(B)).$$

Since F is a positive monotone function, we have $\mu(F(A(1+\varepsilon)^{-1})) = F(\mu(A)(1+\varepsilon)^{-1})$ and $\mu(F(B)) = F(\mu(B))$. Therefore,

$$\mu\left(F\left(\frac{A}{1+\varepsilon}\right)\right) \leq \mu(F(B)),$$

and hence

$$F\left(\frac{A}{1+\varepsilon}\right) \lhd F(B)$$

with $n = 1$. $\square$

Consider a function f of the form (1.1), and where f is also invertible. It follows from Proposition 5.2 that f^{-1} is convex, and hence Proposition 5.3 implies that if $A \lhd B$ then $f^{-1}(\frac{A}{1+\varepsilon}) \lhd f^{-1}(B)$ for all $\varepsilon > 0$. This is the key component in the following result, which is a version of Theorem 2 in Ando's paper [2].

Theorem 5.4. *Let f be an invertible function of the form (1.1). Then for all bounded positive operators A and B and all $\varepsilon > 0$, we have*

$$f^{-1}\left(\frac{|A-B|}{1+\varepsilon}\right) \lhd |f^{-1}(A) - f^{-1}(B)|.$$

Proof. For bounded positive operators X and Y, Theorem 4.5 implies that:

$$|f(X) - f(Y)| \lhd f(|X-Y|).$$

Inserting $X = f^{-1}(A)$ and $Y = f^{-1}(B)$ yields:

$$|A-B| \lhd f(|f^{-1}(A) - f^{-1}(B)|).$$

Since f^{-1} is convex (Proposition 5.2), Proposition 5.3 implies that for every $\varepsilon > 0$ we have

$$f^{-1}\left(\frac{|A-B|}{1+\varepsilon}\right) \lhd f^{-1}(f(|f^{-1}(A) - f^{-1}(B)|)) = |f^{-1}(A) - f^{-1}(B)|. \quad \square$$

Note that the function f^{-1} in the above theorem may be irrational. For example, if

$$f(t) = t + \frac{t}{1+t} = \frac{t^2+2t}{1+t}, \quad t \geq 0$$

then

$$f^{-1}(s) = \frac{1}{2}(s - 2 + (s^2 + 2s + 4)^{1/2}).$$

As in the previous subsection, it follows that for any $\varepsilon > 0$ and any symmetric operator space $(E, \|\cdot\|_E)$ we have the inequality

$$\left\| f^{-1}\left(\frac{|A-B|}{1+\varepsilon}\right)\right\|_E \leq \|f^{-1}(A) - f^{-1}(B)\|_E.$$

5.3. Inequalities for commutators

We now discuss a series of interesting results due to Bhatia and Kittaneh [3] closely related to Ando's inequality.

Theorem 1 of [3] can be restated in our language as the following: if f is an operator monotone function, then for all $X \in M_n(\mathbb{C})$ and all $0 \le A, B \in M_n(\mathbb{C})$, we have the submajorization:

$$f(A)X - Xf(B) \prec\prec \frac{1+\mu(0,X)^2}{2} f\left(\frac{2}{1+\mu(n-1,X)^2}|AX - XB|\right).$$

We note the following property of $\mu(\infty, T)$: if $f : [0,\infty) \to [0,\infty)$ is a decreasing function and $T \ge 0$, then we have:

$$\mu(0, f(T)) = f(\mu(\infty, T)). \tag{5.1}$$

This follows from (4.2).

Remark 5.5. Strictly speaking the results of [3] were stated as an inequality for all unitarily invariant norms on $M_n(\mathbb{C})$. However, in the finite dimensional setting all unitarily invariant norms are fully symmetric, and therefore the result can equivalently be restated as a submajorization.

The following lemma is a variant of [3, Lemma 4], stated for uniform submajorization.

Lemma 5.6. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra, and let f be a function of the form (1.1). Let $0 \le A, B \in \mathcal{M}$, and let $U \in \mathcal{M}$ be unitary.*

$$|f(A)U - Uf(A)| \lhd f(|AU - UA|).$$

Proof. Let $B = UAU^*$. Then by Theorem 4.5, we have:

$$|f(A) - f(B)| \lhd f(|A - B|) \Rightarrow f(A) - f(UAU^*) \lhd f(|A - UAU^*|).$$

Since $f(UAU^*) = Uf(A)U^*$ and $|A - UAU^*| = |AU - UA|$, it follows that:

$$|f(A) - Uf(A)U^*| \lhd f(|AU - UA|)$$

and therefore $|f(A)U - Uf(A)| \lhd f(|AU - UA|)$. □

Now we provide a variant of [3, Theorem 1]. The proof is essentially the same as that given in [3].

Corollary 5.7. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. For all functions f of the form (1.1), and all $0 \le A, B \in \mathcal{M}$ and $X \in \mathcal{M}$, we have:*

$$|f(A)X - Xf(B)| \lhd \frac{2}{1+\mu(0,X)^2} f\left(\frac{2}{1+\mu(\infty,X)^2}|AX - XB|\right).$$

Proof. Initially consider the case $X = X^*$ and $A = B$, and define:

$$U = \frac{X+i}{X-i} = 1 + 2i(X-i)^{-1}.$$

That is, $X = i\frac{1+U}{1-U} = 2i(1-U)^{-1} - i$. Since U is unitary, Lemma 5.6 entails:

$$f(A)U - Uf(A) \lhd f(|AU - UA|). \tag{5.2}$$

We have:

$$\begin{aligned} f(A)X - Xf(A) &= 2if(A)(1-U)^{-1} - 2i(1-U)^{-1}f(A) \\ &= 2i(1-U)^{-1}(Uf(A) - f(A)U)(1-U)^{-1}. \end{aligned}$$

Therefore,

$$\mu(f(A)X - Xf(A)) \leq 2\|(1-U)^{-1}\|^2 \mu(f(A)U - Uf(A)).$$

However, we can compute the norm of $(1-U)^{-1}$ as:

$$\|(1-U)^{-1}\| = \left\| \frac{X+i}{2} \right\|.$$

Since X is self-adjoint, we have:

$$\left\| \frac{X+i}{2} \right\|^2 = \frac{\|X\|^2 + 1}{4} = \frac{\mu(0,X)^2 + 1}{4}.$$

Therefore,

$$\mu(f(A)X - Xf(A)) \leq \frac{\mu(0,X)^2 + 1}{2} \mu(f(A)U - Uf(A)).$$

In particular,

$$f(A)X - Xf(A) \lhd \frac{\mu(0,X)^2 + 1}{2} (f(A)U - Uf(A)).$$

Combined with (5.2), we have:

$$f(A)X - Xf(A) \lhd \frac{\mu(0,X)^2 + 1}{2} f(|AU - UA|).$$

Using $U = 1 + 2i(X-i)^{-1}$, the commutator $AU - UA$ becomes

$$\begin{aligned} AU - UA &= 2iA(X-i)^{-1} - 2i(X-i)^{-1}A \\ &= 2i(X-i)^{-1}(XA - AX)(X-i)^{-1}. \end{aligned}$$

Therefore,

$$\mu(AU - UA) \leq 2\|(X-i)^{-1}\|^2 \mu(AX - XA).$$

By the C^*-identity:

$$\|(X-i)^{-1}\| = \|(X+i)^{-1}(X-i)^{-1}\|^{1/2} = \|(X^2+1)^{-1}\|^{1/2}.$$

However, note that (5.1) implies

$$\|(|X|^2+1)^{-1}\| = \frac{1}{\mu(\infty, X)^2 + 1}.$$

Therefore,

$$\mu(AU - UA) \leq \frac{2}{\mu(\infty, X)^2 + 1} \mu(AX - XA).$$

Since f is monotone,

$$\begin{aligned} \mu(f(|\mu(AU - UA)|)) &\leq f(\mu(AU - UA)) \\ &\leq f\left(\frac{2}{1+\mu(\infty, X)^2}\mu(AX - XA)\right) \\ &= f\left(\frac{2}{1+\mu(\infty, X)^2}|AX - XA|\right). \end{aligned}$$

This completes the proof in the case where $X = X^*$ and $A = B$.

The case $A \neq B$ may be deduced by substituting for A and X the elements

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in M_2(\mathbb{C}) \otimes \mathcal{M}$$

respectively. □

Declaration of competing interest

No competing interest.

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